

TIGHTNESS FOR A STOCHASTIC ALLEN–CAHN EQUATION

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ABSTRACT. We study an Allen–Cahn equation perturbed by a multiplicative stochastic noise that is white in time and correlated in space. Formally this equation approximates a stochastically forced mean curvature flow. We derive a uniform bound for the diffuse surface area, prove the tightness of solutions in the sharp interface limit, and show the convergence to phase-indicator functions.

1. INTRODUCTION

The Allen–Cahn equation

$$\varepsilon \partial_t u_\varepsilon = \varepsilon \Delta u_\varepsilon - \frac{1}{\varepsilon} F'(u_\varepsilon) \quad (1.1)$$

is an important prototype for phase separation processes in melts or alloys that is of fundamental interest both for theory and applications. It describes an evolution of non-conserved phases driven by the surface area reduction of their common interface. The Allen–Cahn equation is a *diffuse interface* model, *i.e.* phases are indicated by smooth fields, assuming a partial mixing of the phases. It is well-known [13, 19, 25] that in the *sharp interface limit* $\varepsilon \rightarrow 0$ solutions of the Allen–Cahn equation converge to an evolution of hypersurfaces $(\Gamma_t)_{t \in (0, T)}$ by *mean curvature flow* (MCF)

$$v(t, \cdot) = H(t, \cdot), \quad (1.2)$$

where v describes the velocity vector of the evolution and $H(t, \cdot)$ denotes the mean curvature vector of Γ_t .

Our goal is to introduce a stochastic perturbation of the Allen–Cahn equation that formally approximates a stochastic mean curvature type flow

$$v(t, \cdot) = H(t, \cdot) + X(t, \cdot), \quad (1.3)$$

where X now is a random vector-field in the ambient space. More specifically what we are considering is the following Stratonovich stochastic partial differential equation (SPDE):

$$du_\varepsilon = \left(\Delta u_\varepsilon - \frac{1}{\varepsilon^2} F'(u_\varepsilon) \right) dt + \nabla u_\varepsilon \cdot X(x, \circ dt), \quad (1.4)$$

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where X is a vectorfield valued Brownian motion. A particular case of such a Brownian motion is

$$X(t, x) = X^0(x)t + \sum_{k=1}^N X^k(x)B_k(t), \quad (1.5)$$

where the X^k are fixed vectorfields and W_k are independent standard Brownian motions. In this case (1.4) reduces to the Stratonovich SPDE

$$du_\varepsilon = \left(\Delta u_\varepsilon - \frac{1}{\varepsilon^2} F'(u_\varepsilon) + \nabla u_\varepsilon \cdot X^0 \right) dt + \sum_{k=1}^N \nabla u_\varepsilon \cdot X^k \circ dB_k(t). \quad (1.6)$$

Our setting is more general as it allows for infinite sums of Brownian motions. See below for a more detailed discussion. We complement (1.4) by deterministic initial and zero Neumann-boundary data,

$$u_\varepsilon(0, \cdot) = u_\varepsilon^0 \quad \text{in } U, \quad (1.7)$$

$$\nabla u_\varepsilon \cdot \nu_U = 0 \quad \text{on } (0, T) \times \partial U, \quad (1.8)$$

where ν_U denotes the outer normal vector to ∂U .

Our main result is the tightness of the solutions $(u_\varepsilon)_{\varepsilon>0}$ of (1.4) and the convergence to an evolution of (random) phase indicator functions $u(t, \cdot) \in BV(U)$. In particular we prove a uniform control (in $\varepsilon > 0$) of the diffuse surface area of u_ε .

In the next sections we briefly review the analysis of the deterministic Allen–Cahn equation and report on stochastic extensions. In Section 3 we state our main assumptions and recall some notations for stochastic flows. Our main results are stated in Section 4. In Section 5 we prove an existence result for (1.4). The main estimates for the diffuse surface area and the tightness of solutions are proved in Section 6. In the last section we present a discussion of the relation to a stochastically perturbed mean curvature flow by a formal passage to the sharp interface limit in a localized surface energy equality.

