Fiddly Lemma

$\mathcal{C}, \mathcal{D}$ ab cats, $F: \mathcal{C} \to \mathcal{D}$ additive

For any $\varphi: X \to Y$ in $\mathcal{C}$, $\exists$ map $F(\ker\varphi) \to \ker(F\varphi)$.

Lemma F is left-exact $\iff$ this map is an iso $\forall \varphi$.

Proof Assume F left exact

For any $\varphi: X \to Y$

$$0 \to \ker\varphi \to X \to \operatorname{Im}\varphi \to 0$$

$$0 \to \operatorname{Im}\varphi \to Y \to \operatorname{coker}\varphi \to 0$$

both exact.

Hence $0 \to F(\ker\varphi) \to FX \to F(\operatorname{Im}\varphi)$

& $0 \to F(\operatorname{Im}\varphi) \to FY \to F(\operatorname{coker}\varphi)$

exact.

2nd seq says that $F(\operatorname{Im}\varphi) \to FY$ is a monomorphism

Hence $\ker(FX \to F\operatorname{Im}\varphi)$ and $\ker(FX \to FY)$ are the same.

So exactness of F(sequence 1) gives

$$0 \to F(\ker\varphi) \to FX \xrightarrow{F\varphi} FY$$

exact, ie $\ker(F\varphi) = F(\ker\varphi)$.

Converse implication: exercise. $\square$

Apply this to $F^{op} : \mathcal{C}^{op} \to \mathcal{D}^{op}$
to see that right-exact $\iff$
preserves cokernels.

Hence exact functors preserve all exact seqs (not just short ones).

Notation If $F : \mathcal{C} \to \mathcal{D}$ contravariant,
say F is left exact
if $F : \mathcal{C}^{op} \to \mathcal{D}$ left exact.

Prop If $\mathcal{C}$ ab. cat, $X \in Ob(\mathcal{C})$
the functors $Hom(X, -) : \mathcal{C} \to \underline{Ab}$
$Hom(-, X) : \mathcal{C} \to \underline{Ab}$ (contra)
are left exact.

Proof suffices to prove 1st claim
Let $0 \to Y \xrightarrow{\alpha} Y' \xrightarrow{\beta} Y''$ exact in $\mathcal{C}$
Need to show: i) if $f \in Hom(X, Y)$ st $\alpha \circ f = 0$
then $f = 0$. This is exactly
claim that α is a monomorphism. ✓
ii) if $f : X \to Y'$ st $\beta \circ f = 0$ then
$\exists!$ lifting to $X \to Y$.
= universal property of $Ker(\beta) = Y$. ✓
$\square$

This gives lots of left-exact functors.

<u>Key Example</u> G group,

$\mathcal{C}$ = cat of G-modules

[abelian gps with left G-action that is $\mathbb{Z}$-linear]

<u>Claim</u> $\mathcal{C} \longrightarrow \underline{Ab}$

$M \longmapsto M^G$ <u>invariants</u> functor

is left-exact.

<u>Pf</u> consider $\mathbb{Z} \in \mathrm{Obj}(\mathcal{C})$ with G acting trivially

Have $\mathrm{Hom}_{\mathcal{C}}(\mathbb{Z}, -) = (-)^G$.

so $(-)^G$ is left exact.

It is <u>not</u> exact: eg let G = infinite cyclic $\langle g \rangle$

$$0 \to \begin{pmatrix} \mathbb{Z} \text{ with} \\ G \text{ triv}^l \end{pmatrix} \longrightarrow \begin{pmatrix} \mathbb{Z}^2 \text{ with} \\ g \text{ acting} \\ \text{as } \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \end{pmatrix} \longrightarrow \mathbb{Z} \to 0$$

short exact in $\mathcal{C}$.

$$0 \to \mathbb{Z} \xrightarrow{id} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$$

not exact at RH end.

(will understand this later using <u>group cohomology</u>.)

Lemma let $\mathcal{C}, \mathcal{D}$ ab. cats
$L : \mathcal{C} \to \mathcal{D}$ additive, $R : \mathcal{D} \to \mathcal{C}$
right adjt to L
Then L is right exact, & R is left exact.

