TCC Homological Algebra: Assignment #1

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1/11/19

This is the first of 3 problem sheets. Solutions should be submitted to me (by email, or via my pigeonhole for Warwick students) by **noon on 22nd November**. This problem sheet will be marked out of a total of 20; the number of marks available for each question is indicated. Questions marked [*] are optional and not assessed.

Note that rings are not necessarily commutative, but are always assumed to be unital (i.e. having a multiplicative identity element 1), and ring homomorphisms are assumed to map 1 to 1.

Categories, functors, and natural transformations

1. [3 points] Let $\mathcal{C}$ be a category admitting a faithful functor $F : \mathcal{C} \to \underline{\mathrm{Set}}$, and $\phi : X \to Y$ a morphism in $\mathcal{C}$.
   (a) Show that:
      i. If $F(\phi)$ is injective, then ϕ is a monomorphism.
      ii. If $F(\phi)$ is surjective, then ϕ is an epimorphism.
   (b) Show that if $\mathcal{C} = \underline{\mathrm{Ring}}$ and F is the forgetful functor, then the converse of (i) is true: if ϕ is a monomorphism, then ϕ is set-theoretically injective. *(Hint:* $\mathrm{Hom}_{\underline{\mathrm{Ring}}}(\mathbf{Z}[X], R) = R$.*)*
   (c) By considering the inclusion map $\mathbf{Z} \to \mathbf{Q}$, or otherwise, show that the converse of (ii) is false in this case.

2. Let $\underline{\mathrm{Ring}}$ be the category of rings and ring homomorphisms, and $u : \underline{\mathrm{Ring}} \to \underline{\mathrm{Grp}}$ the functor sending a ring R to the group $R^\times$ of invertible elements of R (and acting on morphisms in the obvious way).
   (a) [1 point] Is u full?
   (b) [*] Is u faithful?

3. Let k be a field. Let $\mathcal{C}$ denote the category with objects $\{0, 1, \dots\}$ and $\mathrm{Hom}_{\mathcal{C}}(m, n)$ defined to be the space of $n \times m$ matrices over k (with composition defined as matrix multiplication); and let $\mathcal{D}$ denote the category of all finite-dimensional k-vector spaces.
   (a) [1 point] Verify that there is a functor $F : \mathcal{C} \to \mathcal{D}$ sending n to $\mathbb{R}^n$ (you should explain what it does to morphisms).
   (b) [*] Show (directly, without quoting the general criterion from §1.4) that F has a quasi-inverse.

Limits and adjunctions

4. [2 points] Prove the following result (part B of a lemma stated in §1.6): if $f_1, f_2 : X \to Y$ are two morphisms in a category $\mathcal{C}$, and $g : Y \to Z$ is a monomorphism in $\mathcal{C}$, then the pair $(g \circ f_1, g \circ f_2)$ has an equaliser if and only if (f_1, f_2) has an equaliser, and the two equalisers coincide.

5. Let $(\mathcal{P}, \succ)$ be a partially ordered set, and $\mathcal{J}$ the category with objects $\mathcal{P}$ and a single homomorphism $x \to y$ if $x \succ y$.
   (a) [2 points] Show that if there exists a greatest element in $\mathcal{P}$, then $\mathcal{J}$-diagrams have limits in any category. Formulate and prove a similar statement for colimits.

(b) [1 point] Suppose $(\mathcal{P}, \succ) = (\mathbb{N}, \geq)$. Show that a $\mathcal{J}$-diagram in Ab is equivalent to the data of a collection of abelian groups A_i and morphisms $\phi_i : A_{i+1} \to A_i$ for all $i \in \mathbf{N}$, and that the inverse limit

$$\varprojlim_i A_i = \left\{ (x_i) \in \prod_{i \in I} A_i : \phi_i(x_{i+1}) = x_i \ \forall i \right\}$$

satisfies the universal property of the category-theoretic limit.

6. [2 points] Show that the forgetful functor Top $\to$ Set has both a left and a right adjoint, and describe both of these explicitly.

7. [3 points] Let $L : \mathcal{C} \to \mathcal{D}$ and $R : \mathcal{D} \to \mathcal{C}$ be a pair of adjoint functors (L the left adjoint, R the right adjoint).

   (a) Show that if $\mathcal{J}$ is a small category, and D is a $\mathcal{J}$-diagram in $\mathcal{D}$ which has a limit, then $R(D)$ also has a limit and $R(\lim D) = \lim R(D)$ (i.e. R is *continuous*).
   [Hint: Use L to construct a map from cones of $R(D)$ to cones of D.]

   (b) Show that the opposite functor $L^{\text{op}} : \mathcal{C}^{\text{op}} \to \mathcal{D}^{\text{op}}$ is **right** adjoint to R^{op}. By applying part (a) to the opposite functors, or otherwise, show that L is *cocontinuous*, i.e. preserves all colimits.

Additive categories, kernels, cokernels

8. [2 points] Let $\mathcal{C}$ be an additive category. Show that there is an isomorphism $X_1 \oplus (X_2 \oplus X_3) \cong (X_1 \oplus X_2) \oplus X_3$ for any three objects X_1, X_2, X_3 of $\mathcal{C}$.

9. [3 points] Let $\mathcal{C}$ be the full subcategory of Ab consisting of torsion-free abelian groups. Show that all morphisms in $\mathcal{C}$ have kernels and cokernels. Give an example to show that the cokernel of a morphism in $\mathcal{C}$ may not coincide with the cokernel of the same morphism in Ab. Is $\mathcal{C}$ an abelian category?