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2. BACKGROUND

2.1. Deterministic sharp interface limit. As many other diffuse interface models the Allen–Cahn equation (1.1) is based on the *Van der Waals–Cahn–Hilliard* energy

$$E_\varepsilon(u_\varepsilon) := \int_U \left(\frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} F(u_\varepsilon) \right) dx \quad \text{for } u_\varepsilon : U \rightarrow \mathbb{R}. \quad (2.1)$$

The energy E_ε favors a decomposition of the spatial domain U into two regions (phases) where $u_\varepsilon \approx -1$ and $u_\varepsilon \approx 1$, separated by a transition layer (diffuse interface) with a thickness of order ε . Modica and Mortola [36, 34] proved that E_ε Gamma-converges (with respect to L^1 -convergence) to a constant multiple of the perimeter

functional \mathcal{P} , restricted to phase indicator functions,

$$E_\varepsilon \rightarrow c_0 \mathcal{P}, \quad \mathcal{P}(u) := \begin{cases} \frac{1}{2} \int_U d|\nabla u| & \text{if } u \in BV(U, \{-1, 1\}), \\ \infty & \text{otherwise.} \end{cases}$$

\mathcal{P} measures the surface-area of the phase boundary $\partial^*\{u = 1\} \cap U$. In this sense E_ε describes a diffuse approximation of the surface-area functional.

The Allen–Cahn equation (1.1) in fact is the (accelerated) L^2 -gradient flow of E_ε . It is proved in different formulations [13, 19, 25] that (1.1) converges to motion by mean curvature. Since mean curvature flow in general allows for the formation of singularities in finite time it is necessary to consider suitable generalized formulations of (1.2), as for example in the sense of viscosity solutions [2, 9, 10, 19, 20], De Giorgi’s barriers [3, 4, 8, 14], or geometric measure theory. The first approaches rely on the maximum principle, the latter was pioneered by Brakke [6] and is based on the *localized* energy equality

$$\begin{aligned} & \frac{d}{dt} \int_{\Gamma_t} \eta(x) d\mathcal{H}^{n-1}(x) \\ &= \int_{\Gamma_t} \nabla \eta(t, x) \cdot V(t, x) d\mathcal{H}^{n-1}(x) - \int_{\Gamma_t} H(t, x) \cdot V(t, x) \eta(t, x) d\mathcal{H}^{n-1}(x) \end{aligned} \quad (2.2)$$

that holds for arbitrary $\eta \in C_c^1(U)$ and for any classical solutions $(\Gamma_t)_{t \in (0, T)}$ of mean curvature flow. Ilmanen [25] proved the convergence of the Allen–Cahn equation to mean curvature flow in the formulation of Brakke, using a diffuse analog of the (localized) energy equality (2.2). By similar methods Mugnai and the first author [38] proved the convergence of (deterministically) perturbed Allen–Cahn equations.

One of the key results of the present paper is an energy inequality for the stochastic Allen–Cahn equation (1.4). By Itô’s formula the stochastic drift produces some extra terms in the time-derivative of the diffuse surface energy $E_\varepsilon(u_\varepsilon)$. These ‘bad’ terms are exactly compensated by the additional terms in (1.4) which are hidden in the Stratonovich formalism.

2.2. Stochastic perturbations of the Allen–Cahn equation and MCF. Additive perturbations of the Allen–Cahn equation were studied in the one-dimensional case in [22, 7] and in the higher-dimensional case in [23, 40, 32]. Note that perturbation results such as [38] do not apply to the stochastic case as one typically perturbs with a white noise i.e. the time-derivative of a C^α function for $\alpha < \frac{1}{2}$, which is not covered by most techniques.

