Bolzano's Revival of the Classical Model of Science

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I. Introduction

Clearly the most consistent theme throughout the works of Bolzano, from his first publication in 1804 to his magisterial *Theory of Science* (TS) in 1837, is the distinction between proofs that prove and proofs that explain. While scientific and mathematical proofs in Bolzano's time proved scientific conclusions against standards of certainty and obviousness, Bolzano called for a new presentation of these conclusions that explained their grounds from a more objective point of view. Throughout, Bolzano mentions the classical model of science, with primary reference to Aristotle, as a key source for the types of scientific presentation needed in his day.

The classical heritage of Bolzano's theory of science is underdeveloped by most commentators. An appreciation of the classical heritage of Bolzano's work helps to reveal and explain two critical elements of Bolzano's logic that are generally presented in terms of Fregean concerns alien to Bolzano. These two elements are subjective, psychologistic proofs and propositions-in-themselves. First, incorporation of subjective proofs into, and not expelling them from, logic was Bolzano's mission. This is contrary to most interpretations of Bolzano that see him attacking and expelling subjective, psychologistic proofs from logic, an interpretation that is truer of Frege than it is of Bolzano.

Second, this incorporation of subjective proofs into logic occurs when one analyzes and transforms the intuitively presented propositions of such proofs into propositions which have the character of propositions-in-themselves. Bolzano is often accused of Platonism with regard to his doctrine of propositions-in-themselves, ascribing to them a role similar to Frege's *Gedanken*. However, propositions-in-themselves are not different propositions from spoken propositions; rather, they are the intended meanings that constitute spoken propositions, whose full meanings are generally inaccessible to their speakers.

Both misunderstandings of Bolzano reflect a Fregean reading of Bolzano that does not take seriously the classical references made by Bolzano thoughout his corpus to help the reader better understand his doctrines of subjective proofs and objective propositions-in-themselves. This paper seeks to present these two doctrines in the context of a thorough exploration of the classical references made by Bolzano for these doctrines.

II. Bolzano's Revival of the Classical Model of Science

Bolzano opens the preface to his 1804 work, *Considerations on Some Objects of Elementary Geometry*, with a statement of the two rules that would govern his presentation. We find in these two rules one of the first entrees into modern philosophy of the concern about psychologism as well as the response, an objective science of necessary connections grounded in axioms. "*Firstly*, I propose for myself the rule that the *obviousness of a proposition* does not free me from the obligation to continue searching for a proof of it…*Secondly*, I must point out that I believed I could never be satisfied with a completely strict proof *if it were not derived from the same concepts* which the thesis to be proved contained, but rather made use of some fortuitous, alien, *intermediate concept*, which is always an erroneous *metabasis eis allo genos* [crossing to another kind]".[1]

Bolzano's first rule is found in all of Bolzano's works including TS, which distinguishes "between proofs of the mere *hoti* (establishing certainty) and of the *dioti* (grounding)"[2]. This rule is not intended to be a criticism of mathematical science which aims at certainty, but a concern that modern mathematical science is incomplete. Proofs that establish certainty call for a subsequent formulation that reveals their objective grounds.

This objectivity criterion of any *a priori* science leads to Bolzano's second rule that proofs not cross to another kind. Bolzano's use of the Greek, in reference to Aristotle's requirement that "[o]ne cannot, therefore, prove by crossing from another kind – e.g. something geometrical by arithmetic"[3], is repeated in his subsequent works on mathematics in 1810 and 1817.[4] In other words, objective causal connections can only be made between

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[1] Bernard Bolzano, "Considerations on Some Objects of Elementary Geometry," in *The Mathematical Works of Bernard Bolzano*, ed. Steve Russ (Oxford: Oxford University Press, 2004), 31-32.

[2] Quoted in Paolo Mancosu, "On Mathematical Explanation," in *The Growth of Mathematical Knowledge*, ed. Emily Grosholz and Herbert Breger (Dordrecht, The Netherlands: Kluwer Academic Publishers, 2000), 115. Paolo Mancosu's studies of mathematical explanation across the history of mathematics has done much to reveal the close relation between Aristotle and Bolzano in their discussions of explanatory, causal proofs. This is fundamentally because Mancosu "take[s] seriously Bolzano's assertion that the distinction between certain and grounding proofs is the same as that between *hoti* and *dioti* proofs in Aristotle." Paolo Mancosu, "Bolzano and Cournot on Mathematical Explanation," *Revue d'Histoire des Sciences* 52 (1999): 429-455. See Mancosu's *Philosophy of Mathematics and Mathematical Practice in the Seventeenth Century* (New York, NY: Oxford University Press, 1996).

