An Eyring–Kramers law for the stochastic Allen–Cahn equation in dimension two

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Abstract

We study spectral Galerkin approximations of an Allen–Cahn equation over the two-dimensional torus perturbed by weak space-time white noise of strength $\sqrt{\varepsilon}$. We introduce a Wick renormalisation of the equation in order to have a system that is well-defined as the regularisation is removed. We show sharp upper and lower bounds on the transition times from a neighbourhood of the stable configuration -1 to the stable configuration 1 in the asymptotic regime $\varepsilon \to 0$. These estimates are uniform in the discretisation parameter N, suggesting an Eyring–Kramers formula for the limiting renormalised stochastic PDE. The effect of the "infinite renormalisation" is to modify the prefactor and to replace the ratio of determinants in the finite-dimensional Eyring–Kramers law by a renormalised Carleman–Fredholm determinant.

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1 Introduction

Metastability is a common physical phenomenon in which a system spends a long time in metastable states before reaching its equilibrium. One of the most classical mathematical models where this phenomenon has been studied rigorously is the overdamped motion of a particle in a potential V, given by the Itô stochastic differential equation

$$\mathrm{d}x(t) = -\nabla V(x(t))\,\mathrm{d}t + \sqrt{2\varepsilon}\,\mathrm{d}w(t)\;. \tag{1.1}$$

For small noise strength ε solutions of (1.1) typically spend long stretches of time near local minima of the potential V with occasional, relatively quick transitions between these local minima. The mean transition time between minima is then governed by the Eyring–Kramers law [12, 21]: If τ denotes the expected hitting time of a neighbourhood of a local minimum y of the solution of (1.1) started in another minimum x, and under suitable assumptions on the potential V, the Eyring–Kramers law gives the asymptotic expansion

$$\mathbb{E}[\tau] = \frac{2\pi}{|\lambda_0(z)|}\sqrt{\frac{|\det D^2V(z)|}{\det D^2V(x)}}\,\mathrm{e}^{[V(z)-V(x)]/\varepsilon}\big[1+\mathcal{O}_\varepsilon(1)\big]\;, \tag{1.2}$$

where z is the *relevant saddle* between x and y, and $\lambda_0(z)$ is the (by assumption) unique negative eigenvalue of the Hessian $D^2V(z)$ (more precise bounds on the error term $\mathcal{O}_\varepsilon(1)$ are available, see below). The right exponential rate in this formula was established rigorously using large deviation theory [14]. Rigorous proofs including the prefactor were given in [28, 8, 19], see [4] for a recent survey.

It is natural to study metastability in high- or infinite-dimensional systems and to seek an extension of the Eyring–Kramers law. In this direction the Allen–Cahn equation perturbed by a small noise term is a natural model to study. It is given by the stochastic PDE

$$\partial_t \phi(t,x) = \Delta\phi(t,x) - (\phi(t,x)^3 - \phi(t,x)) + \sqrt{2\varepsilon}\xi(t,x) , \tag{1.3}$$

where ξ is a noise term to be described below. Just like in (1.1) the deterministic part of this equation (set $\xi = 0$ in (1.3)) has gradient form and the relevant potential is given by

$$V(\phi) = \int \Big(\frac{1}{2}|\nabla\phi|^2 - \frac{1}{2}\phi^2 + \frac{1}{4}\phi^4\Big)\,\mathrm{d}x . \tag{1.4}$$

The natural choice of noise term ξ is (at least formally) given by space-time white noise because this choice is compatible with the scalar product used to define the deterministic gradient flow and it makes the dynamics given by (1.3) reversible. The constant profiles $\phi_\pm(x) = \pm 1$ are stable solutions of the deterministic system and it is natural to ask how long a small noise term typically needs to move the system from one of these stable profiles to the other one.

In the case where equation (1.3) is solved over a $1+1$-dimensional time-space domain $(t,x) \in [0,\infty) \times [0,L]$ this question was studied in [3, 2, 6] and an infinite-dimensional version of the Eyring–Kramers formula was established. Let τ denote the first-hitting time of a solution of (1.3) starting near the constant profile ϕ_- of a neighbourhood of the constant profile $\phi_+ = 1$. In [6] it was shown, for example in the case where (1.3) is endowed with periodic boundary conditions on $[0,L]$ and $L < 2\pi$, that

$$\mathbb{E}[\tau] = \frac{2\pi}{|\lambda_0|}\prod_{k\in\mathbb{Z}}\sqrt{\frac{|\lambda_k|}{\nu_k}}\,\mathrm{e}^{[V(\phi_0)-V(\phi_-)]/\varepsilon}[1+\mathcal{O}_\varepsilon(1)] . \tag{1.5}$$

The purpose of the condition $L < 2\pi$ is to ensure that the constant profile $\phi_0 = 0$ is the relevant saddle between the two stable minima $\phi_\pm$; but situations for longer intervals and different choices of boundary conditions are also described in [6]. The sequences λ_k, ν_k appearing in this expression are the eigenvalues of the linearisation of V around ϕ_0 and ϕ_-, the operators $-\partial_x^2 - 1$ and $-\partial_x^2 + 2$, both endowed with periodic boundary condition. All of these eigenvalues are strictly positive, except for $\lambda_0 = -1$. Leaving out the factor $k = 0$, the infinite product in (1.5) can be written as

$$\prod_{k\neq 0}\sqrt{\frac{\lambda_k}{\nu_k}} = \prod_{k\neq 0}\sqrt{\Big(1+\frac{\nu_k-\lambda_k}{\lambda_k}\Big)^{-1}} = \frac{1}{\sqrt{\det(\mathrm{Id}+3P_\perp(-\partial_x^2+1)^{-1})}} , \tag{1.6}$$

where $P_\perp$ projects on the complement of the $k = 0$ mode. This expression converges, because $P_\perp(-\partial_x^2+1)^{-1}$ is a trace-class operator, so that the infinite-dimensional (Fredholm) determinant is well-defined (see for instance [13, 22]).

When trying to extend this result to higher spatial dimensions two problems immediately present themselves. First, for spatial dimension $d \geqslant 2$ the Allen–Cahn equation as stated in (1.3) fails to be well-posed: In this situation already the linear stochastic heat equation (drop the non-linear term $-(\phi^3-\phi)$ in (1.3)) has distribution-valued solutions due to the irregularity of the white noise ξ. In this regularity class $-(\phi^3-\phi)$ does not have a canonical definition. As an illustration for the problems caused by this irregularity, it was shown in [17] that for fixed noise strength ε finite-dimensional Galerkin approximations[1]

$$\partial_t \phi_N = \Delta \phi_N - (P_N \phi_N^3 - \phi_N) + \sqrt{2\varepsilon}\xi_N \tag{1.7}$$

defined over a two-dimensional torus converge to a trivial limit as the approximation parameter N goes to ∞ (precise definitions of the finite-dimensional noise ξ_N and the projection operator P_N are given in Section 2 below). A second related problem is, that for $d \geqslant 2$ the infinite product appearing in (1.5) converges to 0, corresponding to the fact that for $d \geqslant 2$ the operator $3P_\perp(-\Delta+1)^{-1}$ fails to be trace-class so that the Fredholm determinant $\det(\mathrm{Id} - 3P_\perp(-\Delta+1)^{-1})$ is not well-defined.

On the level of the $N \to \infty$ limit for fixed ε the idea of renormalisation, inspired by ideas from Quantum Field Theory (see e.g. [15]), has been very successful over the last years. Indeed, in [9] it was shown that in the two-dimensional case, if the approximations in (1.7) are replaced by

$$\partial_t \phi_N = \Delta \phi_N - (P_N \phi_N^3 - 3\varepsilon C_N \phi_N - \phi_N) + \sqrt{2\varepsilon}\xi_N \;, \tag{1.8}$$

for a particular choice of logarithmically divergent constants (see (2.3) below), the solutions do converge to a non-trivial limit which can be interpreted as *renormalised solutions* of (1.3). This result (for a different choice of C_N) was spectacularly extended to three dimensions in Hairer's pioneering work on regularity structures [16]. For spatial dimension $d \geqslant 4$, equation (1.3) fails to satisfy a *subcriticality condition* (see [16]) and non-trivial renormalised solutions are not expected to exist.

Note that formally the extra term C_N moves the stable solutions further apart to $\pm\sqrt{3\varepsilon C_N + 1}$ (and ultimately to $\pm\infty$ as $N \to \infty$). Note furthermore that while the constants C_N diverge as N goes to ∞, for fixed N they are multiplied with a small factor ε. This suggests that in the small-noise regime the renormalised solutions may still behave as perturbations of the Allen-Cahn equation, despite the presence of the infinite renormalisation constant. In [18] this intuition was confirmed on the level of large deviations. There it was shown that both in two and three dimensions the renormalised stochastic PDE satisfies a large-deviation principle as $\varepsilon \to 0$, with respect to a suitable topology and with rate functional given by

$$\mathcal{I}(\phi) = \int_0^T \int \left(\partial_t \phi - \left(\Delta\phi - (\phi^3 - \phi)\right)^2 \mathrm{d}x\,\mathrm{d}t \;, \tag{1.9}$$

which is exactly the "correct" rate functional one would obtain by formally applying Freidlin–Wentzell theory to (1.3) without any regard to renormalisation. Results in a similar spirit had previously been obtained in [20, 1].

The purpose of this article is to show that the renormalised solutions have the right asymptotic small-noise behaviour even beyond large deviations, and to establish an Eyring–Kramers formula in this framework. As remarked also in [26] nothing seems to be known at this level so far. The key observation is that the introduction of the infinite constant not only permits to define the dynamics, but that it also fixes the problem of diverging prefactor in the Eyring–Kramers law (1.5). More precisely, we argue that in two dimensions the correct Eyring–Kramers formula for the renormalised SPDE is

$$\mathbb{E}[\tau] = \frac{2\pi}{|\lambda_0|}\sqrt{\prod_{k\in\mathbb{Z}^2} \frac{|\lambda_k|}{\nu_k} \exp\left\{\frac{\nu_k - \lambda_k}{|\lambda_k|}\right\}}\, \mathrm{e}^{[V(\phi_0) - V(\phi_-)]/\varepsilon}\left[1 + \mathcal{O}_\varepsilon(1)\right] \;, \tag{1.10}$$

[1] In fact, in [17, 9] the nonlinearity ϕ^3 is not projected onto a finite dimensional space, but this does not affect the result.

where as above the λ_k and ν_k are the eigenvalues of $-\Delta - 1$ and $-\Delta + 2$, now indexed by $k \in \mathbb{Z}^2$. In functional-analytic terms this means that due to the presence of the infinite renormalisation constant the regular determinant from (1.5) is replaced by a renormalised or Carleman–Fredholm determinant of the operator $\mathrm{Id} - 3P_\perp(-\Delta-1)^{-1}$. Unlike the "usual" determinant, the Carleman–Fredholm determinant is defined for the class of Hilbert–Schmidt perturbations of the identity and not only for the smaller class of trace-class perturbations of the identity. Recall, that $(-\Delta - 1)^{-1}$ is Hilbert–Schmidt both in two and three dimensions, but not for $d \geqslant 4$. It is striking to note that these are exactly the dimensions in which the Allen–Cahn (or Φ^4) equation can be solved.

In order to illustrate our result in the easiest possible situation we only consider the case of the Allen–Cahn equation in a small domain $\mathbb{T}^2 = [0, L]^2$ of size $L < 2\pi$ with periodic boundary conditions. As in the one-dimensional case this assumption guarantees that the constant profile ϕ_0 is the relevant saddle. We make use of the ± 1 symmetry of the system to simplify some arguments. Throughout the article, we work in the framework of the finite-dimensional Galerkin approximation (1.8) and derive asymptotic bounds for the expected transition time as $\varepsilon \to 0$ which are uniform in the approximation parameter $N \to \infty$.

On the technical level, our analysis builds on the potential-theoretic approach developed in [8]. As we work in finite dimensions throughout, we can avoid making any use of the analytic tools developed in recent years to deal with singular SPDE. A key argument we use heavily, is the classical Nelson argument which permits to bound expectations of exponentials of Hermite polynomials. Another key argument is the observation from [5] that the field ϕ can be decomposed as $\phi = \bar{\phi} + \phi_\perp$ into its average and the transverse directions and that the (non-convex) potential V is convex in the transverse directions (see Lemma 5.3). An additional key idea, following [11], is to avoid using Hausdorff–Young inequalities in the discussion of Laplace asymptotics.

