§5 Spectral Sequences

§5.1 Setup

Motivation G grp, $H \lhd G$

$M = G$-module

$\rightsquigarrow H^i(H, M) \circlearrowleft G/H$

Question: Can we recover $H^i(G,M)$ from $H^p(G/H, H^q(H,M))$?

Answer: Yes, using spectral seq.

Defn $\mathcal{C}$ = abelian category

$r_0 \in \mathbb{Z}$

A first quadrant sp. seq. in $\mathcal{C}$ starting at r_0 is:

(1) for each $r \geq r_0$ an "rth sheet" E_r "rth page"

- $E_r^{p,q}$ objects in $\mathcal{C}$

$p, q \in \mathbb{Z}$ $\quad E_r^{p,q} = 0$ if $p < 0$ or $q < 0$

- morphisms/differentials

$d_r^{p,q} : E_r^{p,q} \longrightarrow E_r^{p+r, q-r+1}$

such that $d_r^{p,q} \circ d_r^{p-r, q+r-1} = 0$

(2) $\forall r \geq r_0$ isomorphisms

$E_{r+1}^{p,q} \xrightarrow{\simeq}$ cohomology of E_r^{**} at (p,q)

$$\underset{\sim}{E_0} \quad \updownarrow d_0 \qquad \begin{array}{ccc} E_0^{01} & E_0^{11} & \\ E_0^{00} & E_0^{10} & \cdots\cdots \end{array}$$

take cohomology

$$\underset{\sim}{E_1} \qquad \begin{array}{l} E_1^{01} \xrightarrow{p} E_1^{11} \xrightarrow{d^1} \\ E_1^{00} \longrightarrow E_1^{10} \xrightarrow{d^1} \end{array}$$

take cohom.

$$E_2 \qquad \begin{array}{cccc} E_2^{01} & E_2^{02} & d^2 & \\ E_2^{00} & E_2^{10} & E_2^{20} & E_2^{30} \end{array}$$

observation:

$$E_r^{p-r,q+r-1} \longrightarrow E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}$$

$r > \max\{p, q+1\} \implies$ these both $= 0$

fix $p,q \implies E_r^{p,q}$ eventually stabilises as $r \to \infty$

define $E_\infty^{p,q} :=$ stable value of $E_r^{p,q}$

<u>Definition</u> $(X^i)_{i \geq 0}$ objects of $\mathcal{C}$

Say $(E_r^{p,q})_r^{p,q}$ <u>converges</u> to $(X^i)_i$

if each X^i has a filtration

$$X^i = \mathrm{Fil}^0 X^i \supseteq \mathrm{Fil}^1 X^i \supseteq \dots \supseteq \mathrm{Fil}^{i+1} X^i = 0$$

s.t. $$\mathrm{Gr}^p_{\mathrm{Fil}} X^{p+q} := \frac{\mathrm{Fil}^p X^{p+q}}{\mathrm{Fil}^{p+1} X^{p+q}} \cong E_\infty^{p,q}$$

Notation: write "$E_r^{p,q} \Rightarrow X^{p+q}$"

§5.2 The sp. seq. associated to a filtered cmplx

"One example to rule them all"

Defn A positive filtered complex in $\mathcal{C}$ is a complex $C^\cdot$ in $\mathcal{C}$

+ $\mathrm{Fil}^n C^i$ decreasing filtration

(1) $d(\mathrm{Fil}^n C^i) \subseteq \mathrm{Fil}^n(C^{i+1})$

(2) $\mathrm{Fil}^0 C^i = C^i$ (positivity)

$\mathrm{Fil}^{i+1} C^i = 0$ ($\Rightarrow C^i = 0$ if $i < 0$)

Thm $C^\cdot$ +ve filtered cmplx

Then $\exists$ sp. seq. $(E_r^{**})_{r \geq 0}$ converging to $H^*(C^\cdot)$

- induced filtr^n is
$\mathrm{Fil}^n H^*(C^\cdot) = \mathrm{image}\left(H^*(\mathrm{Fil}^n(C^\cdot))\right)$
- $E_0^{p,q} = \mathrm{Gr}^p_{\mathrm{Fil}} C^{p+q}$, $d_0 = d_C$
- $E_1^{p,q} = H^{p+q}(\mathrm{Gr}^p_{\mathrm{Fil}} C^\cdot)$