[3] Aristotle, *Posterior Analytics*, 75a38. trans. Jonathan Barnes, in *The Complete Works of Aristotle Vol 1*, ed. Jonathan Barnes (Princeton: Princeton University Press, 1984).

[4] See Bernard Bolzano, "Contributions to a Better-Grounded Presentation of Mathematics," in *The Mathematical Works of Bernard Bolzano*, ed. Steve Russ (Oxford: Oxford University Press, 2004), 126 and

axioms and propositions within the same science. Bolzano is not opposed to proofs that establish certainty by crossing from one science to another. In fact, crossing from one science to another in a proof is characteristic of proofs that establish a fact with certainty.

These two rules that govern the presentation of his 1804 work thus appeal explicitly to two central principles of Aristotelian logic, the *hoti-dioti* distinction and the observation that *hoti* proofs typically cross from more intuitive sciences to prove the certainty of propositions but *dioti* proofs can never cross from one kind to another.

The first complete statement of Bolzano's intention to reintroduce a classical model of science in order to complete the more psychologistic expositions in contemporary science is found in his 1810 *Contributions to a Better-Grounded Presentation of Mathematics.* [5] Bolzano would make the same type of statement in each of his subsequent works, including TS, in which he makes the further point that just as modern science often overlooks the

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Bernard Bolzano, "Purely Analytic Proof of the Theorem, that between any two Values, which give Results of Opposite Sign, there lies at least one real Root of the Equation," in *The Mathematical Works of Bernard Bolzano*, ed. Steve Russ (Oxford: Oxford University Press, 2004), 254. Russ points out that this is not an exact translation, as "Bolzano has έις (to), where the text in Barnes has εξ (from)" in Bolzano, "Considerations on Some Objects of Elementary Geometry," 32.

[5] "But this much seems to me certain: in the realm of truth, i.e. in the collection of all true judgments, a certain *objective connection* prevails which is independent of our accidental and *subjective recognition* of it. As a consequence of this some of these judgments are the grounds of others and the latter are the consequences of the former. Presenting this objective connection of judgments, i.e. choosing a set of judgments and placing them one after another so that a consequence is represented as such and conversely, seems to me the be the real *purpose* to pursue in a scientific exposition. Instead of this, the purpose of a scientific exposition is *usually* imagined to be the greatest possible *certainty* and *strength of conviction*. It therefore happens that the obligation to prove propositions which, in themselves, are already completely certain, is discounted. This is a procedure which, where we are concerned with the practical purpose of certainty, is quite correct and praiseworthy; but it cannot possibly be tolerated in a scientific exposition because it contradicts its essential aim. However, I believe that *Euclid* and his predecessors were in agreement with me and they did not regard the mere *increase in certainty* as any part of the purpose of their method. This can be seen clearly enough from the trouble which these men took to provide many a proposition (which in itself had complete certainty) with a proper *proof*, although it did not thereby become any more certain." Bolzano, "Contributions to a Better-Grounded Presentation of Mathematics," 103.

importance of objective demonstrations, so classical science often overlooked the importance of preliminary proofs that establish certainty.

> Since one so far does not always distinguish clearly the objective ground of a truth from its subjective means of knowledge so it follows automatically that also the grounding proofs cannot always be distinguished exactly from the purely certain proofs. Indeed Aristotle (An. Post. I, 2 and I, 13) and the Scholastics very diligently advanced the division of proofs into those which only show *that* (hoti) something is, and the ones which also show *why* (dioti) something is. They also maintained, with some exaggeration, that only the latter produce a genuine science. However, the new logicians seem to observe this distinction very little.[6]

The claim that classical science exaggerates in asserting that only objectively grounded demonstrations constitute a science is presumably based on statements such as that by Aristotle, who supposed that we "possess unqualified scientific knowledge of a thing, as opposed to knowing it in the accidental way in which the sophist knows, when we think that we know the cause on which the fact depends as the cause of the fact and of no other, and further, that the fact could not be other than it is."[7] Whether Bolzano's critique is balanced or not is not at issue in this paper (that Bolzano's path to demonstrative science by analyzing subjective proofs parallels a very similar path in Aristotle's *Topics* speaks against Bolzano's critique). The points made here are that Bolzano sought to reintroduce the classical model of science by completing, but not replacing, contemporary scientific proofs with grounding proofs that demonstrate and explain modern scientific achievements.[8]

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[6] Quoted in Mancosu, "On Mathematical Explanation," 104.
[7] Aristotle, *Posterior Analytics*, 71b9-11.
[8] As Bolzano would say later in TS that "it is the case that intuitive truths (experiences) are often helpful in the discovery of a purely conceptual truth; but the objective ground of such a truth cannot lie in them; if they have any ground at all, it must lie in other conceptual truths." WL, Sec. 221, p. 284, George translation. .