The rest of this paper is structured as follows: in Sections 2 and 3 we give the precise assumptions, state our main theorem and provide a brief overview over the arguments used in the proof. Lower bounds on the expected transition time are proved in Section 4, upper bounds are proved in Section 5. Some well-known facts about Hermite polynomials and Wick powers are collected in Appendix A.

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2 Results

We consider the sequence of renormalised Allen–Cahn equations

$$\partial_t \phi = \Delta \phi + \big[1 + 3\varepsilon C_N\big]\phi - P_N \phi^3 + \sqrt{2\varepsilon}\, \xi_N \tag{2.1}$$

for $\phi = \phi(t, x) : \mathbb{R}^+ \times \mathbb{T}^2 \to \mathbb{R}$. Here $\mathbb{T}^2 = \mathbb{R}^2/(L\mathbb{Z})^2$ denotes the two-dimensional torus of size $L \times L$, P_N denotes a Galerkin projection on the space spanned by Fourier modes with wave number $|k| = \max\{|k_1|, |k_2|\} \leqslant N$, and $\xi_N = P_N\xi$ is a Galerkin approximation of space-time white noise ξ.

Furthermore, we assume periodic boundary conditions (b.c.), with a domain size satisfying $0 < L < 2\pi$. This assumption guarantees that the identically zero function plays the rôle of the transition state, which separates the basins of attraction of the two stable solutions $\phi_\pm = \pm 1$ of the deterministic Allen-Cahn equation.

Note that the deterministic system is a gradient system, with potential

$$V_N[\phi] = \frac{1}{2}\int_{\mathbb{T}^2} \big[\|\nabla\phi(x)\|^2 - \phi^2(x)\big]\,\mathrm{d}x + \frac{1}{4}\int_{\mathbb{T}^2}\big[\phi^4(x) - 6\varepsilon C_N\phi^2(x) + 3\varepsilon^2 C_N^2\big]\,\mathrm{d}x\,. \tag{2.2}$$

The measure $\mathrm{e}^{-V_N/\varepsilon}$ is an invariant, stationary measure of the stochastic system (2.1), and we will denote by $\mathcal{Z}_N(\varepsilon)$ its normalisation (the partition function of the system). The constant term $3\varepsilon^2 C_N^2$ in the second integral is of course irrelevant for the dynamics, but it will simplify notations. This is related to the fact that $\phi^4(x) - 6\varepsilon C_N\phi^2(x) + 3\varepsilon^2 C_N^2 = {:}\phi^4(x){:}$ is the so-called Wick renormalisation of $\phi^4(x)$. The renormalisation constant C_N is given by

$$C_N = \frac{1}{L^2}\operatorname{Tr}\big(|P_N[-\Delta - 1]^{-1}|\big) := \frac{1}{L^2}\sum_{k\in\mathbb{Z}^2 : |k|\leqslant N}\frac{1}{|\lambda_k|} \tag{2.3}$$

where $\lambda_k = \Omega^2(k_1^2 + k_2^2) - 1$ and $\Omega = 2\pi/L$. Therefore, C_N diverges logarithmically as

$$C_N \asymp \frac{2\pi}{L^2}\log(N)\,. \tag{2.4}$$

The choice of C_N is somewhat arbitrary, as adding a constant independent of N to C_N will also yield a well-defined limit equation as $N \to \infty$. See Remark 2.3 below for the effect of such a shift on the results.

In the deterministic case $\varepsilon = 0$, the Allen–Cahn equation (2.1) has two stable stationary solutions given by $\phi_-(x) = -1$ and $\phi_+(x) = 1$. We are interested in obtaining sharp asymptotics on the expectation of the first-hitting time τ_B of a neighbourhood B of ϕ_+, when starting near ϕ_-. The neighbourhood B should have a minimal size. More precisely, we decompose any function $\phi : \mathbb{T}^2 \to \mathbb{R}$ into its mean and oscillating part by setting

$$\phi(x) = \bar\phi + \phi_\perp(x) \tag{2.5}$$

where the integral of $\phi_\perp$ over $\mathbb{T}^2$ is zero. Then we define the two symmetric sets

$$\begin{aligned} A &= \big\{\phi\colon \bar\phi \in [-1-\delta, -1+\delta], \phi_\perp \in D_\perp\big\}\,, \\ B &= \big\{\phi\colon \bar\phi \in [1-\delta, 1+\delta], \phi_\perp \in D_\perp\big\}\,, \end{aligned} \tag{2.6}$$

where $0 < \delta < 1$ and $D_\perp$ should be large enough for $A \cup B$ to contain most of the mass of the invariant probability measure $\mathcal{Z}_N^{-1}\,\mathrm{e}^{-V_N/\varepsilon}$ of the equation. A sufficient condition for this to hold is that the Fourier components z_k of $\phi_\perp$ satisfy

$$|z_k| \leqslant \sqrt{\frac{c\varepsilon\log(\varepsilon^{-1})[1 + \log\lambda_k]}{\lambda_k}}\,, \tag{2.7}$$

where c is a sufficiently large numerical constant. (Note that functions with Fourier coefficients satisfying (2.7) belong to the fractional Sobolev space H^s for any $s < 0$.)

Our main result for periodic b.c. is as follows.

Theorem 2.1. *Assume $L < 2\pi$. There exists a sequence $\{\mu_N\}_{N\geqslant 1}$ of probability measures concentrated on ∂A such that in the case of periodic b.c.,*

$$\limsup_{N\to\infty} \mathbb{E}^{\mu_N}\big[\tau_B\big] \leqslant \frac{2\pi}{|\lambda_0|}\sqrt{\prod_{k\in\mathbb{Z}^2} \frac{|\lambda_k|}{\nu_k} \exp\biggl\{\frac{\nu_k - \lambda_k}{|\lambda_k|}\biggr\}}\, \mathrm{e}^{[V(\phi_0)-V(\phi_-)]/\varepsilon}\big[1 + c_+\sqrt{\varepsilon}\,\big] ,$$

$$\liminf_{N\to\infty} \mathbb{E}^{\mu_N}\big[\tau_B\big] \geqslant \frac{2\pi}{|\lambda_0|}\sqrt{\prod_{k\in\mathbb{Z}^2} \frac{|\lambda_k|}{\nu_k} \exp\biggl\{\frac{\nu_k - \lambda_k}{|\lambda_k|}\biggr\}}\, \mathrm{e}^{[V(\phi_0)-V(\phi_-)]/\varepsilon}\big[1 - c_-\varepsilon\big] , \tag{2.8}$$

where the constants $c_\pm$ are uniform in ε.

Since $\nu_k = \lambda_k + 3$ and $V(\phi_0) - V(\phi_-) = L^2/4$, the leading term in (2.8) can also be written as

$$2\pi\biggl(\frac{\mathrm{e}^{3/|\lambda_0|}}{|\lambda_0|(\lambda_0+3)} \prod_{k\neq 0} \frac{\mathrm{e}^{3/\lambda_k}}{1 + 3/\lambda_k}\biggr)^{1/2} \mathrm{e}^{L^2/4\varepsilon} . \tag{2.9}$$

The infinite product indeed converges, since

$$\log\biggl(\frac{\mathrm{e}^x}{1+x}\biggr) = x - \log(1+x) \leqslant \frac{1}{2}x^2 , \tag{2.10}$$

and the sum over $\mathbb{Z}^2$ of λ_k^{-2} converges, unlike the sum of λ_k^{-1} that would arise without the regularising term $\mathrm{e}^{-3/\lambda_k}$. On a more abstract level, as already mentioned in the introduction, this is due to the fact that we have replaced the usual Fredholm determinant $\det(\mathrm{Id} + T)$ by the Fredholm–Carleman determinant

$$\det{}_2(\mathrm{Id} + T) = \det(\mathrm{Id} + T)\,\mathrm{e}^{-\operatorname{Tr} T} , \tag{2.11}$$

which is defined for every Hilbert–Schmidt perturbation T of the identity, without the requirement of T to be trace-class [27, Chapter 5].

Remark 2.2. An analogous result holds for zero-flux Neumann b.c. $\partial_x\phi(t,0) = \partial_x\phi(t,L) = 0$, provided $0 < L < \pi$. The only difference is that one has to replace Ω by π/L in the definition of the λ_k, and that the sums are over $k \in \mathbb{N}_0^2$ instead of $k \in \mathbb{Z}^2$.

Remark 2.3. The definition (2.3) of the renormalisation constant C_N is not unique, and one would still obtain a well-defined limit for (2.1) if C_N were replaced by $C_N + \theta/L^2$ for some constant $\theta \in \mathbb{R}$ (or even by $C_N + \theta_N/L^2$, where θ_N converges to a limit θ as $N \to \infty$). One easily checks that the effect of such a shift in the renormalisation constant is to multiply the expected transition time by a factor $\mathrm{e}^{3\theta/2}$.

Remark 2.4. A possible strategy to extend the above Eyring–Kramers formula to a result for the limiting SPDE would be to use results comparing solutions of an SPDE and its spectral Galerkin approximations, such as those obtained in [7]. This strategy was used in the one-dimensional case in [6] (cf. Proposition 3.4 in that work), where it required an a priori bound on the second moment of the expected transition time (any moment of order $p > 1$ would do).

Remark 2.5. In a similar spirit, one would like to extend the above results to initial conditions concentrated in a single point, instead of a non-explicit distribution on a set ∂A. In dimension 1, this can be done using a coupling argument obtained in [23], cf. [6, Proposition 3.6].

Remark 2.6. The error term in $\sqrt{\varepsilon}$ in the upper bound for the expected transition time is due to our using less sharp approximations in the Laplace asymptotics. It is in principle possible, as was done in the one-dimensional case in [11], to derive further terms in the asymptotic expansion. In particular it is expected that the leading error term has order ε.

3 Outline of the proof

The first step is to consider a Galerkin approximation of the solution of (2.1), obtained by truncating its Fourier series to modes with $|k| \leqslant N$. Let $e_k(x) = L^{-1} \mathrm{e}^{\mathrm{i}\Omega k\cdot x}$ denote L^2-normalised basis vectors of $L^2(\mathbb{T}^2)$, and write

$$\phi_N(t,x) = \sum_{k\in\mathbb{Z}^2:|k|\leqslant N} z_k(t) e_k(x) . \tag{3.1}$$

In order to ensure that ϕ_N is real valued the coefficients z_k are chosen to take values in

$$\{(z_k) \in \mathbb{C}^{(-N,\dots,N)^2} \colon z_{-k} = \overline{z_{-k}} \quad \text{for all } k\} \tag{3.2}$$

which we identify with $\mathbb{R}^{(2N+1)^2}$ throughout. In particular, we will always interpret gradients and integration with respect to Lebesgue measure $\mathrm{d}z$ in terms of this identification. The Galerkin approximation with cut-off N is by definition the system of Itô SDEs

$$\mathrm{d}z(t) = -\nabla V_N(z(t))\,\mathrm{d}t + \sqrt{2\varepsilon}\,\mathrm{d}W_t , \tag{3.3}$$

where the potential, obtained by evaluating (2.2) in ϕ_N, is given by

$$V_N(z) = \frac{1}{2}\sum_{|k|\leqslant N} \lambda_k |z_k|^2 + \frac{1}{4}\Bigg(\frac{1}{L^2}\sum_{\substack{k_1+k_2+k_3+k_4=0\\|k_i|\leqslant N}} z_{k_1}z_{k_2}z_{k_3}z_{k_4} - 6\varepsilon C_N \sum_{|k|\leqslant N} |z_k|^2 + 3L^2\varepsilon^2 C_N^2\Bigg) . \tag{3.4}$$

Arguments based on potential theory [8] show that for any finite N one has the relation

$$\mathbb{E}^{\mu_{A,B}}\big[\tau_B\big] = \frac{1}{\operatorname{cap}_A(B)} \int_{B^c} h_{A,B}(z)\,\mathrm{e}^{-V_N(z)/\varepsilon}\,\mathrm{d}z . \tag{3.5}$$

