LECTURE NOTES 3 FOR CAMBRIDGE PART III COURSE ON "THE RIEMANN ZETA FUNCTION", LENT 2014

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Abstract. These are rough notes covering the third block of lectures in "The Riemann Zeta Function" course. In these lectures we will see how certain Dirichlet polynomials can detect zeros of the zeta function, and we will apply Halász's inequality to those Dirichlet polynomials to obtain a basic zero-density estimate. This has applications to the distribution of primes in short intervals.

(No originality is claimed for any of the contents of these notes. In particular, they borrow from the classic books of Ivić [2] and Titchmarsh [3].)

12. Primes in short intervals

If the Riemann Hypothesis is true then $\Psi(x) = x + O(\sqrt{x}\log^2 x)$, and so

$$\begin{aligned}\Psi(x+\sqrt{x}\log^3 x) - \Psi(x) &= (x+\sqrt{x}\log^3 x + O(\sqrt{x}\log^2 x)) - (x + O(\sqrt{x}\log^2 x)) \\ &= \sqrt{x}\log^3 x + O(\sqrt{x}\log^2 x) \\ &= (1+o(1))\sqrt{x}\log^3 x.\end{aligned}$$

In particular, if x is large enough then the short interval $(x, x+\sqrt{x}\log^3 x]$ will contain prime powers (and, since $\Psi(x) = \sum_{p\leq x}\log p + O(\sqrt{x})$, it will actually contain primes.)

We don't know how to prove the Prime Number Theorem with a power saving error term, but we might still hope to show that intervals of power length (e.g. the interval $(x, x+x^{0.99}]$) always contain primes when x is large. *This is because we might expect the error term in the Prime Number Theorem to vary slowly with x, so when we take the difference $\Psi(x+x^{0.99}) - \Psi(x)$ the error terms partially cancel.*

To explore this idea we will use[1] von Mangoldt's explicit formula (Theorem 6.5), which asserts that

$$\Psi(x) = x - \sum_{\substack{\rho:\ \zeta(\rho)=0, \\ |\Im(\rho)|\leq T}} \frac{x^\rho}{\rho} + O\left(\frac{x\log^2 x}{T}\right)$$

for any choice of $2 \leq T \leq x$. We may assume that the sum is over zeros ρ such that $0 < \Re(\rho) < 1$, since we know from our zero-free regions that the zeta function has no zeros with real part ≥ 1, and we know from the functional equation (Theorem 6.1) that

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[1]One can also use Perron's formula directly to study primes in short intervals, but that would be more complicated.

the only zeros with real part ≤ 0 are the trivial zeros at $s = -2, -4, -6, ...$, whose contribution is

$$\sum_{k=1}^{\infty} \frac{1}{(-2k)x^{2k}} = O(1/x^2) = O\left(\frac{x \log^2 x}{T}\right).$$

With this restriction the sum over ρ is certainly a finite sum, since otherwise the zeros would have a limit point and the zeta function would be identically zero.

Remark 12.1. Note that the sum over ρ forms part of the error term in the Prime Number Theorem, but we see it is actually quite structured.

For any $1 \leq y \leq x$, the explicit formula yields that

$$\begin{aligned} \Psi(x+y) - \Psi(x) &= y - \sum_{\substack{\rho:\, \zeta(\rho)=0, \\ |\Im(\rho)| \leq T}} \frac{(x+y)^\rho - x^\rho}{\rho} + O\left(\frac{x \log^2 x}{T}\right) \\ &= y - \sum_{\substack{\rho:\, \zeta(\rho)=0, \\ |\Im(\rho)| \leq T}} x^\rho \frac{(1+(y/x))^\rho - 1}{\rho} + O\left(\frac{x \log^2 x}{T}\right) \\ &= y + O\left(\frac{y}{x} \sum_{\substack{\rho:\, \zeta(\rho)=0, \\ |\Im(\rho)| \leq T}} x^{\Re(\rho)}\right) + O\left(\frac{x \log^2 x}{T}\right), \end{aligned}$$

since $\frac{(1+(y/x))^\rho - 1}{\rho} = \frac{e^{O((y/x)|\rho|)} - 1}{\rho} = O(y/x)$ if $y/x \leq 1/|\rho|$ (by Taylor expansion), whilst if $1/|\rho| < y/x$ then trivially $\frac{(1+(y/x))^\rho - 1}{\rho} \ll \frac{1}{|\rho|} < y/x$. Here we have already gained something because the size of the first "big Oh" term decreases with the interval length y, unlike in a direct application of the prime number theorem.

