LECTURE NOTES 0 FOR CAMBRIDGE PART III COURSE ON "PROBABILISTIC NUMBER THEORY", MICHAELMAS 2015

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Abstract. These are rough notes explaining some preliminary details, mostly practical arrangements, basic notation, the course synopsis, and a few background facts, for the "Probabilistic Number Theory" course.

1. Practical matters

Lecture notes. As discussed later, this course will be divided into three main chapters. I will produce notes for each block of lectures, and post them on my webpage `https://www.dpmms.cam.ac.uk/~ajh228/` at some point, probably around the end of that block of lectures. I will try to write the notes carefully, but their main purpose is for my own reference when giving the lectures. So please come to the lectures, and take your own notes whilst there!

Books. I don't know of any book that covers everything in the course. Tenenbaum's *Introduction to analytic and probabilistic number theory* will cover much of the first third of the course, and also discusses lots of other topics in analytic number theory. Elliott's two volume work on *Probabilistic Number Theory* is also very relevant. Parts of the middle third of the course are discussed in various general analytic number theory books, such as Iwaniec and Kowalski, *Analytic Number Theory.* Standard books on the Riemann zeta function, such as Titchmarsh's *The Theory of the Riemann Zeta-function*, will be useful for the final third of the course. You should be able to follow the course without access to these books, but they are certainly worth a look.

Example sheets. I expect to write three examples sheets for the course, and probably have two examples classes this term, one at the start of next term, and a revision class in Easter term. I will post the sheets on my webpage `https://www.dpmms.cam.ac.uk/~ajh228/` as I write them, and during the lectures we will agree a time for the examples classes.

2. Notation and conventions

This is an analysis course, and will involve estimating/bounding various quantities that are too complicated or that we don't need to understand exactly. To facilitate this we will need a bit of notation.

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We write $f(x) = O(g(x))$, and say that f is "big Oh" of g, if there exists a constant C such that
$$|f(x)| \leq Cg(x) \quad \forall x.$$
Here we usually want the inequality to hold either for all x for which the functions are defined, or for all sufficiently large x (i.e. all x larger than some fixed constant). Which meaning is desired should always be made clear from the context.

We will also write $f(x) \ll g(x)$, which means the same as $f(x) = O(g(x))$.

We write $f(x) \asymp g(x)$, and say that f is *of order* g, if both
$$f(x) \ll g(x) \quad \text{and} \quad g(x) \ll f(x),$$
in other words if $(1/C)g(x) \leq |f(x)| \leq Cg(x)$ for all relevant x.

For $g(x) \neq 0$, we write $f(x) = o(g(x))$ as $x \to \infty$, and say that f is "little oh" of g as $x \to \infty$, if
$$\frac{f(x)}{g(x)} \to 0 \quad \text{as } x \to \infty.$$
One can similarly say that $f(x) = o(g(x))$ as $x \to 0$, for example.

Finally, for $g(x) \neq 0$ we write $f(x) \sim g(x)$ as $x \to \infty$, and say that f is *asymptotic to* g, if
$$\frac{f(x)}{g(x)} \to 1 \quad \text{as } x \to \infty.$$
Note this is the same as saying that $f(x) = (1 + o(1))g(x)$ as $x \to \infty$.

For example, for $x \geq 1$ we have
$$100x = O(x^2), \quad \log x = o(x) \quad \text{as } x \to \infty, \quad x^2 - 2x + 1 = x^2 + O(x), \quad x^3 - 10 \sim x^3 \quad \text{as } x \to \infty.$$

We will usually write C to mean a large constant, c to mean a small constant, and ϵ to mean a parameter close to zero.

We will sometimes write $[N]$ to mean the set $\{1 \leq n \leq N\}$ of natural numbers less than N. We will also write $\mathcal{P}([N])$ to mean the power set of $[N]$, and $\mathbb{P}_N$ to mean the discrete uniform probability measure on $[N]$ (i.e. the measure for which $\mathbb{P}_N(n) = 1/\lfloor N \rfloor$ for each $1 \leq n \leq N$).

3. What will the course be about?

This course is about various ways in which probabilistic ideas enter into number theory. There are two broad aspects of this: the first is using probabilistic language, ideas and methods to actually formulate and prove results; and the second is developing probabilistic heuristics to guess what might be true. We will see both of these strands in the course. Probabilistic number theory in the first sense began in the 1930s-40s (around the time that Kolmogorov formulated his rigorous theory of probability), and some of what we see will be quite classical results on prime divisors and on the statistical behaviour of the Riemann zeta function (though with updated proofs). We will also see some very modern applications and variants of the ideas.

The course will have three main chapters.

(i) The first chapter will explore the behaviour of *additive functions*, which are a class of functions of which the prototype is the number of prime factors function. We will see that the theory of additive functions is closely connected to the theory of sums of independent (or "almost independent") random variables, and a highlight will be the Erdős–Kac central limit theorem. We will also see how these ideas can be useful in some unexpected situations, such as in an ergodic-theoretic problem and when studying the odd Goldbach conjecture.

(ii) In the second chapter we will investigate heuristics, including for the Riemann Hypothesis and for the distribution of primes in various sets. We will see that these heuristics often give a good first approximation to what is going on, but they can fail in interesting ways when examined in more detail. A highlight will be the work of Maier showing discrepancies from the Cramér model for the number of primes in short intervals.

(iii) The third chapter will explore various issues involving the Riemann zeta function $\zeta(s)$ on the critical line $\Re(s) = 1/2$. It turns out that here also one can find "almost independent" behaviour, and we will see this in Selberg's classical central limit theorem for $\log|\zeta(1/2+it)|$. We will also discuss some very modern work connected with this, including on the moments of the zeta function, random matrix models, and possibly the connection with a probabilistic object called branching random walk.

4. Facts about prime numbers

Before starting the course, it will be helpful to know some basic facts about the distribution of prime numbers. Let $\pi(x)$ denote the number of primes p that are $\leq x$.

Fact 1 (Chebychev, c. 1850)**.** *For any $x \geq 2$ we have*

$$\pi(x) \asymp \frac{x}{\log x}.$$

Fact 2 (Mertens, 1874)**.** *There exists a constant c_1 such that, for any $x \geq 2$, we have*

$$\sum_{p \leq x} \frac{\log p}{p} = \log x + O(1), \quad \textit{and} \quad \sum_{p \leq x} \frac{1}{p} = \log\log x + c_1 + O(\frac{1}{\log x}).$$

Fact 3 (Mertens, 1874)**.** *There exists a constant c_2 such that, for any $x \geq 2$, we have*

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right) = \frac{e^{-c_2}}{\log x} \left(1 + O\left(\frac{1}{\log x}\right)\right).$$

The proofs of the above facts are short and elementary (but clever)— some will appear as exercises on the first problem sheet.

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