Here $h_{A,B}(z)$ is the *committor function* (or *equilibrium potential*)

$$h_{A,B}(z) = \mathbb{P}^z\big\{\tau_A < \tau_B\big\} , \tag{3.6}$$

where τ_A denotes the first-hitting time of a set $A \subset \mathbb{R}^{(2N+1)^2}$. The term $\operatorname{cap}_A(B)$ is the so-called *capacity*, which admits several equivalent expressions:

$$\begin{aligned}
\operatorname{cap}_A(B) &= \varepsilon \int_{(A\cup B)^c} \|\nabla h_{A,B}(z)\|^2\,\mathrm{e}^{-V_N(z)/\varepsilon}\,\mathrm{d}z && (3.7)\\
&= \varepsilon \inf_{h\in\mathcal{H}_{A,B}} \int_{(A\cup B)^c} \|\nabla h\|^2\,\mathrm{e}^{-V_N(z)/\varepsilon}\,\mathrm{d}z && (3.8)\\
&= \int_{\partial A} \mathrm{e}^{-V_N(z)/\varepsilon}\,\rho_{A,B}(\mathrm{d}z) , && (3.9)
\end{aligned}$$

where $\mathcal{H}_{A,B}$ is the set of functions $h \in H^1$ such that $h = 1$ in A and $h = 0$ in B, and $\rho_{A,B}(\mathrm{d}z)$ is a measure concentrated on ∂A, called the *equilibrium measure*. Furthermore, $\mu_{A,B}$ is the probability measure on ∂A obtained by normalising $\rho_{A,B}\,\mathrm{e}^{-V/\varepsilon}$:

$$\mu_{A,B}(\mathrm{d}z) = \frac{1}{\operatorname{cap}_A(B)}\,\mathrm{e}^{-V_N(z)/\varepsilon}\,\rho_{A,B}(\mathrm{d}z) . \tag{3.10}$$

The following symmetry argument allows us to link the expected transition time to the partition function of the system.

Lemma 3.1. *If A and B are symmetric with respect to the plane $z_0 = 0$ then*

$$\int_{B^c} h_{A,B}(z)\,\mathrm{e}^{-V_N(z)/\varepsilon}\,\mathrm{d}z = \frac{1}{2}\int_{\mathbb{R}^{(2N+1)^2}} \mathrm{e}^{-V_N(z)/\varepsilon}\,\mathrm{d}z =: \frac{1}{2}\mathcal{Z}_N(\varepsilon)\;. \tag{3.11}$$

Proof: Consider the reflection S given by

$$S(z_0, z_\perp) = (-z_0, z_\perp)\;.$$

The potential V_N satisfies the symmetry

$$V_N(Sz) = V_N(z)$$

which implies

$$\int_{\{z_0<0\}} \mathrm{e}^{-V_N(z)/\varepsilon}\,\mathrm{d}z = \int_{\{z_0>0\}} \mathrm{e}^{-V_N(z)/\varepsilon}\,\mathrm{d}z = \frac{1}{2}\mathcal{Z}_N(\varepsilon)\;. \tag{3.12}$$

Assuming we choose A and B such that $B = SA$, the committor satisfies

$$h_{A,B}(z) = h_{B,A}(Sz)\;.$$

In addition, we always have

$$h_{A,B}(z) = 1 - h_{B,A}(z)\;.$$

Now observe that we have

$$\begin{aligned}
&\int h_{A,B}(z)\,\mathrm{e}^{-V_N(z)/\varepsilon}\,\mathrm{d}z \\
&\quad= \int_{\{z_0<0\}} h_{A,B}(z)\,\mathrm{e}^{-V_N(z)/\varepsilon}\,\mathrm{d}z + \int_{\{z_0>0\}} (1-h_{B,A}(z))\,\mathrm{e}^{-V_N(z)/\varepsilon}\,\mathrm{d}z \\
&\quad= \int_{\{z_0<0\}} h_{A,B}(z)\,\mathrm{e}^{-V_N(z)/\varepsilon}\,\mathrm{d}z + \int_{\{z_0>0\}} \mathrm{e}^{-V_N(z)/\varepsilon}\,\mathrm{d}z - \int_{\{z_0<0\}} h_{A,B}(z)\,\mathrm{e}^{-V_N(z)/\varepsilon}\,\mathrm{d}z \\
&\quad= \int_{\{z_0>0\}} \mathrm{e}^{-V_N(z)/\varepsilon}\,\mathrm{d}z\;,
\end{aligned}$$

and the conclusion follows from (3.12). □

As a consequence of (3.11), we can rewrite (3.5) in the form

$$\mathbb{E}^{\mu_{A,B}}\bigl[\tau_B\bigr] = \frac{1}{2\,\operatorname{cap}_A(B)}\mathcal{Z}_N(\varepsilon)\;. \tag{3.13}$$

Note that this relation can also be written as

$$\frac{1}{\mathbb{E}^{\mu_{A,B}}\bigl[\tau_B\bigr]} = 2\mathbb{E}^{\pi_N(\varepsilon)}\bigl[\varepsilon\|\nabla h_{A,B}\|^2\bigr]\;, \tag{3.14}$$

where $\pi_N(\varepsilon)$ is the probability measure on $\mathbb{R}^{(2N+1)^2}$ with density $\mathcal{Z}_N(\varepsilon)^{-1}\,\mathrm{e}^{-V_N(z)/\varepsilon}\,\mathrm{d}z$.

The result will follow if we are able to prove the estimate

$$\operatorname{cap}_A(B) = \sqrt{\frac{|\lambda_0|\varepsilon}{2\pi}}\prod_{0<|k|\leqslant N}\sqrt{\frac{2\pi\varepsilon}{\lambda_k}}\bigl[1+r(\varepsilon)\bigr] \tag{3.15}$$

on the capacity with $-c_-\sqrt{\varepsilon} \leqslant r(\varepsilon) \leqslant c_+\varepsilon$, as well as an estimate on the partition function of the form

$$\frac{1}{2}\mathcal{Z}_N(\varepsilon) = \prod_{|k|\leqslant N}\sqrt{\frac{2\pi\varepsilon}{\lambda_k+3}}\,\mathrm{e}^{L^2/4\varepsilon}\,\mathrm{e}^{3L^2C_N/2}\bigl[1-r(\varepsilon)\bigr]\;. \tag{3.16}$$

4 Lower bound on the expected transition time

We will start by deriving a lower bound on the expected transition time, which is somewhat simpler to obtain than the upper bound. The expression (3.13) shows that to this end, we need to obtain an upper bound on the capacity and a lower bound on the partition function.

4.1 Upper bound on the capacity

One can obtain an upper bound on the capacity by inserting any test function in the right-hand side of (3.8). Let $\delta > 0$ be a small constant and define

$$h_+(z) = \begin{cases} 1 & \text{if } z_0 \leqslant -\delta \;, \\ \dfrac{\displaystyle\int_{z_0}^{\delta} \mathrm{e}^{-|\lambda_0|t^2/2\varepsilon} \,\mathrm{d}t}{\displaystyle\int_{-\delta}^{\delta} \mathrm{e}^{-|\lambda_0|t^2/2\varepsilon} \,\mathrm{d}t} & \text{if } -\delta < z_0 < \delta \;, \\ 0 & \text{if } z_0 \geqslant \delta \;. \end{cases} \tag{4.1}$$

Although $|\lambda_0| = 1$, we will keep λ_0 in the notation as it allows to keep track of its influence on the result. Observe that

$$\|\nabla h_+(z)\|^2 = \begin{cases} \dfrac{\mathrm{e}^{-|\lambda_0|z_0^2/\varepsilon}}{\left(\displaystyle\int_{-\delta}^{\delta} \mathrm{e}^{-|\lambda_0|t^2/2\varepsilon} \,\mathrm{d}t\right)^2} & \text{if } -\delta < z_0 < \delta \;, \\ 0 & \text{otherwise} \;. \end{cases} \tag{4.2}$$

Noting that

$$\left(\int_{-\delta}^{\delta} \mathrm{e}^{-|\lambda_0|t^2/2\varepsilon} \,\mathrm{d}t\right)^2 = \frac{2\pi\varepsilon}{|\lambda_0|} \left[1 + \mathcal{O}(\mathrm{e}^{-\delta^2/2\varepsilon})\right] \tag{4.3}$$

and inserting in (3.8) we get

$$\begin{aligned} \operatorname{cap}_A(B) &\leqslant \frac{|\lambda_0|}{2\pi} \int_{\mathbb{R}^{(2N+1)^2}} \mathrm{e}^{-[V_N(z)+|\lambda_0|z_0^2]/\varepsilon} \,\mathrm{d}z \left[1 + \mathcal{O}(\mathrm{e}^{-\delta^2/2\varepsilon})\right] \\ &= \frac{|\lambda_0|}{2\pi} \varepsilon^{\frac{1}{2}(2N+1)^2} \int_{\mathbb{R}^{(2N+1)^2}} \mathrm{e}^{-[V_N(\sqrt{\varepsilon}y)/\varepsilon+|\lambda_0|y_0^2]} \,\mathrm{d}y \left[1 + \mathcal{O}(\mathrm{e}^{-\delta^2/2\varepsilon})\right] . \end{aligned} \tag{4.4}$$

Using the scaling $z = \sqrt{\varepsilon}y$ and $\phi_N = \sqrt{\varepsilon}\hat{\phi}_N$ in (3.4) shows that the exponent can be written in the form

$$\frac{1}{\varepsilon}\left[V_N(\sqrt{\varepsilon}y) + \varepsilon|\lambda_0|y_0^2\right] = g_N(y) + \varepsilon w_N(y) \;, \tag{4.5}$$

where

$$\begin{aligned} g_N(y) &= \frac{1}{2}|\lambda_0|y_0^2 + \frac{1}{2}\sum_{0<|k|\leqslant N} \lambda_k |y_k|^2 \;, \\ w_N(y) &= \frac{1}{4}\int_{\mathbb{T}^2} \left(\hat{\phi}_N^4(x) - 6C_N\hat{\phi}_N^2(x) + 3C_N^2\right) \mathrm{d}x \;. \end{aligned} \tag{4.6}$$

The quadratic form g_N allows us to define a Gaussian probability measure $\mathrm{d}\gamma(y) = \mathcal{N}^{-1}\,\mathrm{e}^{-g(y)}\,\mathrm{d}y$, with normalisation

$$\mathcal{N} = \sqrt{\frac{2\pi}{|\lambda_0|}} \prod_{k\neq 0,|k|\leqslant N} \sqrt{\frac{2\pi}{\lambda_k}}\;. \tag{4.7}$$

We can thus rewrite the upper bound (4.4) in the form

$$\operatorname{cap}_A(B) \leqslant \sqrt{\frac{|\lambda_0|\varepsilon}{2\pi}} \prod_{k\neq 0,|k|\leqslant N} \sqrt{\frac{2\pi\varepsilon}{\lambda_k}} \mathbb{E}^{\gamma}\Big[\mathrm{e}^{-\varepsilon w_N}\Big]\Big[1+\mathcal{O}(\mathrm{e}^{-\delta^2/2\varepsilon})\Big]\;. \tag{4.8}$$

The term $\mathbb{E}^{\gamma}\left[\mathrm{e}^{-\varepsilon w_N}\right]$ can be estimated using the Gaussian calculus developed in Appendix A . Indeed, the law of the field $\hat\phi_N$ under γ is exactly as described there. Furthermore, $C_N = \mathbb{E}^{\gamma}[\hat\phi_N(x)^2]$ for each $x\in\mathbb{T}^2$ so that the term $w_N(y)$, defined in (4.6) can be rewritten as

$$w_N = \frac{1}{4}\int_{\mathbb{T}^2} {:}\hat\phi^4(x){:}\,\mathrm{d}x =: \frac{1}{4} U_{4,N}\;. \tag{4.9}$$

In particular $U_{4,N}$ has zero mean under the Gaussian measure $\mathrm{d}\gamma$ and according to (A.13) all its stochastic moments are uniformly bounded in N.