The Vinogradov–Korobov zero-free region (Corollary 11.3) implies that

$$\Re(\rho) \leq 1 - \frac{c}{\log^{2/3} T (\log\log T)^{1/3}}$$

for every ρ in the sum (if T is large), but this bound is not good enough if there are too many terms in the sum. However, we might hope to show that there *cannot* be many terms in the sum where $\Re(\rho)$ is as large as the Vinogradov–Korobov bound. To make this precise, let $N(T)$ denote the number of zeros ρ of the Riemann zeta function, counted with multiplicity, such that $0 < \Re(\rho) < 1$ and $0 \leq \Im(\rho) \leq T$; and for any $0 < \sigma < 1$, let $N(\sigma, T)$ denote the number of zeros ρ of the Riemann zeta function, counted with multiplicity, such that

$$\Re(\rho) \geq \sigma \quad \text{and} \quad 0 \leq \Im(\rho) \leq T.$$

Lemma 12.2. *Let $0 < \epsilon \leq 1/4$ be a small parameter, let T be large, and let the notation be as above. Then for any $x \geq T$ and any $1 \leq y \leq x$,*

$$\Psi(x+y) - \Psi(x) = y + O\left(y \sum_{0 \leq j \leq 1/(2\epsilon)} \min\{x^{-j\epsilon}, x^{-c/(\log^{2/3} T (\log\log T)^{1/3})}\} N(1-(j+1)\epsilon, T)\right) + O\left(yx^{-1/2}N(T)\right) + O\left(\frac{x\log^2 x}{T}\right).$$

Proof of Lemma 12.2. The lemma really just collects together the foregoing discussion. All that remains is to show that $\frac{y}{x}\sum_{\substack{\rho:\, \zeta(\rho)=0, \\ |\Im(\rho)| \leq T}} x^{\Re(\rho)}$ is

$$O\left(y \sum_{0 \leq j \leq 1/(2\epsilon)} \min\{x^{-j\epsilon}, x^{-c/(\log^{2/3} T (\log\log T)^{1/3})}\} N(1-(j+1)\epsilon, T)\right) + O\left(yx^{-1/2}N(T)\right),$$

or equivalently to show that $\sum_{\substack{\rho:\, \zeta(\rho)=0, \\ |\Im(\rho)| \leq T}} x^{\Re(\rho)}$ is

$$O\left(\sum_{0 \leq j \leq 1/(2\epsilon)} \min\{x^{1-j\epsilon}, x^{1-c/(\log^{2/3} T (\log\log T)^{1/3})}\} N(1-(j+1)\epsilon, T)\right) + O\left(\sqrt{x}N(T)\right).$$

But we see

$$\begin{aligned}\sum_{\substack{\rho:\, \zeta(\rho)=0, \\ |\Im(\rho)| \leq T}} x^{\Re(\rho)} &= \sum_{0 \leq j \leq 1/(2\epsilon)} \sum_{\substack{\rho:\, \zeta(\rho)=0, \\ 1-(j+1)\epsilon \leq \Re(\rho) < 1-j\epsilon, \\ |\Im(\rho)| \leq T}} x^{\Re(\rho)} + O\left(\sum_{\substack{\rho:\, \zeta(\rho)=0, \\ \Re(\rho) \leq 1/2, \\ |\Im(\rho)| \leq T}} x^{\Re(\rho)}\right) \\ &\ll \sum_{0 \leq j \leq 1/(2\epsilon)} \min\{x^{1-j\epsilon}, x^{1-c/(\log^{2/3} T (\log\log T)^{1/3})}\} \sum_{\substack{\rho:\, \zeta(\rho)=0, \\ 1-(j+1)\epsilon \leq \Re(\rho) < 1-j\epsilon, \\ |\Im(\rho)| \leq T}} 1 + \sqrt{x} \sum_{\substack{\rho:\, \zeta(\rho)=0, \\ \Re(\rho) \leq 1/2, \\ |\Im(\rho)| \leq T}} 1,\end{aligned}$$

because of the Vinogradov–Korobov zero-free region. And from the definition of the zeta function we always have $\zeta(\sigma - it) = \overline{\zeta(\sigma + it)}$ (for σ, t real), so the sums over ρ here are $\leq 2N(1-(j+1)\epsilon, T)$ and $\leq 2N(T)$, respectively. □

In the rest of this chapter we will obtain bounds for the functions $N(T)$ and $N(\sigma, T)$, which we can substitute into Lemma 12.2 to study primes in short intervals. Bounding $N(T)$ doesn't require much arithmetic information about the zeta function (it is more or less a complex analysis fact, ultimately depending on very crude upper bounds for the size of the zeta function), but showing that $N(\sigma, T)$ is small when σ is close to 1 is more challenging and interesting. Such results are called *zero-density estimates.* Of course we really think (but cannot prove) that $N(\sigma, T) = 0$ whenever $\sigma > 1/2$, because of the Riemann Hypothesis.