Pf R preserves <u>all</u> limits (see Sheet 1)
$\Rightarrow$ preserves kernels.
Similarly L preserves all colimits.

Corollary $R \xrightarrow{\varphi} S$ morphism of rings

$$\underline{R\text{-Mod}} \longrightarrow \underline{S\text{-Mod}}$$
$$M \longmapsto S \otimes_{R,\varphi} M.$$

This functor is right exact, being left adj^t to the functor
$\varphi^* : \underline{S\text{-Mod}} \longrightarrow \underline{R\text{-Mod}}.$

§2.4 Projective & injective objs

$\mathcal{C}$ ab. cat.

<u>Def</u>

Def^n Say $X \in \mathrm{Obj}(\mathcal{C})$ is

<u>projective</u> if $\mathrm{Hom}(X, -)$ exact
<u>injective</u> if $\mathrm{Hom}(-, X)$ "

Equivalently:

X is proj. if "morphisms from X to quotients always lift"
i.e. if $Y \twoheadrightarrow Y'$, any $X \to Y'$ lifts to Y

X is inj if "morphisms from subobjects to X are extendable"

Basic example: will see that in <u>Ab</u>, $\mathbb{Z}$ is proj, $\mathbb{Q}/\mathbb{Z}$ is inj.

Def^n $\mathcal{C}$ has enough projectives

if every $X \in Ob(\mathcal{C})$ has a surj^n

$P \twoheadrightarrow X$ with P proj.

enough injectives if every X has

$X \hookrightarrow I$, I inj.

R ring.

Thm (i) In R-Mod, free modules

$$R^{(\Sigma)} = \bigoplus_{\sigma \in \Sigma} R \quad (\Sigma \text{ any set})$$

are proj.

(ii) R-Mod has enough projs.

(iii) in R-Mod, M is proj $\Longleftrightarrow$ $\exists$ module N

st $M \oplus N$ is free.

Proof i) Suppose $F = R^{(\Sigma)}$ is free

$Y \twoheadrightarrow Y'$ epi. $^{(*)}$ For each $\sigma \in \Sigma$,

lift $\varphi(\sigma^{th}$ gen. of $F)$ to Y arbitrarily.

$\boxed{(*)\ \varphi : F \to Y'}$ This gives a lifting

$\tilde{\varphi} : F \to Y$.

(ii) For any M, free module on underlying set of M surjects onto M.

(NB: This also shows that $\underline{(R\text{-Mod})^{fg}}$, cat. of finitely gen. R-mods, has enough projs; it does <u>not</u> have enough injs.)

(iii) Suppose P proj and take a surj^n $F \overset{\beta}{\twoheadrightarrow} P$, F free. Then

$id: P \to P$ lifts to a map $P \overset{\alpha}{\to} F$, and can check that $im(\alpha) \cong P$

and $F = \ker(\beta) \oplus im(\alpha)$.

Conversely if $F = X \oplus X'$ with F free, $Y \twoheadrightarrow Y'$, $\varphi : X \to Y'$. Then composite

$F \twoheadrightarrow X \xrightarrow{\varphi} Y'$ lifts to $F \to Y$ & compose it with $X \hookrightarrow F$ to lift φ.

Much harder theorem:

<u>Thm</u> R-Mod has enough injectives.

<u>Ideas of pf</u>: ① reduce to $R = \mathbb{Z}$

② use some kind of "duality"

Define Ω = the R-module $\mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})$
with $(r \cdot \varphi)(-) = \varphi((-)\cdot r)$.

Lemma 1 For any R-module M, $\mathrm{Hom}_R(M, \Omega) \cong \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ (naturally in M).

Proof Given $\varphi : M \to \mathbb{Q}/\mathbb{Z}$ gp hom,
let $\tilde{\varphi} : M \to \Omega$ be def by
$m \mapsto (r \mapsto \varphi(rm))$
conversely φ is image of $\tilde{\varphi}$ under eval^n at $1 \in R$.
check these are inverse operations.

Lemma 2 Ω is injective.