III. Bolzano's Classicism and the Catholic Debate on the Status of Mathematical Physics

The predominant histories of analytic philosophy do not locate its origin in an attempted revival of the classical model of science. While the texts of Bolzano support this claim, it is buttressed once one considers Bolzano's possible motivations. What motivated an Austrian priest who occupied a chair in the Science of Religion in early 19th century Prague to look to classical philosophy in order to make such innovative and influential arguments in the philosophy of science and mathematics?[9] Bolzano's ideas were not completely original in his era, for a debate concerning the status of mathematical science as a classical science had raged throughout Europe, particularly in Catholic areas such as Prague, for the past couple centuries.

As philosophers and scientists across Catholic Europe came to terms with the Scientific Revolution, a debate emerged that came to be known as the "Quaestio de Certitudine

[9] The importance of this question is stressed by Petr Dvořák and Jacob Schmutz in their article on the most prominent philosophical predecessor to Bolzano, Juan Caramuel Lobkowitz (1606-1682). "Recent historians of the philosophical tradition of *Mitteleuropa* have often considered the work of the Prague logician Bernard Bolzano (1781–1848) as its starting point. His work is at the origin of the various roads taken by authors such as Gottlob Frege, Franz Brentano, Edmund Husserl or Alexius Meinong up to the Vienna Circle, and it can be considered as the common root of the two major trends of contemporary philosophy, the so-called 'continental' tradition, mainly inspired by phenomenology, and the Anglo-American 'analytical' tradition. But this obliterates the fact that Bolzano himself, a Roman Catholic priest schooled in the very late scholastic tradition of the *Katholische Aufklärung*, was himself deeply indebted to numerous early-modern sources, and that most of the problems he is often acknowledged to have introduced into philosophy – negative state of affairs, the distinctions between the intensional and extensional analyses of concepts, the paradoxes of the infinite, and above all his attempt to vindicate realism against Kantian subjectivism – have all their sources in the tradition of Caramuel's Prague." Petr Dvořák and Jacob Schmutz, "Caramuel in Prague: The Intellectual Roots of *Mitteleuropa*," in *Juan Caramuel Lobkowitz: The Last Scholastic Polymath*, ed. Petr Dvořák and Jacob Schmutz (Prague: Filosofia, 2008), 26.

Mathematicarum".[10] The question debated by philosophers and scientists, primarily in Italy, Germany and Austria, was whether mathematics can supply explanatory causes, along the Aristotelian model, of material phenomena. Bolzano's position is that mathematics can supply, but in practice has not supplied, explanatory causes. While Bolzano only occasionally refers to these recent predecessors, preferring to refer directly to Aristotle and Euclid, their arguments are so similar that Bolzano's own support of the classical model of a science of explanatory causes actually appears conventional for his time. The conclusion drawn here is that it was precisely this debate on the status of mathematical science in Catholic Europe that paved the way for the work of Bolzano and subsequent analytic philosophers.[11]

Historically, the mathematical sciences central to the Scientific Revolution had been considered mathematical inasmuch as they explained immaterial, and thus immobile and quantitative, aspects of these domains. This approach to these mathematical sciences followed the three-fold division of the theoretical sciences presented by Aristotle, Boethius

[10] Mancosu, "On Mathematical Explanation", 110.

[11] Neurath acknowledges that analytic thought took root in precisely Catholic parts of Europe and attributes this to the independence of logic from dogmatic metaphysics in Catholic philosophy, which enabled a transition to logical analysis of science. "Catholics accept a compact body of dogma and place it at the beginning of their reflections, [thus] they are sometimes able to devote themselves to systematic logical analysis, unburdened by any metaphysical details….Once someone in the Catholic camp begins to have doubts about a dogma, he can free himself with particular ease from the whole set of dogmas and is then left a very effective logical instrument in his possession. No so in the Lutheran camp, where…many philosophers and scholars from all disciplines, while avoiding a commitment to a clear body of dogma, have retained half-metaphysical or quarter-metaphysical turns of speech, the last remnants of a theology which has not yet been completely superseded….This may explain why the linguistic analysis of unified science prevailed least in countries where the Lutheran faith had dealt the hardest blows to the Catholic church, despite the fact that technology and the sciences that go along with it are highly developed in these countries." Quoted in Barry Smith, "Austria and the Rise of Scientific Philosophy," in *Phenomenology and Analysis: Essays on Central European Philosophy*, ed. Arkadiusz Chrudzimski and Wolfgang Huemer (Frankfurt, Germany: Alexander von Humboldt Foundation, 2004), 47.