13. Counting all the zeros

In this section we give an upper bound for $N(T)$, the function which counts all zeros ρ of the zeta function with $0 < \Re(\rho) < 1$ (the so-called *critical strip*) and $0 \leq \Im(\rho) \leq T$. In fact we obtain a bit more precise result, bounding the number of zeros in horizontal strips of width 1.

Lemma 13.1. *For any $t \geq 2$,*

$$N(t+1) - N(t) = O(\log t).$$

Since there are certainly only finitely many zeta zeros ρ with $0 < \Re(\rho) < 1$ and $0 \leq \Im(\rho) \leq 2$, the following is an immediate corollary of the lemma.

Corollary 13.2. *For any $T \geq 2$,*

$$N(T) = O(T \log T).$$

The proof of Lemma 13.1 more or less follows from a result about counting zeros of the zeta function in discs around points $1 + it$. You will prove that counting result as question 4 on Example Sheet 2 [[and a printed solution will be provided after the examples class]], using some tools from Chapter 1. However, to obtain the precise statement of Lemma 13.1 one also needs to use the functional equation once.

Proof of Lemma 13.1. By question 4 on Example Sheet 2, for any $t \geq 2$ the number of zeros of the zeta function in the disc $|s - (1+it)| \leq 0.99$ is $O(\log t)$. In particular, the number of zeros ρ in the box

$$1/2 \leq \Re(\rho) < 1, \qquad t \leq \Im(\rho) \leq t + 1/2$$

is $O(\log t)$, since that box is completely contained in the disc. The same is true in the box where $t + 1/2 \leq \Im(\rho) \leq t+1$, so in total the number of zeros with $1/2 \leq \Re(\rho) < 1$ and $t \leq \Im(\rho) \leq t+1$ is $O(\log t)$.

Finally, if $\rho = \sigma + i\tau$ is a zero of the zeta function with $0 < \Re(\rho) = \sigma \leq 1/2$ then, by the functional equation (Theorem 6.1),

$$\zeta(1-\sigma-i\tau) = \zeta(1-\rho) = \frac{\pi^{-\rho/2}\Gamma(\rho/2)\zeta(\rho)}{\pi^{-(1-\rho)/2}\Gamma((1-\rho)/2)} = 0.$$

And we always have $\zeta(\overline{s}) = \overline{\zeta(s)}$, so we have $\zeta(1-\sigma+i\tau) = \overline{\zeta(1-\sigma-i\tau)} = 0$. Note that $1/2 \leq 1-\sigma < 1$, so the total number of zeros with $0 < \Re(\rho) \leq 1/2$ and $t \leq \Im(\rho) \leq t+1$ is actually the same as the number with $1/2 \leq \Re(\rho) < 1$ and $t \leq \Im(\rho) \leq t+1$, which we saw is $O(\log t)$. $\square$

14. Counting the zeros with large real part

In this section we give an upper bound for $N(\sigma, T)$, when $1/2 \leq \sigma < 1$. We trivially always have the bound $N(\sigma, T) \leq N(T) = O(T \log T)$, but when σ is moderately close to 1 we will improve this bound quite a lot, showing that the zeta function cannot have many zeros with real part close to 1.

When we proved our zero-free regions for the zeta function, we exploited the fact that if $\zeta(s) = 0$ for some s just left of the 1-line then $\zeta(s')$ is small for some s' just to the right of the 1-line. Then we showed, by considering the Euler product, that if $\zeta(s')$ is small then the zeta function must be large at some other point just right of the 1-line, which is impossible. To bound $N(\sigma, T)$ we will also switch from counting zeros to counting "large" values of other Dirichlet polynomials $\sum_{n \leq N} \frac{a_n}{n^s}$. Because we are no longer very close to the 1-line, so don't have access to the Euler product, we won't be able to show this is impossible, but we will show it cannot happen at many points s.