Proof: look in Gelfand–Manin

Key example: double complexes

objects $X^{p,q}$ $\quad p,q \in \mathbb{Z}_{\geq 0}$

differentials $\quad d_h : X^{p,q} \longrightarrow X^{p+1,q}$

$\quad d_v : X^{p,q} \longrightarrow X^{p,q+1}$

s.t. $d_h^2 = 0$, $d_v^2 = 0$

$d_h d_v + d_v d_h = 0$

$\rightsquigarrow$ total complex

$\mathrm{Tot}(X^{\cdot})$: $\quad T^i = \bigoplus_{p+q=i} X^{p,q}$

$d_{Tot} := d_h + d_v$

$\exists$ 2 natural filtrations on T^i:

${}^{I}\mathrm{Fil}^n T^i = \bigoplus_{\substack{p+q=i \\ p \geq n}} X^{p,q}$ $\quad$ filtration by columns

${}^{II}\mathrm{Fil}^n T^i = \bigoplus_{\substack{p+q=i \\ q \geq n}} X^{p,q}$ $\quad$ filtration by rows.

$\rightsquigarrow$ 2 sp. seq.

${}^{I}E_r^{p,q} \Longrightarrow H^{p+q}(T^{\cdot})$

${}^{II}E_r^{p,q} \Longrightarrow H^{p+q}(T^{\cdot})$

${}^{I}E_0^{p,q} = X^{p,q}$ $\quad d_0 = d_v$

${}^{I}E_1^{p,q} = H_v^q(X^{p,\bullet})$ $\quad d_1$ induced by d_h

${}^{I}E_2^{p,q} = H_h^p(H_v^q(X^{\cdot,\cdot}))$

Similarly: $\quad {}^{II}E_0^{p,q} = X^{q,p}$

${}^{II}E_1^{p,q} = H_h^q(X^{\bullet,p})$

${}^{II}E_2^{p,q} = H_v^p H_h^q(X^{\bullet,\bullet})$

§5.3 Hyper derived functors

$F: \mathcal{C} \longrightarrow \mathcal{D}$ left exact

$X \in \mathrm{ob}(\mathcal{C}) \rightsquigarrow (R^iF)(X)$

now: define $(R^iF)(X^\bullet)$

$X^\bullet \in Ch^{\geq 0}(\mathcal{C})$

Defn $X^\bullet \in Ch^{\geq 0}(\mathcal{C})$

A Cartan–Eilenberg resolution of $X^\bullet$ is a double complex $J^{\bullet\bullet}$ w/ a cochain map

$X^\bullet \longrightarrow J^{\bullet,0}$ s.t.:

(1) pth column $J^{p,\bullet}$ is an injective resolution of X^p $\forall p \geq 0$

(2) rows $J^{\bullet,q}$ satisfy a "splitness condition"

$\Rightarrow H^p_h(J^{\bullet\bullet})$ is an injective res^n of $H^p(X^\bullet)$

Fact If $\mathcal{C}$ has enough injectives, then every such $X^\bullet$ has a CE resolution, and these are "functorial up to homotopy"

Defn $(R^iF)(X^\bullet) = H^i(\mathrm{Tot}(F(J^{\bullet\bullet})))$

$$\exists \ 2 \text{ sp. seq.} \Rightarrow (\mathbb{R}^i F)(X^\bullet)$$

$${}^{I}E_0^{p,q} = F(J^{p,q}) \qquad d_0 = d_v$$

$${}^{I}E_1^{p,q} = H^q_v\left(F(J^{p,\bullet})\right) = (R^q F)(X^p) \qquad d_1 = d_X$$