We now derive a uniform-in-N bound on $\mathbb{E}^{\gamma}\Big[\mathrm{e}^{-\varepsilon w_N}\Big]$ following a classical argument due to Nelson (see e.g. [15, Sec 8.6] or [10, Sec. 4]).

Proposition 4.1. *There exists a constant K, independent of N, such that*

$$\mathbb{E}^{\gamma}\Big[\mathrm{e}^{-U_{4,N}}\Big] \leqslant K\;. \tag{4.10}$$

Proof: First note that (4.9) implies for any $M\in\mathbb{N}$

$${:}\hat\phi_M^4(x){:} = (\hat\phi_M^2(x) - 3C_M)^2 - 6C_M^2\;,$$

so that

$$U_{4,M} \geqslant -6L^2C_M^2 =: -D_M\;.$$

Since $\mathbb{E}^{\gamma_1}\left[\mathrm{e}^{-U_{4,N}}\,1_{\{U_{4,N}\geqslant 0\}}\right] \leqslant \mathbb{P}^{\gamma_1}\{U_{4,N}\geqslant 0\}\leqslant 1$, it is sufficient to bound

$$\begin{aligned}\mathbb{E}^{\gamma}\Big[\mathrm{e}^{-U_{4,N}}\,1_{\{U_{4,N}<0\}}\Big] &= 1 + \int_0^\infty \mathrm{e}^t\,\mathbb{P}^{\gamma_1}\Big\{-U_{4,N} > t\Big\}\,\mathrm{d}t \\ &\leqslant \mathrm{e} + \int_1^\infty \mathrm{e}^t\,\mathbb{P}^{\gamma_1}\Big\{-U_{4,N} > t\Big\}\,\mathrm{d}t\;.\end{aligned}$$

If $t\geqslant D_N$, then $\mathbb{P}^{\gamma_1}\Big\{-U_{4,N}>t\Big\} = 0$, otherwise we have for any M

$$\begin{aligned}\mathbb{P}^{\gamma}\Big\{-U_{4,N} > t\Big\} &\leqslant \mathbb{P}^{\gamma}\Big\{U_{4,M} - U_{4,N} > t - D_M\Big\} \\ &\leqslant \mathbb{P}^{\gamma}\Big\{|U_{4,M} - U_{4,N}|^{p(t)} > |t-D_M|^{2p(t)}\Big\}\;,\end{aligned}$$

for any choice of $p(t)\in 2\mathbb{N}$. We apply this inequality for $N = N(t)$ satisfying

$$t - D_{N(t)} \geqslant 1\;, \tag{4.11}$$

which implies that $M < N$.

Then we get by Markov's inequality and Lemma A.4 combined with (A.11)

$$\begin{aligned}\mathbb{P}^\gamma\bigl\{-U_{4,N} > t\bigr\} &\leqslant \mathbb{E}^\gamma\Bigl[|U_{4,M(t)} - U_{4,N}|^{p(t)}\Bigr] \\ &\leqslant \mathit{const}\,(p(t)-1)^{2p(t)}\mathbb{E}^\gamma\Bigl[|U_{4,M(t)} - U_{4,N}|^2\Bigr]^{p(t)/2} \\ &\leqslant \mathit{const}\,\frac{(p(t)-1)^{2p(t)}}{M(t)^{(2-2\eta)p(t)}}\end{aligned}$$

for any $\eta > 0$. The condition (4.11) on $M(t)$ imposes that $\log M(t)$ grows at most as $t^{1/2}$. Choosing for instance $p(t) \sim t^\beta$ for $\beta > \eta$, since

$$\log\Bigl(\mathrm{e}^t\,\mathbb{P}^\gamma\bigl\{-U_{4,N} > t\bigr\}\Bigr) \leqslant t + 2\beta t^\beta \log t - t^{\beta+(1-\eta)}\ ,$$

we obtain a convergent integral. □

This a priori estimate can now quite easily be turned into a sharper estimate. Indeed, we have the following result.

Proposition 4.2. *We have*

$$\mathbb{E}^\gamma\Bigl[\mathrm{e}^{-\varepsilon U_{N,4}/4}\Bigr] = 1 + \mathcal{O}(\varepsilon)\ , \tag{4.12}$$

where the remainder is bounded uniformly in N.

Proof: Introduce the sets

$$\Omega_+ = \Bigl\{\hat\phi_N : U_{4,N} > 0\Bigr\}$$

and $\Omega_- = \Omega_+^c$. Then we have

$$\mathbb{E}^\gamma\Bigl[\mathrm{e}^{-\varepsilon U_{4,N}/4}\,1_{\Omega_+}\Bigr] = \mathbb{P}^{\gamma_1}(\Omega_+) + \mathbb{E}^\gamma\Bigl[(\mathrm{e}^{-\varepsilon U_{4,N}/4} - 1)1_{\Omega_+}\Bigr]\ .$$

Note that on Ω_+, we have $-\frac{\varepsilon}{4}|U_{N,4}| \leqslant \mathrm{e}^{-\varepsilon U_{4,N}/4} - 1 \leqslant 0$, so that

$$\mathbb{P}^\gamma(\Omega_+) - \frac{\varepsilon}{4}\mathbb{E}^\gamma\Bigl[|U_{4,N}|1_{\Omega_+}\Bigr] \leqslant \mathbb{E}^\gamma\Bigl[\mathrm{e}^{-\varepsilon U_{4,N}/4}\,1_{\Omega_+}\Bigr] \leqslant \mathbb{P}^\gamma(\Omega_+)\ .$$

Since $U_{4,N}$ has finite variance bounded uniformly in N, we know by Cauchy–Schwarz that $\mathbb{E}^\gamma\bigl[|U_{4,N}|\bigr]$ is bounded uniformly in N. Similarly, we have

$$\mathbb{E}^\gamma\Bigl[\mathrm{e}^{-\varepsilon U_{4,N}/4}\,1_{\Omega_-}\Bigr] = \mathbb{P}^{\gamma_1}(\Omega_-) + \mathbb{E}^\gamma\Bigl[(\mathrm{e}^{-\varepsilon U_{4,N}/4} - 1)1_{\Omega_-}\Bigr]\ .$$

This time, we use that on Ω_-, one has $0 \leqslant \mathrm{e}^{-\varepsilon U_{4,N}/4} - 1 \leqslant \frac{\varepsilon}{4}|U_{N,4}|\,\mathrm{e}^{-\varepsilon U_{4,N}/4}$. Thus by Cauchy–Schwarz,

$$\begin{aligned}0 \leqslant \mathbb{E}^\gamma\Bigl[(\mathrm{e}^{-\varepsilon U_{4,N}/4} - 1)1_{\Omega_-}\Bigr] &\leqslant \frac{\varepsilon}{4}\mathbb{E}^\gamma\Bigl[|U_{N,4}|\,\mathrm{e}^{-\varepsilon U_{4,N}/4}\,1_{\Omega_-}\Bigr] \\ &\leqslant \frac{\varepsilon}{4}\biggl(\mathbb{E}^\gamma\Bigl[\mathrm{e}^{-2\varepsilon U_{4,N}/4}\,1_{\Omega_-}\Bigr]\mathbb{E}^\gamma\Bigl[|U_{4,N}|^2 1_{\Omega_-}\Bigr]\biggr)^{1/2}\end{aligned}$$

The term $\mathbb{E}^\gamma\bigl[|U_{4,N}|^2\bigr]$ is bounded uniformly in N as before, while the term $\mathbb{E}^\gamma\bigl[\mathrm{e}^{-2\varepsilon U_{4,N}/4}\bigr]$ is bounded uniformly in N for $\varepsilon \leqslant 1/2$ by Proposition 4.1. Summing the two estimates, we get the result. □

Substituting this estimate in (4.8), we immediately get the following upper bound on the capacity.

Corollary 4.3. *There exists a constant c_+, uniform in ε and N, such that the capacity satisfies the upper bound*

$$\operatorname{cap}_A(B) \leqslant \sqrt{\frac{|\lambda_0|\varepsilon}{2\pi}}\prod_{k\neq 0, |k|\leqslant N}\sqrt{\frac{2\pi\varepsilon}{\lambda_k}}\,\bigl[1 + c_+\varepsilon\bigr]\ . \tag{4.13}$$

4.2 Lower bound on the partition function

By symmetry, cf. (3.12), the partition function can be computed using the relation

$$\frac{1}{2}\mathcal{Z}_N(\varepsilon) = \int_{\Omega_+} \mathrm{e}^{-V_N(z)/\varepsilon}\,\mathrm{d}z\;, \qquad \Omega_+ = \{z_0 > 0\}\;. \tag{4.14}$$

A lower bound on $\mathcal{Z}_N(\varepsilon)$ can be obtained quite directly from Jensen's inequality. It will be convenient to shift coordinates to the positive stable stationary solution of the deterministic equation (without the normalisation). That is, we set

$$\phi_N(x) = 1 + \sqrt{\varepsilon}\hat\phi_{N,+}(x)\;, \tag{4.15}$$

with the Fourier decomposition

$$\hat\phi_{N,+}(x) = \sum_{|k|\leqslant N} y_k e_k(x)\;. \tag{4.16}$$

Substituting in (2.2) and using the relation (A.3) yields the following expression for the potential:

$$\begin{aligned} V_N^+(y) &:= \frac{1}{\varepsilon}V_N[1+\sqrt{\varepsilon}\hat\phi_{N,+}(x)] \\ &= -\frac{L^2}{4\varepsilon} + \frac{1}{2}\int_{\mathbb{T}^2}\Big(|\nabla\hat\phi_{N,+}(x)|^2 - \phi_{N,+}^2(x) + 3H_2(\hat\phi_{N,+},C_N)\Big)\,\mathrm{d}x \\ &\quad + \frac{1}{4}\int_{\mathbb{T}^2}\Big(4\sqrt{\varepsilon}H_3(\hat\phi_{N,+}(x),C_N) + \varepsilon H_4(\hat\phi_{N,+}(x),C_N)\Big)\,\mathrm{d}x. \end{aligned} \tag{4.17}$$

Now the relevant Gaussian measure γ_+ is defined by the quadratic form

$$\begin{aligned} g_{N,+}(y) &= \frac{1}{2}\int_{\mathbb{T}^2}|\nabla\hat\phi_{N,+}(x)|^2 - \hat\phi_{N,+}^2(x) + 3\hat\phi_{N,+}^2(x)\,\mathrm{d}x \\ &= \frac{1}{2}\sum_{0<|k|\leqslant N}(\lambda_k+3)|y_k|^2\;. \end{aligned} \tag{4.18}$$

Observe that a term $-\frac{3}{2}C_N L^2$ appears owing to the Hermite polynomial $3H_2(\phi_{N,+},C_N)$. It is precisely this term which is ultimately responsible for the renormalisation of the pre-factor. To bound expectations of the terms appearing in the last line of (4.17) it is convenient to rewrite them as Wick powers with respect to the Gaussian measure defined by $g_{N,+}$. The associated renormalisation constant is

$$C_{N,+} = \frac{1}{L^2}\sum_{0<|k|\leqslant N}\frac{1}{\lambda_k+3}\;. \tag{4.19}$$

Observe in particular that

$$C_N - C_{N,+} = \frac{1}{L^2}\sum_{0<|k|\leqslant N}\frac{3}{|\lambda_k|(\lambda_k+3)} \tag{4.20}$$

is bounded uniformly in N. Now using the relation (A.5) that allows to transform Hermite polynomials with respect to different constants we get

$$\begin{aligned} \sqrt{\varepsilon}H_3(\hat\phi_{N,+},C_N) &= \sqrt{\varepsilon}H_3(\hat\phi_{N,+},C_{N,+}) - \sqrt{\varepsilon}3(C_N-C_{N,+})\hat\phi_{N,+} \\ \frac{\varepsilon}{4}H_4(\hat\phi_{N,+},C_N) &= \frac{\varepsilon}{4}H_4(\hat\phi_{N,+},C_{N,+}) - \frac{3}{2}\varepsilon(C_N-C_{N,+})H_2(\hat\phi_{N,+},C_N) \\ &\quad + \frac{3}{4}\varepsilon(C_N-C_{N,+})^2\;. \end{aligned} \tag{4.21}$$