Definition 14.1. We define the *Möbius function* $\mu(n)$ to be $(-1)^w$ if n has w distinct prime factors, and to be 0 if n has any repeated prime factors.

Lemma 14.2. *Let T and M be large, and suppose that $0 < \sigma \leq 1$ and $T/2 \leq t \leq T$. Then*
$$\zeta(\sigma + it) \sum_{m \leq M} \frac{\mu(m)}{m^{\sigma+it}} = 1 + \sum_{\min\{M,T\} < n \leq MT} \frac{a_n}{n^{\sigma+it}} + O\left(\frac{M \log M}{T^\sigma M^\sigma}\right),$$
where $a_n := \sum_{m|n, m \leq M, (n/m) \leq T} \mu(m)$.

Proof of Lemma 14.2. As we have seen many times before, the Hardy–Littlewood approximation to the zeta function (Theorem 3.3) yields that
$$\zeta(\sigma + it) = \sum_{n \leq T} \frac{1}{n^{\sigma+it}} + \frac{T^{1-\sigma-it}}{\sigma + it - 1} + O(T^{-\sigma}) = \sum_{n \leq T} \frac{1}{n^{\sigma+it}} + O(T^{-\sigma}),$$
since $T/2 \leq t \leq T$. So multiplying out term by term, we find
$$\begin{aligned}
\zeta(\sigma + it) \sum_{m \leq M} \frac{\mu(m)}{m^{\sigma+it}} &= \sum_{n \leq T} \frac{1}{n^{\sigma+it}} \sum_{m \leq M} \frac{\mu(m)}{m^{\sigma+it}} + O\left(T^{-\sigma} \sum_{m \leq M} \frac{1}{m^\sigma}\right) \\
&= \sum_{\substack{nm \leq MT, \\ n \leq T, m \leq M}} \frac{\mu(m)}{(nm)^{\sigma+it}} + O(T^{-\sigma} M^{1-\sigma} \log M) \\
&= \sum_{n \leq MT} \frac{a_n}{n^{\sigma+it}} + O\left(\frac{M \log M}{T^\sigma M^\sigma}\right),
\end{aligned}$$
by definition of the coefficients a_n (and relabelling nm as n).

Finally, if $n \leq \min\{M,T\}$ then $a_n = \sum_{m|n} \mu(m)$. And by definition of the Möbius function $\mu(m)$, we see that

$$\sum_{m|n} \mu(m) = \sum_{\substack{m|n, \\ m \text{ squarefree}}} \mu(m) = \prod_{p|n} (1+\mu(p))$$

is zero unless $n = 1$, in which case it is 1. Therefore $a_1 = 1$, and $a_n = 0$ for all $2 \leq n \leq \min\{M,T\}$. □

If the size of M is such that the "big Oh" term in Lemma 14.2 has absolute value $\leq 1/2$, say, then the Dirichlet polynomial $\sum_{\min\{M,T\}<n\leq MT} \frac{a_n}{n^{\sigma+it}}$ will *detect zeros of the zeta function*, in the sense that it is forced to have absolute value at least 1/2 at any zero $\sigma + it$. This is recorded in the next lemma.

Lemma 14.3 (Zero Detection Lemma)**.** *Let T be large, and let $0 < \sigma \leq 1$. Also let M be large, and suppose that $M^{1-\sigma}\log^2 M \leq T^{\sigma}$. Then there exist real numbers b_n, depending only on M and T, such that $|b_n| \leq \sum_{d|n} 1$ for all n, and such that*

$$\sum_{0\leq k\leq \max\{(\log M)/\log 2, (\log T)/\log 2\}} \left| \sum_{2^k \min\{M,T\}<n\leq 2^{k+1}\min\{M,T\}} \frac{b_n}{n^s} \right| \geq 1/2$$

whenever s is a zero of the zeta function such that $\Re(s) \geq \sigma$ and $T/2 \leq \Im(s) \leq T$.