$${}^{I}E_2^{p,q} = H^p\left((R^q F)(X^\bullet)\right)$$

$${}^{II}E_0^{p,q} = F(J^{q,p}) \qquad d_0 = d_h$$

$${}^{II}E_1^{p,q} = H^q_h\left(F(J^{\bullet,p})\right) \overset{\text{splitness}}{=} F\, H^q_h(J^{\bullet,p})$$

pth term in injective resolution of $H^q(X^\bullet)$

$${}^{II}E_2^{p,q} = (R^p F)\left(H^q(X^\bullet)\right)$$

$\Rightarrow \exists \ 2$ sp. seq

$$\begin{aligned} {}^{I}E_2^{p,q} &= H^p\left((R^q F)(X^\bullet)\right) \\ {}^{II}E_2^{p,q} &= (R^p F)\left(H^q(X^\bullet)\right) \end{aligned} \Rightarrow (\mathbb{R}^{p+q} F)(X^\bullet)$$

application: acyclic resolutions

Defn $F : \mathcal{C} \to \mathcal{D}$ left exact

$X \in \mathrm{Ob}(\mathcal{C})$

An F-acyclic resolution of X is a left resolution $[X] \to A^\bullet$ s.t. $(R^iF)(A^j) = 0$ $\forall i \geq 1, \forall j \geq 0$

Proposition $(R^iF)(X) = H^i(F(A^\bullet))$

Proof Look at ${}^{I}E, {}^{II}E \Rightarrow (R^iF)(A^\bullet)$

$${}^{I}E_2^{p,q} = H^p((R^qF)(A^\bullet)) = \begin{cases} 0 & q \neq 0 \\ H^p(F(A^\bullet)) & q = 0 \end{cases}$$

$$\Rightarrow H^p(F(A)) \cong R^pF(A^\bullet)$$

$${}^{II}E_2^{p,q} = R^pF(H^q(A^\bullet))$$

$$= \begin{cases} 0 & q \neq 0 \\ (R^pF)(X) & q = 0 \end{cases}$$

$$\Rightarrow (R^pF)(A^\bullet) \cong (R^pF)(X) \quad \square$$

§5.4 Examples

① M = real manifold

$X = \underline{\mathbb{R}}$ constant sheaf

Ω^p_M = sheaf of differential p-forms on M

Poincaré lemma $\Longrightarrow$

$$0 \to \underline{\mathbb{R}} \to \Omega^0_M \xrightarrow{d} \Omega^1_M \to \cdots\cdots$$

is exact

$\therefore$ $(\Omega^\bullet_M, d)$ is a resolution of $\underline{\mathbb{R}}$

<u>Fact</u>: each Ω^p_M is $H^0(M, -)$ acyclic

$$\Longrightarrow H^i(M, \underline{\mathbb{R}}) = H^i(\Gamma(\Omega^\bullet_M))$$

"de Rham's thm"
$$= \frac{\{\text{closed } i\text{-forms}\}}{\{\text{exact } i\text{-forms}\}}$$

(1b) X = compact complex manifold

Ω^p_X = holomorphic p-forms on X

Poincaré lemma

$\Longrightarrow (\Omega^\bullet_X, d)$ is a resolution of $\underline{\mathbb{C}}$

$\rightsquigarrow$ 2 sp. seq $\Longrightarrow \mathbb{H}^i(X, \Omega^\bullet_X)$

$$\mathbb{H}^i(X, \Omega^\bullet_X) \overset{!!}{=} H^i_{dR}(X/\mathbb{C})$$

"II" spec. seq $\quad {}_{\mathrm{II}}E_2^{p,q} = \begin{cases} 0 & \text{if } q \neq 0 \\ H^p(X, \underline{\mathbb{C}}) & q = 0 \end{cases}$

$\rightsquigarrow \mathbb{H}^p(X, \Omega^\bullet_X) = H^p(X, \underline{\mathbb{C}})$

"I" sp. seq $\quad E_1^{p,q} = H^q(X, \Omega^p_X)$

$$\Longrightarrow H^{p+q}(X, \underline{\mathbb{C}})$$

"Hodge $\to$ de Rham sp. seq"

induces Hodge filtration $H^i(X, \underline{\mathbb{C}})$