and Aquinas, according to which natural philosophy treats of what exists in matter, mathematics treats of what exists in matter but doesn't require matter to be understood and metaphysics treats of what exists without matter and motion.[12] Astronomy, optics and mechanics were considered middle sciences because they can be treated either at one level of abstraction in terms of principles of matter and motion (as natural philosophy), or at a further level of abstraction in terms of principles that don't include matter or motion (as mathematics).[13] While debates occurred as to whether these sciences were more properly considered natural philosophy or mathematics, the distinction between natural philosophy and mathematics was not in doubt (in fact, it was agreement concerning this distinction that sustained the debate over the proper placement of the middle sciences).

The momentous change that is primarily responsible for bringing about the Scientific Revolution, as historians of science such as Edward Grant have documented, was the merging of the middle sciences into natural philosophy. This integration meant that mathematics was no longer restricted to seeking immaterial and immobile first principles, but was also applied to what exists in virtue of material causes.[14] The result of this

[12] This formulations of the threefold division of the theoretical sciences closely follows that described by Aquinas in his *The Division and Methods of the Sciences: Questions V and VII of his Commentary on the De Trinitate of Boetheus*, ed. Armand Maurer (Toronto: Pontifical Institute of Medieval Studies, 1986), 14-15.

[13] "For mathematics is about forms, for its objects are not said of any underlying subject – for even if geometrical objects are said of some underlying subject, still it is not *as* being said of an underlying subject that they are studied." Aristotle, *Posterior Analytics*, I 13, 79a8-10.

[14] As Grant explains while describing what was revolutionary in Newton's *Mathematical Principles of Natural Philosophy*, "To devote a treatise to ascertaining the mathematical principles of natural philosophy qualified as a virtual contradiction in terms. Why? Because natural philosophy in the medieval Aristotelian tradition did not – and could not – have mathematical principles." Edward Grant, *A History of Natural Philosophy* (Cambridge: Cambridge University Press, 2007), 313.

integration was innumerable scientific projects designed to mathematically reveal the causes of natural phenomena.

However, this integration also prompted Allessandro Piccolomini to deny in an influential 1547 book that mathematics can supply *explanatory causes* of material phenomena.[15] Jesuit scholars argued on both sides of the ensuing debate, a highpoint of which occurred at the Collegio Romano in the late 16th century between philosopher Benito Pereyra, who defended Piccololini, and mathematician Christoph Clavius.[16] Clavius' advocacy extended to the development of the *Ratio studiorum* that was developed in 1599, and that would govern Jesuit education in Austria and elsewhere, but which in the end represented a compromise between the philosophers and the mathematicians reflecting the unresolved nature of the debate.[17]

It stipulated, for example, that the *De caelo* was to be taught in mathematics courses, except for the sections dealing with the elements and the heavens, which were to be taught in physics courses.[18] These stipulations were not always adhered to, and the tensions between those Jesuits who viewed mathematical physics as a science of causes in the classical model (Clavius, Biancani, Barrow, Wallis and Arriaga) and those who denied this was possible (Piccolomini, Pereyra, the Coimbrian commentators and Gassendi)[19] ultimately erupted in

[15] Mancosu, *Philosophy of Mathematics and Mathematical Practice in the Seventeenth Century*, 12.
[16] While Pereyra argued that mathematics, while admired for its certainty, was limited to the accidental form of quantity and thus had no access to the explanatory causes of existing things, Clavius viewed mathematical physics as in complete harmony with classical science.
[17] Marcus Hellyer, *Catholic Physics* (Notre Dame, Indiana: University of Notre Dame Press, 1996), 120-121.
[18] Ibid., 125.
[19] Mancosu, *Philosophy of Mathematics and Mathematical Practice in the Seventeenth Century*, 110

censurae of certain mathematicians from Rome.[20] Barrow, for example, argues in 1683 that Pereyra "attempt[s] to prove that Mathematical Ratiocinations are not *Scientific*, *Causal* and *Perfect*, because the Science of a Thing signifies to know it by its Cause" when in fact "Mathematical Demonstrations are eminently *Causal*, from whence, because they only fetch their Conclusions from Axioms which exhibit the principal and most universal Affections of all quantities, and from Definitions which declare the constitutive Generations and essential Passions of particular Magnitudes."[21] Herein lies the distinction that Bolzano would later clarify between proofs that explain and proofs that establish certainty.