Proof of Lemma 14.3. Note that if $\Re(s) \geq \sigma$ then $M(\log M)/(MT)^{\Re(s)} \leq M(\log M)/(MT)^{\sigma} \leq 1/\log M$, by our hypothesis about the size of M. Therefore if $T/2 \leq \Im(s) \leq T$ and $\zeta(s) = 0$, Lemma 14.2 implies that

$$\left| \sum_{\min\{M,T\}<n\leq MT} \frac{a_n}{n^s} \right| = 1 + O\left(\frac{1}{\log M}\right) \geq 1/2.$$

By the triangle inequality we have

$$\left| \sum_{\min\{M,T\}<n\leq MT} \frac{a_n}{n^s} \right| \leq \sum_{0\leq k\leq \max\{(\log M)/\log 2, (\log T)/\log 2\}} \left| \sum_{2^k \min\{M,T\}<n\leq 2^{k+1}\min\{M,T\}} \frac{b_n}{n^s} \right|,$$

where b_n is equal to a_n if $n \leq MT$, and b_n is equal to zero otherwise. Finally, the bound $|b_n| \leq \sum_{d|n} 1$ follows immediately since $|a_n| \leq \sum_{m|n, m\leq M} 1 \leq \sum_{m|n} 1$, by definition. □

Remark 14.4. The point of multiplying $\zeta(\sigma + it)$ by $\sum_{m\leq M} \mu(m)/m^{\sigma+it}$ in Lemma 14.2 was that it killed off all the terms with $n \leq \min\{M,T\}$ when we multiplied out, apart from the constant term 1. This is not surprising, since we know that $\sum_{m=1}^{\infty} \mu(m)/m^s = 1/\zeta(s)$ when $\Re(s) > 1$. The remaining terms with $n > \min\{M,T\}$ have denominators of relatively large size $n^{\Re(s)} \geq n^{\sigma} > \min\{M^{\sigma}, T^{\sigma}\}$, which (as we shall see and exploit) makes it rare for the sum in the conclusion of Lemma 14.3 to be $\geq 1/2$.

In order to exploit Lemma 14.3 we will need a new tool, a simple but very important inequality of Halász. The following is a fairly special case that will suffice for us.

Lemma 14.5 (Halász's inequality (L^1 version), 1969-1970). *Let $(c_n)_{N \leq n \leq 2N}$ be any complex numbers, and let $(\sigma_1, t_1), ..., (\sigma_R, t_R)$ be any pairs of real numbers. Then*

$$\sum_{1 \leq r \leq R} \left| \sum_{N \leq n \leq 2N} \frac{c_n}{n^{\sigma_r + it_r}} \right| \leq \sqrt{\sum_{N \leq n \leq 2N} |c_n|^2} \sqrt{\sum_{r,s \leq R} \left| \sum_{N \leq n \leq 2N} \frac{1}{n^{\sigma_r + \sigma_s + i(t_r - t_s)}} \right|}.$$

Proof of Lemma 14.5. The proof is just a clever application of the Cauchy–Schwarz inequality. For each $1 \leq r \leq R$, let η_r be the complex number of absolute value 1 such that $\left|\sum_{N \leq n \leq 2N} \frac{c_n}{n^{\sigma_r + it_r}}\right| = \eta_r \sum_{N \leq n \leq 2N} \frac{c_n}{n^{\sigma_r + it_r}}$. Then

$$\begin{aligned}
\sum_{1 \leq r \leq R} \left| \sum_{N \leq n \leq 2N} \frac{c_n}{n^{\sigma_r + it_r}} \right| = \sum_{1 \leq r \leq R} \eta_r \sum_{N \leq n \leq 2N} \frac{c_n}{n^{\sigma_r + it_r}} &= \sum_{N \leq n \leq 2N} c_n \sum_{1 \leq r \leq R} \eta_r \frac{1}{n^{\sigma_r + it_r}} \\
&\leq \sqrt{\sum_{N \leq n \leq 2N} |c_n|^2} \sqrt{\sum_{N \leq n \leq 2N} \left| \sum_{1 \leq r \leq R} \eta_r \frac{1}{n^{\sigma_r + it_r}} \right|^2} \\
&= \sqrt{\sum_{N \leq n \leq 2N} |c_n|^2} \sqrt{\sum_{r,s \leq R} \eta_r \overline{\eta_s} \sum_{N \leq n \leq 2N} \frac{1}{n^{\sigma_r + it_r + \sigma_s - it_s}}}
\end{aligned}$$

by the Cauchy–Schwarz inequality. The lemma follows immediately. □

Halász's inequality is powerful because it lets us pass from working with the sums $\sum_{N \leq n \leq 2N} c_n / n^{\sigma + it}$ to working with more standard sums $\sum_{N \leq n \leq 2N} 1/n^{\sigma + it}$, in which there are no difficult coefficients. Building on it we can prove the following lemma, which is a very special variant of a "large values" inequality of Montgomery.

