TCC Homological Algebra: Assignment #2

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22/11/19

This is the second of 3 problem sheets. Solutions should be submitted to me (by email, or via my pigeonhole for Warwick students) by **noon on 6th December**. This problem sheet will be marked out of a total of 20; the number of marks available for each question is indicated. Questions marked [*] are optional and not assessed.

Note that rings are not necessarily commutative, but are always assumed to be unital (i.e. having a multiplicative identity element 1), and ring homomorphisms are assumed to map 1 to 1. The notation $\underline{\text{Ab}}$ denotes the category of abelian groups, and $\underline{R\text{-Mod}}$ the category of left modules over the ring R.

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1. Let $\underline{\text{FAb}}$ denote the full subcategory of $\underline{\text{Ab}}$ whose objects are finite abelian groups.
   (a) [1 point] Show that $\underline{\text{FAb}}$ is an abelian category. (You may assume that $\underline{\text{Ab}}$ is abelian.)
   (b) [2 points] Show that the only injective object in $\underline{\text{FAb}}$ is 0. (Hint: if G is a non-zero injective, consider homomorphisms from cyclic groups to G.)

2. [3 points] Let $\mathcal{C}$ be an abelian category, Σ a set, and for each $\sigma \in \Sigma$, let M_σ be an object of $\mathcal{C}$. We define $\prod_{\sigma\in\Sigma} M_\sigma$ to be the limit of the diagram consisting of the objects M_σ with no morphisms between them, and $\bigoplus_{\sigma\in\Sigma} M_\sigma$ its colimit, assuming these limits exist.
   (a) Show that if M_σ is projective for all σ, then so is $\bigoplus_{\sigma\in\Sigma} M_\sigma$.
   (b) Show that if M_σ is injective for all σ, then so is $\prod_{\sigma\in\Sigma} M_\sigma$.
   (c) Show that $\operatorname{Hom}_{\mathcal{C}}\left(\bigoplus_{\sigma\in\Sigma} M_\sigma, Z\right) = \prod_{\sigma\in\Sigma} \operatorname{Hom}_{\mathcal{C}}(M_\sigma, Z)$ for any object Z of $\mathcal{C}$.

3. [3 points] Let $\mathcal{C}$ be an abelian category and $A^\bullet, B^\bullet$ cochain complexes over $\mathcal{C}$. Define a complex $\mathcal{H} = \underline{Hom}(A^\bullet, B^\bullet) \in \operatorname{Ch}(\underline{\text{Ab}})$ by $\mathcal{H}^i = \prod_{j\in\mathbf{Z}} \operatorname{Hom}_{\mathcal{C}}(A^j, B^{j+i})$.
   (a) Show that the maps $d^i_{\mathcal{H}} : \mathcal{H}^i \to \mathcal{H}^{i+1}$ defined by
   $$d^i_{\mathcal{H}}\left((f^j)_{j\in\mathbf{Z}}\right) = (f^{j+1}\circ d^j_A - (-1)^i d^{j+i}_B \circ f^j)_{j\in\mathbf{Z}}$$
   are well-defined, and satisfy $d^{i+1}_{\mathcal{H}} \circ d^i_{\mathcal{H}} = 0$.
   (b) Show that $\ker(d^0_{\mathcal{H}}) = \operatorname{Hom}_{\operatorname{Ch}(\mathcal{C})}(A^\bullet, B^\bullet)$.
   (c) Show that $\operatorname{im}(d^{(-1)}_{\mathcal{H}})$ is the null-homotopic maps.

4. [2 points] Let X, Y be two objects in an abelian category $\mathcal{C}$, and $I^\bullet$, $J^\bullet$ injective resolutions of X, Y respectively. Let $f^\bullet : I^\bullet \to J^\bullet$ a morphism of complexes which induces the zero map $X \to Y$ on H^0. Show that $f^\bullet$ is null-homotopic.
   [Hint: We are looking for maps $s^i : I^i \to J^{i-1}$ for all i such that $f = ds + sd$. For $i \le 0$ the target of s^i is the zero object, so the first nontrivial step is to construct $s^1 : I^1 \to J^0$ compatible with f^0. Then look for an opportunity to induct on i.]

5. [2 points] Give an example of a morphism in $\operatorname{Ch}(\underline{\text{Ab}})$ which is a quasi-isomorphism, but not a homotopy equivalence.

6. [2 points] Show that if $F : \mathcal{C} \to \mathcal{D}$ a left-exact functor between abelian categories, and $0 \to A \to B \to C \to 0$ is an exact sequence with A injective, then $0 \to F(A) \to F(B) \to F(C) \to 0$ is exact.
   *[Hint: We are **not** assuming that $\mathcal{C}$ has enough injectives, so it is not enough to say that $R^1(F)(A) = 0$.]*

7. [1 point] Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be abelian categories and $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E}$ additive functors. Assume $\mathcal{C}$ has enough injectives, G is exact, and F is left-exact. Show that $R^i(G \circ F) = G \circ R^i(F)$ for all i, as functors $\mathcal{C} \to \mathcal{E}$.

8. [3 points] Let $G = C_2 = \{1, \sigma\}$.
   (a) Show that
   $$\dots \mathbf{Z}[G] \xrightarrow{\sigma-1} \mathbf{Z}[G] \xrightarrow{\sigma+1} \mathbf{Z}[G] \xrightarrow{\sigma-1} \mathbf{Z}[G]$$
   is a projective resolution of the trivial module $\mathbf{Z}$ as a $\mathbf{Z}[G]$-module.
   (b) Hence compute the cohomology groups of
      i. $\mathbf{Z}$ with the trivial G-action;
      ii. $\mathbf{Z}$ with the generator σ acting as -1.

9. Let R be a ring, A, B objects of <u>R-Mod</u>, and $\sigma \in \mathrm{Ext}^1(A, B)$, represented by a homomorphism $f \in \mathrm{Hom}(A, Z^1(I^\bullet))$ where $I^\bullet$ is an injective resolution of B.
   (a) [1 point] Show that the module
   $$E = \{(x, a) \in I^0 \oplus A : d(x) = f(a)\},$$
   with the obvious maps from B and to A, defines an extension of A by B; and show that the equivalence class of this extension depends only on σ and not on the representative f.
   (b) [*] Show that this construction an inverse of the map
   $$\text{(equivalence classes of extensions)} \to \mathrm{Ext}^1(A, B)$$
   that we defined in lectures.