These tensions were far less present in non-Catholic Europe, where it was more common to dismiss the classical model of natural philosophy as an impure form of science.[22] However, it is the presence of this debate in Catholic Europe that laid the groundwork for a future synthesis of the classical model of science and mathematical science in Bolzano's work and for subsequent analytic philosophy.

These tensions continued into the generation of Bolzano's teachers and were reflected in the untenable positions maintained by the General Councils of the Jesuit Order. The

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[20] "The Society of Jesus' teaching enterprise developed at a time when the status of mathematics and its relationship with physics were undergoing thorough reevaluation….[T]he integration of mathematics into the Society's pedagogical program did not by any means follow a smooth path. The tensions between Jesuit mathematicians and philosophers can be seen in numerous questions along the porous border between mathematics and natural philosophy. For example, when mathematicians treated natural philosophical questions, a task they increasingly took upon themselves in the seventeenth century, they often did not adhere to the strictures of peripatetic philosophy. Consequently, they encountered problems with the *censurae* of the Roman revisers." Hellyer, *Catholic Physics*, 121.

[21] Quoted in Mancosu, *Philosophy of Mathematics and Mathematical Practice in the Seventeenth Century*, 20-21.

[22] "We have yet to find a pure natural philosophy; so far it has been infected and corrupted: in Aristotle's school by logic." *The Oxford Francis Bacon*, vol. xi (Oxford: Oxford University Press, 2004), 153-155.

Sixteenth General Congregation in 1731 decreed there to be no opposition between Aristotelian philosophy and “the more attractive style of learning in physics…with which the more notable natural phenomena are explained and illustrated by mathematical principles”, a decree which was repeated by the Seventeenth General Congregation in 1751. These tensions ultimately erupted in the Suppression of the Jesuit Order in 1773, which was permitted by a monarchy eager to invest in applied science over natural philosophy for the further legitimization of its rule.

In Prague itself, the pre-Suppression period witnessed this tension in large measure, beginning with a proliferation of work in what has come to be known as Jesuit mathematics. The groundwork for this was laid by the students of Suárez who were sent to Prague in the 17th century, particularly Arriaga, who viewed mathematical physics as an Aristotelian science of causes and wrote that such doctrines “are completely accepted here at the University of Prague”.[23] In fact, Prague is considered by some to be unmatched in all of Europe in the advances made by Jesuit mathematicians before the Suppression.[24] Monarchical enthusiasm for the *Ratio studiorum* began to wane in the mid 18th century as the prospect of applied sciences to further legitimize the monarchy began to have a hold in Vienna[25]. Joseph Stepling, a philosophy instructor who began lecturing in analytical

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[23] Hellyer, *Catholic Physics*, 47.

[24] J. Kasparova & K. Macak, *Utilitas matheseos. Jesuit Mathematics in the Clementinum (1602-1773)* (Prague: National Library, 2002) 66.

[25] “When tracing the development of Czech technical and industrial education, the last third of the 18th century and the early 19th century stand out as the major hallmark era. Many legal and educational measures were then carried out to meet the needs of the manufacturing industries, trade and agriculture. During that period, at the beginning of the first industrial revolution, the Czech lands, recognized as the most economically advanced part of the Hapsburg monarchy, witnessed the formation of their iron-making, textile, glass-making,

geometry and differential and integral calculus in 1754[26], was appointed "director of studies" in the philosophy faculty by the monarchy as an attempt to extend the influence of the court into philosophy education.[27] This influence was subsequently felt in the attempt to replace traditional philosophical curricula with the exact sciences.[28] While it is well documented that Prague Jesuit intellectuals attempted to support and assimilate the developments in mathematical physics[29], as is attested by the disputations Prague students conducted in 1766 on the works of Newton which had been translated into Czech[30], the ensuing faculty debates always concluded with a decision to retain the Aristotelian framework for study into the theoretical sciences.[31]

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ceramic and porcelain manufactories, coupled with an upsurge in ore and coal mining and an emerging food processing industry." Marcela Efmertova , "Czech Technical Education: The Educational Reforms of Franz Joseph Von Gerstner and his Relationship with the Paris Ecole Polytechnique," *ICON* 3 (1998): 202-223.

[26] "Beginning in 1754, Stepling delivered lectures on analytical geometry and differential and integral calculus, subjects that had not traditionally found their way into the Jesuit university curriculum." Paul Shore, *The Eagle and the Cross: Jesuits in Late Baroque Prague*, (St Louis: Institute of Jesuit Sources, 2002) 188-189.