Now we define the random variables

$$U^+_{n,N} = \int_{\mathbb{T}^2} :\hat{\phi}^n_{N,+}(x): \mathrm{d}x = \int_{\mathbb{T}^2} H_n(\hat{\phi}_{N,+}(x), C_{N,+})\,\mathrm{d}x \tag{4.22}$$

which have zero mean under γ_+ as well as a variance bounded uniformly in N. Substituting (4.21) in (4.17), we get

$$V^+_N(y) = q + g_{N,+}(y) + w_{N,+}(y) \;, \tag{4.23}$$

where

$$\begin{aligned} q &= -\frac{L^2}{4\varepsilon} - \frac{3}{2}L^2 C_N + \frac{3}{4}L^2\varepsilon(C_N - C_{N,+})^2 \\ w_{N,+}(y) &= \sqrt{\varepsilon}\,U^+_{3,N} + \frac{1}{4}\varepsilon U^+_{4,N} - 3(C_N - C_{N,+})\big(\frac{\varepsilon}{2}U^+_{2,N} + \sqrt{\varepsilon}U^+_{1,N}\big). \end{aligned} \tag{4.24}$$

It follows by a similar argument as in the previous section that

$$\frac{1}{2}\mathcal{Z}_N(\varepsilon) = \prod_{|k|\leqslant N}\sqrt{\frac{2\pi\varepsilon}{\lambda_k + 3}}\,\mathrm{e}^{-q}\,\mathbb{E}^{\gamma_+}\big[\mathrm{e}^{-w_{N,+}}\,1_{\Omega_+}\big]\,. \tag{4.25}$$

Proposition 4.4. *There exists a constant c_-, independent of N and ε, such that*

$$\mathbb{E}^{\gamma_+}\big[\mathrm{e}^{-w_{N,+}}\,1_{\Omega_+}\big] \geqslant 1 - c_-\varepsilon\;. \tag{4.26}$$

PROOF: Recall that $w_{N,+}$ has zero expectation under γ_+. Jensen's inequality yields

$$\begin{aligned} \mathbb{E}^{\gamma_+}\big[\mathrm{e}^{-w_{N,+}}\,1_{\Omega_+}\big] &= \mathbb{P}^{\gamma_+}(\Omega_+)\mathbb{E}^{\gamma_+}\big[\mathrm{e}^{-w_{N,+}}\,\big|\,\Omega_+\big] \\ &\geqslant \mathbb{P}^{\gamma_+}(\Omega_+)\,\mathrm{e}^{-\mathbb{E}^{\gamma_+}[w_{N,+}|\Omega_+]} \\ &= \mathbb{P}^{\gamma_+}(\Omega_+)\,\mathrm{e}^{-\mathbb{E}^{\gamma_+}[w_{N,+}1_{\Omega_+}]/\mathbb{P}^{\gamma_+}(\Omega_+)}\;. \end{aligned}$$

Since the first marginal of γ_+ is a Gaussian distribution, centred at the positive stationary solution, standard tail estimates show that there is a constant $c_0 > 0$ such that

$$\mathbb{P}^{\gamma_+}(\Omega_+) \geqslant 1 - \mathrm{e}^{-c_0/\varepsilon}\;.$$

Furthermore, there is a constant K such that uniformly in N and for $n = 1,2,3,4$

$$\begin{aligned} \big|\mathbb{E}^{\gamma_+}\big[U^+_{n,N}1_{\Omega_+}\big]\big| &= \big|\mathbb{E}^{\gamma_+}\big[U^+_{n,N}1_{\Omega^c_+}\big]\big| \leqslant \mathbb{E}^{\gamma_+}\big[|U^+_{n,N}|1_{\Omega^c_+}\big] \\ &\leqslant \mathbb{E}^{\gamma_+}\big[(U^+_{n,N})^2\big]^{1/2}\mathbb{P}^{\gamma_+}(\Omega^c_+)^{1/2} \leqslant K\,\mathrm{e}^{-c_0/2\varepsilon}\;. \end{aligned}$$

It thus follows that

$$\mathbb{E}^{\gamma_+}\big[w_{N,+}1_{\Omega_+}\big] \geqslant -\mathcal{O}(\mathrm{e}^{-c_0/2\varepsilon}),$$

which yields the required estimate $\mathbb{E}^{\gamma_+}\left[\mathrm{e}^{-w_{N,+}}\,1_{\Omega_+}\right] \geqslant 1 - c_1\varepsilon$. □

Combining this result with (4.25) and Corollary 4.3, we finally obtain the following lower bound on the expected transition times.

Proposition 4.5. *There exists a constant C_-, uniform in N and ε, such that*

$$\mathbb{E}^{\mu_{A,B}}\big[\tau_B\big] \geqslant 2\pi\Bigg(\frac{\mathrm{e}^{3/|\lambda_0|}}{|\lambda_0|(\lambda_0+3)}\prod_{0<|k|\leqslant N}\bigg[\frac{\mathrm{e}^{3/\lambda_k}}{1+3/\lambda_k}\bigg]\Bigg)^{1/2}\mathrm{e}^{L^2/4\varepsilon}\big[1 - C_-\varepsilon\big] \tag{4.27}$$

holds for all $N \geqslant 1$.

PROOF: Plugging (4.26) into (4.25), using the upper bound (4.13) on the capacity and substituting in (3.13), we obtain

$$\mathbb{E}^{\mu_{A,B}}\big[\tau_B\big] \geqslant \sqrt{\frac{2\pi}{|\lambda_0|\varepsilon}}\sqrt{\frac{2\pi\varepsilon}{\lambda_0+3}}\prod_{0<|k|\leqslant N}\sqrt{\frac{\lambda_k}{\lambda_k+3}}\,\mathrm{e}^{L^2/4\varepsilon}\,\mathrm{e}^{3L^2C_N/2}\big[1-\mathcal{O}(\varepsilon)\big]\;.$$

Using the fact that

$$\frac{3}{2}L^2C_N = \frac{3}{2}\biggl(1+\sum_{0<|k|\leqslant N}\frac{1}{\lambda_k}\biggr)$$

yields the result. □

5 Upper bound on the expected transition time

5.1 Longitudinal-transversal decomposition of the potential

In several computations that follow, it will be useful to decompose the field ϕ_N into its mean and its fluctuating part, setting

$$\phi_N(x) = \frac{z_0}{L} + \sqrt{\varepsilon}\hat\phi_{N,\perp}(x)\;, \tag{5.1}$$

where

$$\hat\phi_{N,\perp}(x) = \sum_{0<|k|\leqslant N} y_k e_k(x)\;. \tag{5.2}$$

Note in particular the Parseval identity

$$\int_{\mathbb{T}^2}\hat\phi_{N,\perp}^2(x)\,\mathrm{d}x = \sum_{0<|k|\leqslant N}|y_k|^2\;. \tag{5.3}$$

Similarly to (4.17) the potential can be written in the form

$$\begin{aligned}\frac{1}{\varepsilon}V_N(z_0,y_\perp) = \frac{1}{\varepsilon}q(z_0) + g_{N,\perp}(z_0,y_\perp) &+ \frac{1}{4}\int_{\mathbb{T}^2}\frac{6z_0^2}{L^2}H_2(\hat\phi_{N,\perp}(x),C_N)\,\mathrm{d}x\\ &+\frac{1}{4}\int_{\mathbb{T}^2}\Bigl(\frac{4z_0}{L}\sqrt{\varepsilon}H_3(\hat\phi_{N,\perp}(x),C_N)+\varepsilon H_4(\hat\phi_{N,\perp}(x),C_N)\Bigr)\,\mathrm{d}x\;,\end{aligned} \tag{5.4}$$

where this time

$$\begin{aligned}q(z_0) &= \frac{1}{4L^2}z_0^4 - \frac{|\lambda_0|}{2}z_0^2\;,\\ g_{N,\perp}(z_0,y_\perp) &= \frac{1}{2}\sum_{0<|k|\leqslant N}\lambda_k|y_k|^2\;.\end{aligned} \tag{5.5}$$

Here we have used the fact that by assumption $\int_{\mathbb{T}^2}\hat\phi_{N,\perp}(x)\,\mathrm{d}x = 0$, so that the corresponding term drops. The quadratic form $g_{N,\perp}$ defines a Gaussian measure $\gamma_0^\perp$ with normalisation

$$\mathcal{N}_0^\perp = \prod_{0<|k|\leqslant N}\sqrt{\frac{2\pi}{\lambda_k}}\;. \tag{5.6}$$

The associated renormalisation constant is given by

$$\mathbb{E}^{\gamma_0^\perp}\big[\hat\phi_{N,\perp}^2(x)\big] = \frac{1}{L^2}\sum_{0<|k|\leqslant N}\frac{1}{\lambda_k} =: C_N^\perp = C_N - \frac{1}{L^2}\;. \tag{5.7}$$

As before, we define Wick powers with respect to this measure

$$:\hat{\phi}^n_{N,\perp}: = H_n(\hat{\phi}_{N,\perp}, C^\perp_N) \tag{5.8}$$

and set

$$U^\perp_{n,N} = \int_{\mathbb{T}^2} :\hat{\phi}^n_{N,\perp}(x): \mathrm{d}x\;, \qquad U^\perp_n = \lim_{N\to\infty} U^\perp_{n,N}\;. \tag{5.9}$$

By construction, these random variables have (under the measure $\gamma^\perp_0$) zero mean and a variance bounded uniformly in N. Furthermore we see that

$$\int_{\mathbb{T}^2} H_3(\hat{\phi}_{N,\perp}(x), C_N)\,\mathrm{d}x = U^\perp_{3,N}\;, \tag{5.10}$$

owing to the fact that $\hat{\phi}_{N,\perp}$ has zero mean, and

$$\int_{\mathbb{T}^2} H_4(\hat{\phi}_{N,\perp}(x), C_N)\,\mathrm{d}x = U^\perp_{4,N} - \frac{6}{L^2}U^\perp_{2,N} + \frac{3}{L^2}\;. \tag{5.11}$$

The following expression for the potential then follows immediately from (5.4).

Proposition 5.1. *The potential can be decomposed as*

$$\frac{1}{\varepsilon}V_N(z_0,y_\perp) = \frac{1}{\varepsilon}q(z_0) + q_1(z_0,\varepsilon) + g_{N,\perp}(y_\perp) + w_{N,\perp}(z_0,y_\perp)\;, \tag{5.12}$$

where $q(z_0)$ and $g_{N,\perp}(y_\perp)$ are given in (5.5)*, and*

$$\begin{aligned} q_1(z_0,\varepsilon) &= -\frac{3z_0^2}{2L^2} + \frac{3\varepsilon}{4L^2}\;,\\ w_{N,\perp}(z_0,y_\perp) &= \frac{3(z_0^2-\varepsilon)}{2L^2}U^\perp_{2,N} + \frac{z_0}{L}\sqrt{\varepsilon}U^\perp_{3,N} + \frac{1}{4}\varepsilon U^\perp_{4,N}\;. \end{aligned} \tag{5.13}$$

5.2 Upper bound on the partition function

In order to obtain an upper bound on $\mathcal{Z}_N(\varepsilon)$, we will first perform the integration over the fluctuating modes $y_\perp$, and then the integration over the mean value z_0. The basic observation is the following rewriting of $\mathcal{Z}_N(\varepsilon)$.