Lemma 14.6. *Let $\sigma > 0$ and let T be large. Also let $(\sigma_1, t_1), ..., (\sigma_R, t_R)$ be any pairs of real numbers such that $\sigma_r \geq \sigma$ and $|t_r| \leq T$ for all r, and such that $|t_r - t_s| \geq 1$ whenever $r \neq s$. Finally, let $N \geq T$ and let $(c_n)_{N \leq n \leq 2N}$ be any complex numbers. Then*

$$\sum_{1 \leq r \leq R} \left| \sum_{N \leq n \leq 2N} \frac{c_n}{n^{\sigma_r + it_r}} \right| \ll \frac{\sqrt{RN \log T}}{N^{\sigma}} \sqrt{\sum_{N \leq n \leq 2N} |c_n|^2}.$$

Proof of Lemma 14.6. For any $r \leq R$ and $s \leq R$ we see

$$\left| \sum_{N \leq n \leq 2N} \frac{1}{n^{\sigma_r + \sigma_s + i(t_r - t_s)}} \right| \ll \frac{1}{N^{\sigma_r + \sigma_s}} \max_{N \leq N' \leq 2N} \left| \sum_{N \leq n \leq N'} n^{-i(t_r - t_s)} \right| \leq \frac{1}{N^{2\sigma}} \max_{N \leq N' \leq 2N} \left| \sum_{N \leq n \leq N'} n^{-i(t_r - t_s)} \right|,$$

by Abel's summation lemma (as in the proof of Lemma 3.4) and our assumption that $\sigma_r, \sigma_s \geq \sigma$. Moreover, note that $n^{-i(t_r - t_s)} = e(f(n))$ where $f(x) = -(1/2\pi)(t_r - t_s) \log x$, and $f'(x)$ is continuous, monotonic, and satisfies $|f'(x)| = |t_r - t_s|/(2\pi x) \leq 2T/(2\pi N) \leq$

$1/\pi$. Therefore Van der Corput's Lemma (Lemma 3.4) is applicable, and shows that

$$\sum_{N \le n \le N'} n^{-i(t_r - t_s)} = \int_N^{N'} x^{-i(t_r - t_s)} dx + O(1) = O\left(\frac{N}{1 + |t_r - t_s|}\right).$$

Substituting all of this into Lemma 14.5, we obtain that

$$\sum_{1 \le r \le R} \left| \sum_{N \le n \le 2N} \frac{c_n}{n^{\sigma_r + it_r}} \right| \ll \frac{\sqrt{N}}{N^\sigma} \sqrt{\sum_{N \le n \le 2N} |c_n|^2} \sqrt{\sum_{r,s \le R} \frac{1}{1 + |t_r - t_s|}}.$$

For any fixed $r \le R$, the numbers $t_r - t_s$ are all at least one apart as s varies, since we assumed that is true for the numbers t_s. Moreover they are all at most $2T$ in absolute value. Therefore

$$\sum_{r,s \le R} \frac{1}{1 + |t_r - t_s|} \ll \sum_{r \le R} \sum_{1 \le s \le 2T} \frac{1}{s} \ll R \log T,$$

from which the lemma follows. □

Finally, by combining Lemma 14.6 with our Zero Detection Lemma (Lemma 14.3) we can show that very few zeta zeros have large real part.

Theorem 14.7 (Basic Zero-Density Estimate)**.** *For any large T, and any $3/4 \le \sigma < 1$, we have the bound*

$$N(\sigma, T) = O(T^{4(1-\sigma)} \log^8 T).$$

Proof of Theorem 14.7. Let $\mathcal{N}$ denote the multiset of all zeta zeros ρ with $\Re(\rho) \ge \sigma$ and $T/2 \le \Im(\rho) \le T$, counted with multiplicity. We will show that $\#\mathcal{N} = O(T^{4(1-\sigma)} \log^7 T)$, which suffices to prove the theorem on replacing T by $T/2^k$ and summing over k.

Let $\mathcal{N}_{\text{even}} \subseteq \mathcal{N}$ denote the subset consisting of zeros ρ such that $\lfloor \Im(\rho) \rfloor$ is even, and similarly define $\mathcal{N}_{\text{odd}} \subseteq \mathcal{N}$. We will in fact show that $\#\mathcal{N}_{\text{even}} = O(T^{4(1-\sigma)} \log^7 T)$, and an exactly similar argument applies to $\mathcal{N}_{\text{odd}}$.