[27] Shore, *The Eagle and the Cross*, 170.

[28] "In the second half of the 18th century, the Czech lands, like other countries in Europe (other parts of the Hapsburg monarcy, German lands, Poland, etc), displayed efforts to integrate their university studies in natural and technical sciences with faculties at the existing universities. At Prague University's Faculty of Philosophy, lectures were given in practical applications, such as surveying and geodesy, mechanics, civil and military engineering, technology associated with natural sciences, mathematics and physics branches, as well as the fundamentals of agriculture, mining, cameralist theories, etc." (Efmertova, "Czech Technical Education", 207). In 1747 a professorship in experimental physics was established in Prague and in 1763 a chair in mining was created in the Philosophy faculty. The former position was that of professor *physicae experimentalis et metaphysicae*. Ibid. The latter is discussed in Shore, *The Eagle and the Cross,* 35.

[29] "As for the actual research activities of Jesuit natural scientists [in Bohemia], it is clear that many of them disregarded the Holy See's expectation that they confine themselves to Ptolemaic cosmology." Shore, 2001, p. 74.

[30] Shore 2002, p. 159 ff.

[31] "Nevertheless, the university was in a less than entirely secure position in the second half of the [eighteenth] century, in that its curriculum, particularly in the Faculty of Philosophy, was beginning to seem archaic and far removed from curricular developments embraced elsewhere. The decision of the Society, after a rancorous internal battle, to remain committed to Aristotelian philosophy and physics only added to the

How does Bolzano's position relate to those of the interlocutors of this debate? His position is a unique synthesis of each side of the debate. Bolzano reveals himself to be a passionate advocate of the mathematical sciences in his affirmation of the discoveries of mathematicians and the type of proofs through which those discoveries are presented. His contribution to this debate, however, is to also affirm that these proofs are not explanatory of the causes of their objects. A subsequent presentation is required for this. His distinction between proofs that prove and proofs that explain is unique for its affirmation of both styles of proof in an overarching theory of science that remains consistent throughout his career. This distinction would eventually develop into the distinction in analytic philosophy between psychologistic explanations and axiomatic systems.

IV. From Subjective to Objective Proofs

This final section of the paper articulates the path to an objective exposition of *a priori* science given by Bolzano, a path whose starting points are the merely subjective, psychologistic proofs that establish the certainty of their conclusions. With the classical framework for Bolzano's work before us, we can more clearly interpret two often misinterpreted doctrines that are critical turns in this path from subjective to objective proofs. First, this section presents the evidence that incorporating subjective proofs into, and not expelling them from, logic was Bolzano's mission. This is contrary to most interpretations of Bolzano that see him attacking and expelling subjective proofs from

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conviction that the Society's philosophical program was out of date and even hostile to the new developments." (Shore, "Universalism, Rationalism, and Nationalism", 172).

logic, an interpretation that is more true of Frege than it is of Bolzano. Second, this section presents the semantic analysis through which Bolzano proceeds from subjective to objective proofs. Semantic analysis of subjective proofs resulted from Bolzano's intense analysis of mathematical proofs to uncover propositions with any implicit reference to a science foreign to the conclusion.

Bolzano's Incorporation of Subjective Proofs into *A Priori* Science

Bolzano writes of subjective proofs in 1810: "This is a procedure which, where we are concerned with the practical purpose of certainty, is quite correct and praiseworthy; but it cannot possibly be tolerated in a scientific exposition because it contradicts its essential aim."[32] Bolzano does not wish to expel subjective proofs, which are "quite correct and praiseworthy" when one's purpose is "certainty", from logic. One instead finds Bolzano making a distinction between the predominant mode of subjective proof and the "discounted" mode of objective proofs. This distinction, we have seen, is one Bolzano consciously parallels with the Aristotelian distinction between two modes of proof.

> I have been strengthened in the view that there is a special relation of ground and consequence between truths by noticing that some of the most penetrating thinkers have been of the same opinion. It is known that Aristotle already (*Analytica Posteriora*, 78a22) and for many centuries after him the scholastics have distinguished two kinds of proofs, namely those which show only the hoti, i.e. which show *that* something is the case, and those which show the dioti, i.e. *why* something is the case.[33]

[32] Ibid., 103.
[33] WL, Sec 198, p. 272. George translation.

Aristotle received the title ‘Philosopher of Common Sense’ precisely because he did *not* seek to expel proofs from science that fell short of the model of science as a demonstration of necessary causes. Rather, proofs which show *that* something is the case formed the starting point for his derivation of proofs which show *why* something is the case. Bolzano included this type of “derivation”, or “special consideration”, in his “Mathematical Method” in 1810 and in his mathematical works. His objective proof of the Intermediate Value Theorem that was so critical to analysis is not intended to replace geometric demonstrations of the theorem, but to complete it.