Proposition 5.2. *The partition function is given by the integral*

$$\mathcal{Z}_N(\varepsilon) = \int_{-\infty}^{\infty} \mathrm{e}^{-q(z_0)/\varepsilon} g(z_0,\varepsilon)\,\mathrm{d}z_0\;, \tag{5.14}$$

where

$$g(z_0,\varepsilon) = \mathrm{e}^{-q_1(z_0,\varepsilon)} \prod_{0<|k|\leqslant N} \sqrt{\frac{2\pi\varepsilon}{\lambda_k}}\,\mathbb{E}^{\gamma^\perp_0}\big[\mathrm{e}^{-w_{N,\perp}(z_0,\cdot)}\big]\;. \tag{5.15}$$

By standard, one-dimensional Laplace asymptotics, we expect the integral (5.14) to be close to $2\sqrt{\pi\varepsilon}\,\mathrm{e}^{L^2/4\varepsilon}g(L,\varepsilon)$. Our aim is thus to bound the expectation in (5.15). In order to apply a Nelson estimate, we will need a lower bound on $w_{N,\perp}(z_0,y_\perp)$. In fact, for later use we will derive a lower bound for the slightly more general quantity

$$w^{(\mu)}_{N,\perp}(z_0,y_\perp) = \frac{3z_0^2}{2L^2}U^\perp_{2,N} + \mu\bigg(-\frac{3\varepsilon}{2L^2}U^\perp_{2,N} + \frac{z_0}{L}\sqrt{\varepsilon}U^\perp_{3,N} + \frac{1}{4}\varepsilon U^\perp_{4,N}\bigg)\;, \tag{5.16}$$

where μ is a real parameter. Note in particular that $w^{(1)}_{N,\perp}(z_0,y_\perp) = w_{N,\perp}(z_0,y_\perp)$. The proof of the following simple but useful lower bound is inspired by Proposition 3.2 in [5].

Lemma 5.3. *For any $N \in \mathbb{N}$, $z_0 \in \mathbb{R}$ and $\mu \in (0, \frac{3}{2})$,*

$$w^{(\mu)}_{N,\perp}(z_0, y_\perp) \geqslant -D_N(z_0, \mu, \varepsilon) \;, \tag{5.17}$$

where

$$D_N(z_0, \mu, \varepsilon) = \frac{3z_0^2}{2L^2} C_N + \frac{3}{4}\mu\varepsilon C_N^2 L^2 \biggl(1 + \frac{3}{1 - 2\mu/3}\biggr) \;. \tag{5.18}$$

Proof: Using the definition (5.8) of Wick powers, we see that

$$\begin{aligned} w^{(\mu)}_{N,\perp}(z_0, y_\perp) = \frac{1}{4}\int_{\mathbb{T}^2} \hat\phi^2_{N,\perp}(x) \biggl[\mu\varepsilon\hat\phi^2_{N,\perp}(x) + 4\mu\frac{z_0}{L}\sqrt{\varepsilon}\hat\phi_{N,\perp}(x) + 6\frac{z_0^2}{L^2} - 6\mu\varepsilon C_N\biggr] \mathrm{d}x \\ - \frac{3(z_0^2 - \mu\varepsilon)}{2L^2} C_N^\perp - \frac{3}{4}\mu\varepsilon (C_N^\perp)^2 L^2 \;, \end{aligned} \tag{5.19}$$

where we have used the fact that $C_N^\perp + \frac{1}{L^2} = C_N$. A completion-of-squares argument shows that the term in square brackets in (5.19) is bounded below by

$$\mu\varepsilon\biggl(1 - \frac{2}{3}\mu\biggr)\hat\phi^2_{N,\perp}(x) - 6\mu\varepsilon C_N \;.$$

Performing a second completion of squares shows that the integral in (5.19) is bounded below by

$$\frac{\mu\varepsilon}{4}\int_{\mathbb{T}^2}\biggl[\hat\phi^4_{N,\perp}(x)\Bigl(1 - \frac{2}{3}\mu\Bigr) - 6C_N\hat\phi^2_{N,\perp}(x)\biggr]\mathrm{d}x \geqslant -\frac{9\mu\varepsilon C_N^2 L^2}{1 - 2\mu/3} \;.$$

The result follows, bounding $C_N^\perp$ above by C_N. □

We are now in a position to imitate the proof of Proposition 4.1, to show the following upper bound.

Proposition 5.4. *There exist constants $M(\mu)$ and $\varepsilon_0(\mu)$, uniform in N, ε and z_0, such that*

$$\mathbb{E}^{\gamma_0^\perp}\bigl[\mathrm{e}^{-w^{(\mu)}_{N,\perp}(z_0,\cdot)}\bigr] \leqslant M(\mu)\bigl[1 + \sqrt{\varepsilon}\,\mathrm{e}^{M(\mu) z_0^2 \log(z_0)/\sqrt{\varepsilon}}\bigr] \tag{5.20}$$

holds for any $\mu \in (0, \frac{3}{2})$ and all $\varepsilon < \varepsilon_0(\mu)$.

Proof: We will give the proof for $w^{(\mu)}_\perp = \lim_{N\to\infty} w^{(\mu)}_{N,\perp}(z_0, \cdot)$, since the same proof applies for any finite N. For any $t_0 \in \mathbb{R}$, we have the integration-by-parts formula

$$\begin{aligned} \mathbb{E}^{\gamma_0^\perp}\bigl[\mathrm{e}^{-w^{(\mu)}_\perp}\bigr] &\leqslant \mathrm{e}^{t_0} + \int_{t_0}^\infty \mathrm{e}^t\,\mathbb{P}^{\gamma_0^\perp}\bigl\{-w^{(\mu)}_\perp > t\bigr\}\,\mathrm{d}t \\ &= \mathrm{e}^{t_0}\biggl[1 + \int_0^\infty \mathrm{e}^t\,\mathbb{P}^{\gamma_0^\perp}\bigl\{-w^{(\mu)}_\perp > t + t_0\bigr\}\,\mathrm{d}t\biggr] \;. \end{aligned}$$

We thus have to estimate $\mathbb{P}^{\gamma_0^\perp}\{-w^{(\mu)}_\perp > t\}$ when $t > t_0$. Assuming $t_0 \geqslant 1$, we pick for any $t \geqslant t_0$ an $N(t) \in \mathbb{N}$ such that

$$t - D_{N(t)}(z_0, \mu, \varepsilon) \geqslant 1 \;.$$

Note that by (5.18), there exists a constant $M_0(\mu)$, uniform in N, ε and z_0, such that

$$D_N(z_0, \mu, \varepsilon) \leqslant M_0(\mu)\bigl[\varepsilon(\log N)^2 + z_0^2 \log N\bigr] \;.$$

The condition on $N(t)$ is thus satisfied if we impose the condition

$$\varepsilon(\log N(t))^2 + z_0^2 \log N(t) \leqslant \frac{t-t_0}{M_0(\mu)} . \tag{5.21}$$

By Lemma 5.3 and the above condition, we have

$$\mathbb{P}^{\gamma_0^\perp}\bigl\{-w_\perp^{(\mu)} > t\bigr\} \leqslant \mathbb{P}^{\gamma_0^\perp}\bigl\{|w_{N(t),\perp}^{(\mu)} - w_\perp^{(\mu)}| > 1\bigr\} .$$

Now observe (c.f. (5.16)) that

$$w_{N(t),\perp}^{(\mu)} - w_\perp^{(\mu)} = \sum_{j=2}^{4} a_j(z_0,\varepsilon)(U_{j,N(t)}^\perp - U_j^\perp) ,$$

with

$$a_2 = \frac{3}{2L^2}(z_0^2 - \mu\varepsilon) , \qquad a_3 = \frac{z_0}{L}\sqrt{\varepsilon} , \qquad a_4 = \frac{1}{4}\varepsilon .$$

It follows that

$$\mathbb{P}^{\gamma_0^\perp}\bigl\{|w_{N(t),\perp}^{(\mu)} - w_\perp^{(\mu)}| > 1\bigr\} \leqslant \sum_{j=2}^{4} P_j , \qquad P_j = \mathbb{P}^{\gamma_0^\perp}\biggl\{|a_j||U_{j,N(t)}^\perp - U_j^\perp| > \frac{1}{3}\biggr\} .$$

For any choice of $p_j(t) \in 2\mathbb{N}$, we have by Markov's inequality

$$P_j \leqslant |3a_j(z_0,\varepsilon)|^{p_j(t)} E_j , \qquad E_j = \mathbb{E}^{\gamma_0^\perp}\bigl[|U_{j,N(t)}^\perp - U_j^\perp|^{p_j(t)}\bigr]$$

where by Nelson's estimate (A.12)

$$E_j \lesssim (p_j(t)-1)^{jp_j(t)/2}\mathbb{E}^{\gamma_0^\perp}\bigl[|U_{j,N(t)}^\perp - U_j^\perp|^2\bigr]^{p_j(t)/2} \lesssim \frac{(p_j(t)-1)^{jp_j(t)/2}}{N(t)^{p_j(t)}} .$$

A possible choice is to take (where $a \wedge b := \min\{a,b\}$ and $\lfloor a \rfloor$ is the integer part of a)

$$\begin{aligned} p_j(t) &= 2\biggl\lfloor (t-t_0)^{1/2} \wedge \frac{z_0^2}{\sqrt{\varepsilon}} \biggr\rfloor \\ \log N(t) &= \frac{1}{M_1}\biggl[\biggl(\frac{t-t_0}{\varepsilon}\biggr)^{1/2} \wedge \frac{t-t_0}{z_0^2}\biggr] , \end{aligned}$$

with M_1 large enough to satisfy (5.21). Indeed, this yields $\log(N(t)^{p_j(t)}) \simeq c(t-t_0)/\sqrt{\varepsilon}$ for some $c = c(\mu) > 0$, and thus

$$\mathbb{P}^{\gamma_0^\perp}\bigl\{-w_\perp^{(\mu)} > t\bigr\} \lesssim \mathrm{e}^{c' \log(z_0) z_0^2/\sqrt{\varepsilon}}\, \mathrm{e}^{-c(t-t_0)/\sqrt{\varepsilon}}$$

for some $c' > 0$, where the first exponential is due to the term $|3a_2|^{p_2(t)}$. This shows that

$$\int_0^\infty \mathrm{e}^t\, \mathbb{P}^{\gamma_0^\perp}\bigl\{-w_\perp^{(\mu)} > t + t_0\bigr\}\, \mathrm{d}t \leqslant \mathrm{e}^{c' \log(z_0) z_0^2/\sqrt{\varepsilon}}\Bigl(\frac{c}{\sqrt{\varepsilon}} - 1\Bigr)^{-1}$$

if $\varepsilon < c^2$. Substituting in the integration-by-parts formula proves the claim. □

Our aim is now to sharpen this bound by applying a similar trick as in the proof of Proposition 4.2. To this end, it will be convenient to work with Gaussian measures $\gamma_{z_0}^\perp$, defined by the quadratic form

$$g_{N,\perp,z_0}(y_\perp) = \sum_{0<|k|\leqslant N} \biggl[\lambda_k + \frac{3z_0^2}{L^2}\biggr] |y_k|^2 . \tag{5.22}$$

The following result allows converting between expectations with respect to $\gamma_0^\perp$ and $\gamma_{z_0}^\perp$.

Lemma 5.5. *For any random variable $X = X(y_\perp)$ integrable with respect to $\gamma_0^\perp$,*

$$\mathbb{E}^{\gamma_0^\perp}[X] = K(z_0)\mathbb{E}^{\gamma_{z_0}^\perp}\Bigl[X\,\mathrm{e}^{3z_0^2 U_{2,N}^\perp/2L^2}\Bigr] , \tag{5.23}$$

where

$$K(z_0) = \biggl[\prod_{0<|k|\leqslant N} \frac{\mathrm{e}^{3z_0^2/L^2\lambda_k}}{1 + 3z_0^2/L^2\lambda_k}\biggr]^{1/2} .$$

Proof: This follows from a short computation, writing out explicitly the density of $\gamma_0^\perp$ and expressing $\sum_k |y_k|^2$ in terms of $U_{2,N}^\perp$. □

Remark 5.6. Writing $\zeta_k = 3z_0^2/L^2\lambda_k$ and using the fact that the Taylor series of $\log(1+\zeta_k)$ is alternating, we obtain

$$2\log K(z_0) = \sum_{0<|k|\leqslant N} \bigl[\zeta_k - \log(1+\zeta_k)\bigr] \leqslant \frac{1}{2}\sum_{0<|k|\leqslant N} \zeta_k^2 \lesssim z_0^4 . \tag{5.24}$$

This shows that $K(z_0) \leqslant \mathrm{e}^{M_1 z_0^4}$ for some constant M_1, independent of N and z_0.

We can now state the sharper bound on the expectation of $\mathrm{e}^{-w_{N,\perp}}$.