Pf By Lemma 1 suffices to show that $\mathbb{Q}/\mathbb{Z}$ is inj. in Ab

Induction step: if $G \supset H$ ab gps,
$\varphi : H \to \mathbb{Q}/\mathbb{Z}$, G/H cyclic gen. by some g.
If g has ∞ order in G/H, then $G \cong H \oplus \mathbb{Z}$
and can extend φ to G by letting $\varphi(g)$ be arbitrary.
If g has order $n < \infty$, then $\varphi(g^n) \in \mathbb{Q}/\mathbb{Z}$
is given, & since $\mathbb{Q}/\mathbb{Z}$ is divisible can
define a hom $G \to \mathbb{Q}/\mathbb{Z}$ by choosing
$\varphi(g)$ st $n \cdot \varphi(g) = \varphi(g^n)$.
Zorn's lemma $\Rightarrow$ our lemma follows.

Lemma 3 For any R-module M, and
$m \neq 0 \in M$, $\exists$ hom $M \xrightarrow{\varphi} \Omega$
st $\varphi(m) \neq 0$.

Pf Again can assume $R = \mathbb{Z}$ wlog by Lem 1.
Since $\mathbb{Q}/\mathbb{Z}$ inj, can assume M generated
by m. If m has order n send it to $\frac{1}{n}$
∞, (anything $\neq 0$).

P.

Pf of Thm

Define $D(M)$, for M in R-Mod,
as $\mathrm{Hom}_R(M, \Omega)$ (with some R-mod str.)

For any M have nat'l map

$$M \to D(D(M))$$
$$m \mapsto (\varphi \mapsto \varphi(m))$$

By L3 this is an injection.

Will show $D(D(M))$ injects into an injective.

Take surj^n $F \to D(M)$ $F \cong R^{(\Sigma)}$ free

thus get map $D(D(M)) \to D(F)$
and $D(F)$ is a direct sum of copies of Ω, hence injective. $\Box$

3. Complexes and Homotopies

3.1 The category of complexes

$\mathcal{C}$ ab. cat.

Def^n $Ch(\mathcal{C})$ = category whose objs are cochain complexes in $\mathcal{C}$, $(X^i)_{i \in \mathbb{Z}}$ + morphisms being collections $(f^i : A^i \to B^i)_{i \in \mathbb{Z}}$ compat. w. the differentials

$$\begin{tikzcd} A^i \arrow[r, "d_A^i"] \arrow[d, "f^i"'] & A^{i+1} \arrow[d, "f^{i+1}"] \\ B^i \arrow[r, "d_B^i"] & B^{i+1} \end{tikzcd}$$ commutes $\forall i$.

Prop (easy): $Ch(\mathcal{C})$ is an ab. cat. with obvious notions of kernel, cokernel.

Def^n the "n^{th} cohomology functor"

$Ch(\mathcal{C}) \to \mathcal{C}$ sends $A^\cdot$ to $H^n(A^\cdot)$

Problem: not actually well-def! — kernels + cokernels not unique. Want to pick a ker & coker for every morphism in $\mathcal{C}$.

In practice most ab cats come with "standard choices" of ker / coker.

Def^n A morphism f in $Ch(\mathcal{C})$ is a quasi-isomorphism if it induces isomorphisms on H^i for all $i \in \mathbb{Z}$.

Lemma ("Snake lemma") for any diagram

$$
\begin{tikzcd}
 & \ker f \arrow[r] \arrow[d] & \ker g \arrow[r] \arrow[d] & \ker h \arrow[d] \arrow[ddddll, out=0, in=180, looseness=1.5] & \\
 & A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & C' \arrow[d] & \\
 & \operatorname{cok} f \arrow[r] & \operatorname{cok} g \arrow[r] & \operatorname{cok} h & \\
 & & & &
\end{tikzcd}
$$

in $\mathcal{C}$, commutative with exact rows,

$\exists$ arrow $\ker(h) \to \operatorname{coker}(f)$

st $\ker(f) \to \ker(g) \to \ker(h) \to \operatorname{coker} f \to \cdots$ is exact

Can prove this directly for R-Mod, see next time

For gen' $\mathcal{C}$, can reduce to this via Freyd-Mitchell embedding.