> For while the geometrical truth to which we refer here is (as we have already admitted) extremely obvious and therefore needs no proof in the sense of confirmation, it none the less does need a grounding.[34]

Bolzano is often interpreted, in contrast to what has been said here, as attacking psychologistic proofs and arguing for their replacement with objective proofs. Rolf George begins one of his treatments of Bolzano and psychologism with the claim: “It must be granted that Bolzano did not think much of grounding logic in psychological self-consciousness”.[35] While this is certainly true, it is not the whole story of psychologism’s role in Bolzano’s logic. For it is also true that, as George himself writes later in the article,

[34] Bolzano, “Purely Analytic Proof of the Theorem, that between any two Values, which give Results of Opposite Sign, there lies at least one real Root of the Equation,” 254. Bolzano even favors the use of subjective proofs in objective expositions for less important propositions when such concessions would facilitate the explanation of the proof: “What I have set out to accomplish in this book is to present the most important theses in such a way that one will be able to recognize the objective connection between them; although in the case of many less important propositions I will content myself with mere certifications in order to avoid prolixity”. Bolzano, “On the Mathematical Method,” 71.

[35] Rolf George, “Psychologism in Logic: Bacon to Bolzano,” *Philosophy and Rhetoric* 30 (1997): 213.

"In contrast to Frege and Husserl, [in Bolzano] there is no polemic to speak of against those who had thought to found logic upon psychology."[36]

George's contrast of Bolzano with Frege is revealing, for what one finds in secondary literature on Bolzano is often a Fregean Bolzano who allegedly attacked psychologism and in response postulated an independent realm of propositions-in-themselves (playing the same role as Frege's *Gedanken*) that exist whether they are thought or not. Drozdek, for example, argues that, "[i]n his battle against one-sidedness of subjectivism and psychologism, Bolzano fell into one-sidedness of objectivism by concentrating on the objectivity of truth, on its being truth in itself, and overlooking the fact that it should also be truth for us, known to us, and used by us." Nothing, it seems from Bolzano's texts, could be further from the truth. Such claims are difficult to square with remarks of Bolzano such as his 1810 claim, "The history of mathematics shows increasingly that whatever has been accepted at first merely from experience is subsequently derived from concepts and therefore comes to be treated as a part of pure *a priori* mathesis".[37]

Bolzano's Semantic Analysis of Subjective Proofs

This section presents in greater detail the path that Bolzano advocates from subjective proofs to objective proofs in science. The hallmark of this transition from subjective to objective proofs is a semantic analysis of subjective proofs. This semantic analysis was not formalized until TS, but was carried out in all of Bolzano's works on mathematics.

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[36] Ibid., 234.
[37] Bolzano, "Contributions to a Better-Grounded Presentation of Mathematics," 100.

The most important feature of an objective presentation of an *a priori* science for Bolzano is necessary connections of grounds and consequents, and the most common sin against *a priori* science is proofs that cross to more intuitive, but alien, genera. Thus, the first step is to intensely analyze the prevailing proofs to uncover such foreign, intuitive elements. Bolzano describes this semantic analysis of subjective proofs in TS.

> We think a certain representation in itself, i.e. we have a corresponding mental representation, only if we think all the parts of which it consists, i.e. if we also have mental representations of these parts. But it is not necessarily the case that we are always clearly conscious of, and able to disclose, what we think. Thus it may occur that we think a complex representation in itself, and are conscious that we think it, without being conscious of the thinking of its individual parts or beings able to indicate them.[38]

Bolzano's description of this process of semantic analysis in TS is presented as the transformation of propositions into the character of propositions-in-themselves. Bolzano's semantic analysis would lay the broad outline for future axiomatic methods in analytic philosophy that begin with linguistic analysis of what is being said, and then proceed to situate a transformed, symbolic restatement of what is said into an axiomatic system of similarly transformed propositions.

Bolzano begins his exposition of this method with his famous distinction between thought propositions and propositions in themselves. By a proposition-in-itself, Bolzano means

[38] WL, Sec 56, p. 69. George translation.

"any assertion that something is or is not the case, regardless whether or not somebody has put it into words, and regardless even whether or not it has been thought".[39]

When one speaks of spoken propositions, or of thought propositions, one tacitly recognizes propositions-in-themselves as differentiated from their articulation. This revelation is critical for Bolzano, for whom this distinction reveals that the intended meaning, and thus the basis in truth, of judgments is usually more than one is aware of. Spoken propositions or thought propositions (judgments) are thus the subjective counterparts to propositions-in-themselves, which are objective propositions.