Next, choose a subset $\mathcal{N}' \subseteq \mathcal{N}_{\text{even}}$ by throwing away all but one zero with imaginary part in each strip $2k \le \Im(\rho) < 2k+1$ (and, if there are no zeros with imaginary part in a given strip, doing nothing). In particular, notice the zeros in $\mathcal{N}'$ are counted *without* multiplicity, since we throw away repeated copies of a single zero. By Lemma 13.1, in each strip there were at most $O(\log T)$ zeros, counted with multiplicity, so it will now suffice to show that

$$\#\mathcal{N}' = O(T^{4(1-\sigma)} \log^6 T).$$

Having completed the above tidying up, we can apply Lemma 14.3 with the choice $M = T$, obtaining that

$$\sum_{\rho \in \mathcal{N}'} \sum_{0 \le k \le (\log T)/\log 2} \left| \sum_{2^k T < n \le 2^{k+1} T} \frac{b_n}{n^\rho} \right| \ge (1/2)\#\mathcal{N}'.$$

On the other hand, since the points in $\mathcal{N}'$ have imaginary parts differing by at least one, and since the lower endpoint of each sum is $\geq T$, we can apply Lemma 14.6 to obtain that

$$\sum_{\rho \in \mathcal{N}'} \left| \sum_{2^k T < n \leq 2^{k+1} T} \frac{b_n}{n^\rho} \right| \ll \frac{\sqrt{(\#\mathcal{N}') 2^k T \log T}}{(2^k T)^\sigma} \sqrt{\sum_{2^k T < n \leq 2^{k+1} T} |b_n|^2} \quad \forall 0 \leq k \leq (\log T)/\log 2.$$

We know from Lemma 14.3 that $|b_n| \leq \sum_{d|n} 1$, and we see

$$\sum_{N < n \leq 2N} \left(\sum_{d|n} 1 \right)^2 = \sum_{d_1 \leq 2N} \sum_{d_2 \leq 2N} \sum_{\substack{N < n \leq 2N, \\ [d_1, d_2] | n}} 1 = \sum_{h \leq 2N} \sum_{\substack{e_1, e_2 \leq 2N, \\ (e_1, e_2) = 1}} \sum_{\substack{N < n \leq 2N, \\ h e_1 e_2 | n}} 1 \ll N \sum_{h \leq 2N} \frac{1}{h} \sum_{\substack{e_1, e_2 \leq 2N, \\ (e_1, e_2) = 1}} \frac{1}{e_1 e_2}$$
$$\ll N \log^3 N,$$

on dividing up pairs of divisors $d_1 = e_1 h$, $d_2 = e_2 h$ according to their highest common factor h. Inserting this in our bound, and summing over $0 \leq k \leq (\log T)/\log 2$, we conclude that

$$\#\mathcal{N}' \leq 2 \sum_{\rho \in \mathcal{N}'} \sum_{0 \leq k \leq (\log T)/\log 2} \left| \sum_{2^k T < n \leq 2^{k+1} T} \frac{b_n}{n^\rho} \right| \ll \sqrt{\#\mathcal{N}'} \log^2 T \sum_{0 \leq k \leq (\log T)/\log 2} (2^k T)^{1-\sigma}$$
$$\ll \sqrt{\#\mathcal{N}'} (\log^3 T) T^{2(1-\sigma)}.$$

The claimed bound $\#\mathcal{N}' = O(T^{4(1-\sigma)} \log^6 T)$ follows immediately. □

Notice that Theorem 14.7 is trivial if $\sigma < 3/4$ (since we always have $N(\sigma, T) \leq N(T) = O(T \log T)$), so we may apply the theorem on that range as well.

15. Back to primes in short intervals

[[We ran out of time to finish off this chapter in lectures, so the following theorem will become a question on Example Sheet 3.]]

Having proved our zero-density estimate (Theorem 14.7), we can immediately deduce a fairly impressive result about primes in short intervals.