Proposition 5.7. *There exists a constant $M > 0$, uniform in N, ε and z_0, such that*

$$\mathbb{E}^{\gamma_0^\perp}\Bigl[\mathrm{e}^{-w_{N,\perp}(z_0,\cdot)}\Bigr] \leqslant K(z_0)\Bigl[1 + M\sqrt{\varepsilon}(1+|z_0|)\bigl(1 + \sqrt{\varepsilon}\,\mathrm{e}^{Mz_0^2\log(z_0)/\sqrt{\varepsilon}}\bigr)\Bigr] . \tag{5.25}$$

Proof: By Lemma 5.5, we have

$$\mathbb{E}^{\gamma_0^\perp}\Bigl[\mathrm{e}^{-w_{N,\perp}(z_0,\cdot)}\Bigr] = K(z_0)\mathbb{E}^{\gamma_{z_0}^\perp}\Bigl[\mathrm{e}^{-\hat{w}_{N,\perp}(z_0,\cdot)}\Bigr]$$

where

$$\hat{w}_{N,\perp}(z_0,\cdot) = -\frac{3\varepsilon}{2L^2}U_{2,N}^\perp + \frac{z_0}{L}\sqrt{\varepsilon}U_{3,N}^\perp + \frac{1}{4}\varepsilon U_{4,N}^\perp .$$

As in the proof of Proposition 4.2, we write

$$\begin{aligned}
\mathbb{E}^{\gamma_{z_0}^\perp}\Bigl[\mathrm{e}^{-\hat{w}_{N,\perp}(z_0,\cdot)}\Bigr] &\leqslant 1 + \mathbb{E}^{\gamma_{z_0}^\perp}\Bigl[(\mathrm{e}^{-\hat{w}_{N,\perp}(z_0,\cdot)} - 1)1_{\{\hat{w}_{N,\perp}(z_0,\cdot)<0\}}\Bigr] \\
&\leqslant 1 + \mathbb{E}^{\gamma_{z_0}^\perp}\Bigl[|\hat{w}_{N,\perp}(z_0,\cdot)|\,\mathrm{e}^{-\hat{w}_{N,\perp}(z_0,\cdot)}\,1_{\{\hat{w}_{N,\perp}(z_0,\cdot)<0\}}\Bigr] \\
&\leqslant 1 + \mathbb{E}^{\gamma_{z_0}^\perp}\Bigl[(\hat{w}_{N,\perp}(z_0,\cdot))^p\Bigr]^{1/p}\mathbb{E}^{\gamma_{z_0}^\perp}\Bigl[\mathrm{e}^{-q\hat{w}_{N,\perp}(z_0,\cdot)}\Bigr]^{1/q} .
\end{aligned}$$

In the last line, we have used Hölder's inequality, and $p,q \geqslant 1$ are Hölder conjugates. It follows from standard Wick calculus that

$$\mathbb{E}^{\gamma_{z_0}^\perp}\Bigl[(\hat{w}_{N,\perp}(z_0,\cdot))^p\Bigr]^{1/p} \lesssim \sqrt{\varepsilon}(1+|z_0|) ,$$

while another application of Lemma 5.5 yields

$$\mathbb{E}^{\gamma_{z_0}^\perp}\Bigl[\mathrm{e}^{-q\hat{w}_{N,\perp}(z_0,\cdot)}\Bigr] \leqslant \frac{1}{K(z_0)}\mathbb{E}^{\gamma_0^\perp}\Bigl[\mathrm{e}^{-w_{N,\perp}^{(q)}(z_0,\cdot)}\Bigr] .$$

Applying Proposition 5.4 for some $q \in (1, \frac{3}{2})$ and combining the different estimates yields the result. □

5.3 Lower bound on the capacity

Assume that $A = -I \times A_\perp$ and $B = I \times A_\perp$ where $\pm I = \pm[L-\delta, L+\delta]$ (with $0 < \delta < L$) are small intervals around the two stationary solutions. Let $D = J \times D_\perp$ where $J = [-\rho,\rho]$ is an interval joining $-I$ and I, with $\rho = L-\delta$, and $D_\perp \subset A_\perp$. Then we have

$$\begin{aligned}\operatorname{cap}_A(B) &\geqslant \varepsilon \int_D \left|\frac{\partial h_{A,B}(z)}{\partial z_0}\right|^2 \mathrm{e}^{-V_N(z)/\varepsilon}\,\mathrm{d}z \\ &\geqslant \varepsilon \int_{D_\perp}\left[\inf_{f:f(-\rho)=1,f(\rho)=0}\int_{-\rho}^{\rho} \mathrm{e}^{-V_N(z_0,z_\perp)/\varepsilon} f'(z_0)^2\,\mathrm{d}z_0\right]\mathrm{d}z_\perp\,. \end{aligned}\qquad(5.26)$$

Writing the Euler–Lagrange equations, it is easy to see that the minimiser for the term in brackets is such that

$$f'(z_0) = \frac{\mathrm{e}^{V_N(z_0,z_\perp)/\varepsilon}}{\displaystyle\int_{-\rho}^{\rho}\mathrm{e}^{V_N(y,z_\perp)/\varepsilon}\,\mathrm{d}y}\,. \qquad(5.27)$$

This yields the lower bound

$$\operatorname{cap}_A(B) \geqslant \varepsilon \int_{D_\perp}\frac{1}{\displaystyle\int_{-\rho}^{\rho}\mathrm{e}^{V_N(z_0,z_\perp)/\varepsilon}\,\mathrm{d}z_0}\,\mathrm{d}z_\perp\,. \qquad(5.28)$$

Proposition 5.8. *There exists a constant $c_- > 0$, uniform in ε and N, such that*

$$\operatorname{cap}_A(B) \geqslant \sqrt{\frac{\varepsilon|\lambda_0|}{2\pi}}\prod_{0<|k|\leqslant N}\sqrt{\frac{2\pi\varepsilon}{\lambda_k}}\Big[\mathbb{P}^{\gamma_0^\perp}(\hat D_\perp) - c_-\sqrt{\varepsilon}\,\Big]\,, \qquad(5.29)$$

where $\hat D_\perp = D_\perp/\sqrt{\varepsilon}$.

Proof: We start by obtaining a lower bound on V_N in which z_0 is decoupled from the transverse coordinates. Using the expression (5.12) obtained in Proposition 5.1 and the elementary inequality $2|ab| \leqslant (a^2/c + cb^2)$ for $c > 0$, we obtain

$$\frac{1}{\varepsilon}V_N(z_0,y_\perp) \leqslant \frac{1}{\varepsilon}q(z_0) + q_1(z_0,\varepsilon) + \frac{z_0^2}{2L^2} + \frac{3z_0^4}{4L^2\sqrt{\varepsilon}} + g_{N,\perp}(y_\perp) + \sqrt{\varepsilon}R(y_\perp,\varepsilon)\,,$$

where

$$R(y_\perp,\varepsilon) = \frac{3}{4L^2}(U^\perp_{2,N})^2 + \frac{1}{2}\sqrt{\varepsilon}(U^\perp_{3,N})^2 - \frac{3\sqrt{\varepsilon}}{2L^2}U^\perp_{2,N} + \frac{1}{4}\sqrt{\varepsilon}U^\perp_{4,N}\,.$$

Substituting in (5.28) (and taking into account the scaling $z_\perp = \sqrt{\varepsilon y_\perp}$) yields

$$\operatorname{cap}_A(B) \geqslant \frac{\varepsilon}{\mathcal{J}}\prod_{0<|k|\leqslant N}\sqrt{\frac{2\pi\varepsilon}{\lambda_k}}\,\mathbb{E}^{\gamma_0^\perp}\big[\mathrm{e}^{-\sqrt{\varepsilon}R}\,1_{\hat D_\perp}\big]\,,$$

where

$$\mathcal{J} = \int_{-\rho}^{\rho}\exp\left\{\frac{1}{\varepsilon}q(z_0) + q_1(z_0,\varepsilon) + \frac{z_0^2}{2L^2} + \frac{3z_0^4}{4L^2\sqrt{\varepsilon}}\right\}\mathrm{d}z_0\,.$$

Since $q(z_0)$ has a quadratic maximum on $[-\rho,\rho]$ at 0, standard one-dimensional Laplace asymptotics (see for instance [25, Chapter 3, Theorems 7.1 and 8.1]) show that

$$\mathcal{J} = \sqrt{\frac{2\pi\varepsilon}{|\lambda_0|}}\big[1 + \mathcal{O}(\sqrt{\varepsilon}\,)\big]\,.$$

Furthermore, Jensen's inequality implies that

$$\begin{aligned}
\mathbb{E}^{\gamma_0^\perp}\Big[\mathrm{e}^{-\sqrt{\varepsilon}R}\,1_{\hat{D}_\perp}\Big] &= \mathbb{E}^{\gamma_0^\perp}\Big[\mathrm{e}^{-\sqrt{\varepsilon}R}\;\Big|\;\hat{D}_\perp\Big]\mathbb{P}^{\gamma_0^\perp}\{\hat{D}_\perp\} \\
&\geqslant \mathrm{e}^{-\sqrt{\varepsilon}\mathbb{E}^{\gamma_0^\perp}[R|\hat{D}_\perp]}\,\mathbb{P}^{\gamma_0^\perp}\{\hat{D}_\perp\} \\
&\geqslant \left(1-\sqrt{\varepsilon}\frac{\mathbb{E}^{\gamma_0^\perp}[|R|]}{\mathbb{P}^{\gamma_0^\perp}\{\hat{D}_\perp\}}\right)\mathbb{P}^{\gamma_0^\perp}\{\hat{D}_\perp\}\;.
\end{aligned}$$

Since $\mathbb{E}^{\gamma_0^\perp}[|R|]$ is bounded uniformly, the result follows. □

The lower bound on the capacity is thus complete, provided we take $D_\perp$ large enough to capture almost all the mass of $\gamma_0^\perp$. A possible choice is as follows.

Lemma 5.9. *Assume*

$$D_\perp \supset \prod_{0<|k|\leqslant N}[-a_k,a_k] \qquad \textit{with } a_k = \sqrt{\frac{4\varepsilon\log(\varepsilon^{-1})[1+\log\lambda_k]}{\lambda_k}}\;. \tag{5.30}$$

Then for sufficiently small ε, one has

$$\mathbb{P}^{\gamma_0^\perp}\Big\{\hat{D}_\perp^c\Big\} = \mathcal{O}(\varepsilon)\;. \tag{5.31}$$

Proof: Standard Gaussian tail estimates show that

$$\begin{aligned}
\mathbb{P}^{\gamma_0^\perp}(\hat{D}_\perp^c) &\leqslant \sum_{0<|k|\leqslant N} 2\,\mathrm{e}^{-a_k^2\lambda_k/2} \\
&= 2\sum_{0<|k|\leqslant N}\exp\Big\{-2\log(\varepsilon^{-1})[1+\log\lambda_k]\Big\} \\
&\leqslant 2\varepsilon^2\sum_{0<|k|\leqslant N}\lambda_k^{-2\log(\varepsilon^{-1})}\;.
\end{aligned}$$

The last sum is bounded uniformly in N if $\varepsilon \leqslant \mathrm{e}^{-1}$. □

5.4 Laplace asymptotics and transition times

Combining the results from the last two subsections, we finally obtain the following upper bound on the expected transition time.

Proposition 5.10. *There exists a constant C_+, uniform in N and ε, such that*

$$\mathbb{E}^{\nu_{A,B}}\big[\tau_B\big] \leqslant 2\pi\left(\frac{\mathrm{e}^{3/|\lambda_0|}}{|\lambda_0|(\lambda_0+3)}\prod_{0<|k|\leqslant N}\left[\frac{\mathrm{e}^{3/\lambda_k}}{1+3/\lambda_k}\right]\right)^{1/2}\mathrm{e}^{L^2/4\varepsilon}\big[1+C_+\sqrt{\varepsilon}\,\big] \tag{5.32}$$

holds for all $N \geqslant 1$.