The transformation of subjective propositions to propositions-in-themselves is a process of abstracting from figurative associations which adhere to a judgment in order to understand what is really being said, what is really intended. Essentially, according to Bolzano, all propositions can be rewritten in the objective form, 'Idea A has Idea B'. While this act of transformation is limited and, in fact, was also performed by Aristotle for whom the forms 'Idea A is Idea B' and 'Idea B is said of Idea A' can express all judgments, Bolzano also maintains that parts of propositions can themselves be propositions. The idea of a right triangle, for instance, is the idea of a 'triangle which has a right angle' which contains the idea, triangle, and the proposition, 'has a right angle'.[40]

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[39] *WL*, Sec. 19, p. 20, George translation. "[T]he source of most errors in logic has been the lack of distinction between thought truths and truths in themselves, and between thought propositions and thought concepts on the one hand, and propositions and concepts in themselves on the other." WL, Sec. 12, p. 13, George translation.

[40] *WL*, Sec. 63, p. 79, George translation.

Bolzano is often accused of Platonism with regard to his doctrine of propositions-in-themselves and ideas-in-themselves. This accusation misunderstands Bolzano's argument. First, the concept of an idea-in-itself is derivative of the concept of a proposition-in-itself for Bolzano, so any accusation of Platonism concerns only propositions-in-themselves. Second, propositions-in-themselves are not different propositions from spoken propositions; rather, they are the intended meanings that constitute spoken propositions, whose full meanings are generally inaccessible to their speakers. In this regard, Bolzano answers this accusation as follows: "What is meant, then, when we assert that there are such truths? Nothing, I answer, but that certain propositions have the character of truths in themselves."[41] It is through the act of transformation that a spoken proposition takes on this character and discloses its full meaning. Third, while subjective propositions have existence, Bolzano goes out of his way to deny existence to propositions-in-themselves. They have neither *Sein*, nor *Dasein*, nor *Existenz*, nor *Wirklichkeit*.

This denial of the existence of propositions-in-themselves has perplexed interpreters of Bolzano who then ask what sense of 'are' is meant in the statement, 'Propositions-in-themselves are the matter of actual spoken propositions'. However, the Aristotelian and Thomistic tradition to which Bolzano often appeals walks a similar middle road between Platonism and nominalism, but with regard to the natures of things. Both substantial and accidental natures exist only in individuals for Aristotle and Thomas, yet the existence of

[41] *WL*, Sec. 30, p. 39, George translation.

such individuals is accidental to the natures as such.[42] While Bolzano does not appeal to Aristotle or Thomas in his denial that propositions-in-themselves exist, it seems consistent with Bolzano's Aristotelian approach to natures to suppose that Bolzano took the same approach to propositions-in-themselves.

Coffa, who viewed Bolzano as the first of a tradition of philosophers who employed semantic analysis, has presented the clearest articulation of Bolzano's purpose. Coffa answers the question of "the sense and purpose of foundationalist or reductionist projects such as the reduction of mathematics to arithmetic, or of arithmetic to logic".

> It is widely thought that the principle inspiring such reconstructive efforts was epistemological, that they were basically a search for certainty. This is a serious error. It is true, of course, that most of those engaging in these projects believed in the possibility of achieving something in the neighborhood of Cartesian certainty for those principles of logic or arithmetic on which a priori knowledge was to be based. But it would be a gross misunderstanding to see in this belief the basic aim of the enterprise. A no less important purpose was the clarification of what was being said.[43]

While it is very true that the essence of the grounding of a proposition is for Bolzano essentially a "clarification of what was being said", what is missing from this summary is the initial problem that led Bolzano to such semantic clarifications, avoidance of crossing to another kind within a proof. This Aristotelian impetus for semantic analysis was soon

[42] "That which is prior is always the reason for what is posterior, and when the posterior is removed, the prior remains but not conversely. Thence it is that what is attributed to a nature according to an absolute consideration is the reason for its being attributed to some nature according to the existence which it has in a singular, and not conversely. For Socrates is rational because man is rational, and not conversely. So if Socrates and Plato did not exist, rationality, would still be attributable to human nature." St. Thomas Aquinas, *Quodl*. VIII, I, I, ed. R Spiazzi (Marietti, 1956), 159.
[43] Coffa, "The Semantic Tradition from Kant to Carnap," 26.

found missing in the tradition Bolzano founded, as it is from most interpretations of Bolzano himself.

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