Theorem 15.1 (Hoheisel type prime number theorem). *Let* $0.75 < \theta \leq 1$ *be fixed. Then*

$$\Psi(x + x^\theta) - \Psi(x) = (1 + o(1)) x^\theta \quad \text{as } x \to \infty.$$

Proof of Theorem 15.1. We apply Lemma 12.2 with the choices $y = x^\theta$, $\epsilon = 1/\log x$ and $T = x^{1-\theta} \log^3 x$ (to neutralise the third error term there). The lemma implies that

$\Psi(x+x^{\theta}) - \Psi(x)$ is

$$= x^{\theta} + O\left(x^{\theta} \sum_{0 \leq j \leq 1/(2\epsilon)} \min\{x^{-j\epsilon}, x^{-c/(\log^{2/3} x (\log\log x)^{1/3})}\} N(1-(j+1)\epsilon, x^{1-\theta}\log^3 x) \right)$$
$$+ O\left(x^{\theta-1/2} N(x^{1-\theta}\log^3 x)\right) + O\left(\frac{x^{\theta}}{\log x}\right),$$

and by our result about counting all the zeros (Corollary 13.2) the second error term is $O(x^{1/2}\log^4 x)$, which is certainly $o(x^{\theta})$.

Finally, our zero-density estimate (Theorem 14.7) shows that

$$N(1-(j+1)\epsilon, x^{1-\theta}\log^3 x) \ll (x^{1-\theta}\log^3 x)^{4(j+1)\epsilon}\log^8 x \ll (x^{1-\theta}\log^3 x)^{4j\epsilon}\log^8 x$$
$$\ll x^{4(1-\theta)j\epsilon}\log^{14} x,$$

since $\epsilon = 1/\log x$ (so the difference between $j\epsilon$ and $(j+1)\epsilon$ in the exponent is negligible), and since $4j\epsilon \leq 2$ for $j \leq 1/(2\epsilon)$. Note that $4(1-\theta) = 1 - 2\delta(\theta)$ for a certain constant $\delta(\theta) > 0$, by hypothesis about θ. Thus if $j\epsilon$ is a large enough multiple of $(\log\log x)/\log x$ (depending on θ) we will certainly have

$$N(1-(j+1)\epsilon, x^{1-\theta}\log^3 x) \ll x^{(1-\delta(\theta))j\epsilon}.$$

On the other hand, if $j\epsilon$ is smaller then the logarithmic part of the bound is dominant, and so

$$N(1-(j+1)\epsilon, x^{1-\theta}\log^3 x) \ll \log^{C(\theta)} x,$$

for a certain large constant $C(\theta)$. Thus, remembering that $\epsilon = 1/\log x$, we have in any case that

$$x^{\theta} \sum_{0 \leq j \leq 1/(2\epsilon)} \min\{x^{-j\epsilon}, x^{-c/(\log^{2/3} x (\log\log x)^{1/3})}\} N(1-(j+1)\epsilon, x^{1-\theta}\log^3 x)$$
$$\ll x^{\theta} \sum_{0 \leq j \leq \log\log x} x^{-c/(\log^{2/3} x (\log\log x)^{1/3})} \log^{C(\theta)} x +$$
$$+ x^{\theta} \sum_{\log\log x < j \leq (\log x)/2} \left(x^{-\delta(\theta) j\epsilon} + x^{-c/(\log^{2/3} x (\log\log x)^{1/3})} \log^{C(\theta)} x \right)$$
$$\ll x^{\theta} x^{-c/(\log^{2/3} x (\log\log x)^{1/3})} \log^{C(\theta)+1} x + x^{\theta} \sum_{\log\log x < j \leq (\log x)/2} e^{-\delta(\theta) j}.$$

Both of these terms are $o(x^{\theta})$ as $x \to \infty$, as required. □

The first ever result like Theorem 15.1, proving the existence of primes in intervals of length x^{θ} for some $\theta < 1$, was obtained by Hoheisel in 1930. Hoheisel could only handle the case where $\theta > 1 - 1/33000$, but this is mainly because he did not have access to the Vinogradov–Korobov zero-free region to efficiently handle small j (at the time the best known zero-free region, due to Littlewood, was of the form $\{\sigma \geq 1 - c(\log\log(|t|+3))/\log(|t|+2)\}$).

In the proof of our zero-density estimate, we chose to take $M = T$ so we could apply Lemma 14.6. But there are very many other inequalities like Lemma 14.6, which allow lots of flexibility in the choice of M, and by using those one can show that $N(\sigma, T) \ll T^{(12/5)(1-\sigma)} \log^9 T$ for all σ. That was first done by Huxley [1] in 1972, and lets one prove a Hoheisel type prime number theorem for all $\theta > 7/12$. By introducing many more ideas, one can prove a Hoheisel type theorem when $\theta = 7/12$, and one can show the existence of primes (but not an asymptotic formula) whenever $\theta \geq 0.525$, which is not too far from the squareroot type intervals that can be handled assuming the Riemann Hypothesis.

References

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