Proof: If follows from Proposition 5.2, Proposition 5.7 and Proposition 5.8 that

$$\frac{\mathcal{Z}_N(\varepsilon)}{2\operatorname{cap}_A(B)} \leqslant \sqrt{\frac{2\pi}{\varepsilon|\lambda_0|}}\int_0^\infty \mathrm{e}^{-q(z_0)/\varepsilon}\,\hat{g}(z_0,\varepsilon)\,\mathrm{d}z_0\;,$$

where

$$\hat{g}(z_0,\varepsilon) \leqslant \mathrm{e}^{-q_1(z_0,\varepsilon)}\,K(z_0)\Big[1+M'\sqrt{\varepsilon}(1+|z_0|)\big(1+\sqrt{\varepsilon}\,\mathrm{e}^{Mz_0^2\log(z_0)/\sqrt{\varepsilon}}\big)\Big]$$

for a constant $M' \geqslant M$. In particular, we have

$$\hat{g}(L,0) \leqslant \mathrm{e}^{3/2} K(L) = \left(\mathrm{e}^3 \prod_{0<|k|\leqslant N} \left[\frac{\mathrm{e}^{3/\lambda_k}}{1+3/\lambda_k}\right]\right)^{1/2} .$$

Since q reaches its minimum $-L^2/4$ on $\mathbb{R}_+$ in $z_0 = L$, writing

$$\frac{1}{\sqrt{\varepsilon}} \int_0^\infty \mathrm{e}^{-[q(z_0)-q(L)]/\varepsilon} \hat{g}(z_0,\varepsilon)\,\mathrm{d}z_0 \leqslant \mathcal{I}_0 + \sqrt{\varepsilon}\mathcal{I}_1 + \varepsilon\mathcal{I}_2$$

and applying one-dimensional Laplace asymptotics, we obtain

$$\mathcal{I}_0 \leqslant \sqrt{\pi}\hat{g}(L,0)\big(1 + C_+\sqrt{\varepsilon}\big) ,$$

for the leading term, while $\mathcal{I}_1$ and $\mathcal{I}_2$ are bounded. □

A Hermite Polynomials and Wick Powers

In this appendix we recall some well-known facts about Hermite polynomials $H_n = H_n(X,C)$ we use throughout the article. Recall that they are defined recursively by setting

$$\begin{cases} H_0 = 1 , \\ H_n = XH_{n-1} - C\,\partial_X H_{n-1} \quad n \in \mathbb{N} . \end{cases} \tag{A.1}$$

In particular, we have

$$\begin{aligned} H_1(X,C) &= X & H_2(X,C) &= X^2 - C \\ H_3(X,C) &= X^3 - 3CX & H_4(X,C) &= X^4 - 6CX^2 + 3C^2 . \end{aligned} \tag{A.2}$$

The following binomial identity for Hermite polynomials is well-known.

Lemma A.1 (Binomial formula for Hermite polynomials, [9, Lem. 3.1])**.** *We have for any $n \in \mathbb{N}$ and $X, v, C \in \mathbb{R}$*

$$H_n(X+v,C) = \sum_{k=0}^{n} \binom{n}{k} H_{n-k}(X,C)\, v^k . \tag{A.3}$$

We will mostly be interested in Hermite polynomials of centered Gaussian random variables X and we will typically choose $C = \mathbb{E}[X^2]$. In this case the random variable $H_n(X,\mathbb{E}[X^2])$ is sometimes referred to as the n-th Wick power of X and denoted by $:X^n:$. The following identity is one of the key properties of Wick powers.

Lemma A.2 ([24, Lemma 1.1.1])**.** *Let X, Y be centered jointly Gaussian random variables. Then*

$$\mathbb{E}\big[(:X^n:)(:Y^m:)\big] = \begin{cases} \mathbb{E}[XY]^n & \text{if } n = m \\ 0 & \text{else} . \end{cases} \tag{A.4}$$

Note that this implies in particular, that for $n \geqslant 1$ we have $\mathbb{E}[:X^n:] = 0$. In some calculations it is convenient for us to change the value of the constant C appearing in H_n.

This will be relevant in some calculations when changing the Gaussian reference measure. The following transformation rule, valid for any real $X, C, \bar{C}$ is easy to check:

$$
\begin{aligned}
H_1(X,C) &= H_1(X,\bar{C}) \\
H_2(X,C) &= H_2(X,\bar{C}) - (C-\bar{C}) \\
H_3(X,C) &= H_3(X,\bar{C}) - 3(C-\bar{C})H_1(X,\bar{C}) \\
H_4(X,C) &= H_4(X,\bar{C}) - 6(C-\bar{C})H_2(X,\bar{C}) + 3(C-\bar{C})^2 \; .
\end{aligned}
\tag{A.5}
$$

We now use (A.4) to derive some classical facts about (Galerkin approximations of) the two dimensional massive Gaussian free field and its Wick powers. For any N and for $x \in \mathbb{T}^2 = \mathbb{R}^2/(L\mathbb{Z})^2$ we consider the random field

$$
\phi_N(x) = \sum_{|k|\leqslant N} \frac{z_k}{\sqrt{|\lambda_k + m^2|}} e_k(x) \;, \tag{A.6}
$$

where for $k \in \mathbb{Z}^2$ we have set $|k| = \max\{|k_1|,|k_2|\}$, $e_k(x) = \frac{1}{L}\mathrm{e}^{\mathrm{i}\,\Omega k\cdot x}$, $\Omega = 2\pi/L$ and $\lambda_k = \Omega^2(k_1^2+k_2^2) - 1$. The z_k are complex-valued Gaussian random variables which are independent up to the constraint $z_k = \overline{z_{-k}}$ which makes ϕ_N a real-valued field and which satisfy

$$
\mathbb{E}[z_k z_{-\ell}] = \begin{cases} 1 & \text{if } k = \ell \\ 0 & \text{else .} \end{cases} \tag{A.7}
$$

Note that due to the constraint $z_k = \overline{z_{-k}}$ these $(2N+1)^2$ dependent complex-valued Gaussian random variables can be represented in terms of $(2N+1)^2$ independent real-valued random variables. The mass $m^2 \geqslant 0$ is a parameter of the model, which in our case only takes either the value 0 or the value 3.

For fixed x we get

$$
\mathbb{E}[\phi_N(x)^2] = \sum_{|k|\leqslant N} \frac{1}{|\lambda_k + m^2|} =: C_N \; . \tag{A.8}
$$

Note that C_N diverges logarithmically, which suggests that the random variables $\phi_N(x)$ for a fixed x do not converge to a meaningful limit as N goes to ∞. However, it is well-known that for any test-function ψ the random variables $\int \phi_N(x)\psi(x)\,\mathrm{d}x$ converge in L^2 (with respect to probability) to a Gaussian limiting random variable. We will not make use of this general fact, but only use that the integrals of $\phi_N(x)$ as well as its Wick powers $:\phi_N^n(x): = H_n(\phi_N(x), C_N)$ have a uniformly-in-N bounded variance. To see this we write for $M > N$

$$
\begin{aligned}
&\mathbb{E}\Big[\int_{\mathbb{T}^2} :\phi_M^n(x):\mathrm{d}x \int_{\mathbb{T}^2} :\phi_N^n(y):\mathrm{d}y \Big] \\
&= \int_{\mathbb{T}^2}\int_{\mathbb{T}^2} \mathbb{E}\big[(:\phi_M^n(x):)(:\phi_N^n(y):)\big]\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{T}^2}\int_{\mathbb{T}^2} \mathbb{E}\big[\phi_M(x)\phi_N(y)\big]^n\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{\mathbb{T}^2}\int_{\mathbb{T}^2} \Big(\frac{1}{L}\sum_{|k|\leqslant N} \frac{e_k(x-y)}{|\lambda_k+m^2|}\Big)^n \mathrm{d}x\,\mathrm{d}y = L^{2-n}\int_{\mathbb{T}^2}\Big(\sum_{|k|\leqslant N}\frac{e_k(x)}{|\lambda_k+m^2|}\Big)^n \mathrm{d}x \\
&= L^{2-2n} \sum_{\substack{k_1+k_2+\cdots+k_n=0 \\ |k_i|\leqslant N}} \frac{1}{|\lambda_{k_1}+m^2|}\cdots\frac{1}{|\lambda_{k_n}+m^2|} \; .
\end{aligned}
\tag{A.9}
$$

This calculation has several immediate consequences. First of all we can conclude that as announced above the variances of $\int_{\mathbb{T}^2} :\phi_N^n(x): \mathrm{d}x$ are uniformly bounded as N goes to ∞:

$$\sup_N \mathbb{E}\Big[\Big(\int_{\mathbb{T}^2} :\phi_N^n(x): \mathrm{d}x\Big)^2\Big] = L^{2-2n} \sum_{\substack{k_1+k_2+\cdots+k_n=0\\ k_i\in\mathbb{Z}^2}} \frac{1}{|\lambda_{k_1}+m^2|}\cdots\frac{1}{|\lambda_{k_n}+m^2|} < \infty\,. \tag{A.10}$$

Indeed, the convergence of this sum can be checked easily (e.g. as in [29, Lem. 3.10]). Furthermore, we get for $M > N$

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\int_{\mathbb{T}^2} :\phi_M^n(x): \mathrm{d}x - \int_{\mathbb{T}^2} :\phi_N^n(x): \mathrm{d}x\Big)^2\Big]\\
&= L^{4-2n} \sum_{\substack{k_1+k_2+\cdots+k_n=0\\ |k_i|\leqslant M}} \frac{1}{|\lambda_{k_1}+m^2|}\cdots\frac{1}{|\lambda_{k_n}+m^2|} - L^{4-2n} \sum_{\substack{k_1+k_2+\cdots+k_n=0\\ |k_i|\leqslant N}} \frac{1}{|\lambda_{k_1}+m^2|}\cdots\frac{1}{|\lambda_{k_n}+m^2|}\\
&\leqslant C_{n,L}\frac{(\log N)^{n-2}}{N^2}\,,
\end{aligned} \tag{A.11}$$

for a constant $C_{n,L}$ which depends on n, L but not on N, M.

Finally, we recall the definition of Wiener chaos which in this finite dimensional context is the following:

Definition A.3. *For $n \in \mathbb{N}_0$ the n-th (inhomogeneous) Wiener chaos generated by the random variables $(z_k)_{|k|\leqslant N}$ is the vector space of real-valued random variables X which can be written as polynomials of degree at most n in the finitely many random variables z_k.*

As stated this definition depend on the number of independent Gaussians used to define the Wiener chaos. However, the following classical and important estimate holds true uniformly in that number. See e.g. [10, Thm. 4.1] for a direct proof. This Theorem can also be deduced immediately from the hyper-contractivity of the Ornstein-Uhlenbeck semigroup [24, Thm. 1.4.1].

Lemma A.4 (Equivalence of moments)**.** *Let X be a random variable, belonging to the n-th inhomogeneous Wiener chaos. Then for any $m \geqslant 1$ one has*

$$\mathbb{E}\big[X^{2m}\big]^{\frac{1}{2m}} \leqslant C_n(2m-1)^{\frac{n}{2}}\mathbb{E}\big[X^2\big]^{\frac{1}{2}} \tag{A.12}$$

where C_n only depends on n.

Remark A.5. The n-th *homogeneous* Wiener chaos is defined as the orthogonal complement (with respect to the L^2 scalar product) of the $n-1$-st inhomogeneous Wiener chaos in the n-th inhomogeneous Wiener chaos. If the previous Lemma, X takes values in the homogeneous Wiener chaos, then the estimate holds true with constant $C_n = 1$.

Now combining Lemma A.4 with (A.10) we obtain for $m \geqslant 1$ that

$$\sup_N \mathbb{E}\Big[\Big(\int_{\mathbb{T}^2} :\phi_N^n(x): \mathrm{d}x\Big)^{2m}\Big] < \infty \tag{A.13}$$

and combining Lemma A.4 with (A.11) we see that for $M > N$ and any $m \geqslant 1$

$$\mathbb{E}\Big[\Big(\int_{\mathbb{T}^2} :\phi_M^n(x): \mathrm{d}x - \int_{\mathbb{T}^2} :\phi_N^n(x): \mathrm{d}x\Big)^{2m}\Big]^{\frac{1}{m}} \leqslant C_{n,L}(2m-1)\frac{(\log N)^{n-2}}{N} \tag{A.14}$$

for a constant $C_{n,L}$ which depends on n, L but not on m, M, N.

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