In the one-dimensional case the equation was studied with an additive space-time white noise and at least for the case where the interface consists of a single kink the sharp interface limit was described rigorously [22, 7]. In higher dimensions the picture is much less complete. For instance, the Allen–Cahn equation with space-time white noise is in general not well-posed: the noise term is so rough that for $n \geq 2$ solutions to the stochastic heat equation attain values only in Sobolev spaces of negative order, and on such spaces the nonlinear potential can a priori not be defined. This existence problem can be avoided if one introduces spatial correlations as we do in (1.4). In all of the above papers conditions on the stochastic perturbations

are much more restrictive than in our approach. In fact it is always assumed that the noise is constant in space and smoothened in time with a correlation length that is coupled to the interface width ε and goes to zero for $\varepsilon \downarrow 0$. All of these papers rely on a construction of the limit dynamics by different means and then an explicit construction of sub- and supersolutions making use of the maximum principle. Our approach is based only on energy estimates. On the other hand, we only prove tightness of the approximations and do not obtain an evolution law for limit points.

The restriction on spatially constant noise in previous papers and our problem to derive the stochastic motion law in the limit are closely related to the lack of existence results and generalized formulations for stochastically forced mean curvature flow. Up to now there are only results for spatially constant forcing [15, 23, 32] or in the case of evolution of graphs in $1 + 1$ dimensions [16].

Our approach is closely related to Yip's construction [42] of a time-discrete stochastically forced mean curvature flow. Yip follows the deterministic scheme of [1, 33], where for a given time step $\delta > 0$ a sequence of sets of bounded perimeter is constructed iteratively. The heart of the construction is the minimization of a functional that is given by the perimeter plus a suitable distance from the previous set. Yip [42] introduces randomness to this scheme by performing a stochastic flow in between two minimization steps. For the resulting time-discrete evolution of sets Yip proves uniform bounds (in δ) for the perimeter and shows tightness of the time-discrete solutions with $\delta \rightarrow 0$. As in our case, a characterization of the limiting evolution is not given. If one applies Yip's scheme to the Allen–Cahn equation (substituting the perimeter functional by the diffuse surface area energy and using a rescaled L^2 -distance between phase fields) one in fact would obtain our stochastic Allen–Cahn equation (1.4) in the limit $\delta \rightarrow 0$.

Noisy perturbations of the Allen–Cahn equation were studied from a different point of view in [27]. There the authors study the action functional which appears if one first applies Freidlin–Wentzel theory to the Allen–Cahn equation with an additive noise that is white in time and spatially correlated, and then formally takes the spatial correlation to zero. Then the sharp interface limit $\varepsilon \downarrow 0$ is studied on the level of action functionals and a *reduced action functional* as a possible Γ -limit is derived. See [41, 37] for a subsequent analysis.

3. ASSUMPTIONS AND STOCHASTIC FLOWS

3.1. Notation and assumptions. Let $U \subset \mathbb{R}^n$ be an open bounded subset of \mathbb{R}^n with smooth boundary, let $T > 0$, and set $U_T := (0, T) \times U$. We denote by $x \in U$ and $t \in (0, T)$ the space- and time-variables respectively; ∇ and Δ denote the spatial gradient and Laplacian. The space of functions of bounded variation in U with values in $\{-1, 1\}$ almost everywhere is denoted with $BV(U; \{-1, 1\})$. $C^{k, \alpha}(\bar{U})$ denotes the space of k -times differentiable functions such that all k^{th} -partial derivatives are bounded and Hölder-continuous with exponent α .

We assume the potential F to be smooth and verify the following assumptions:

$$\left. \begin{aligned} F(r) &\geq 0 \quad \text{and} \quad F(r) = 0 \quad \text{iff } r = \pm 1, \\ F' &\text{ admits exactly three zeros } \{\pm 1, 0\} \text{ and } F''(0) < 0, F''(\pm 1) > 0, \\ F &\text{ is symmetric, } \forall r \geq 0 \quad F(r) = F(-r), \\ F(r) &\geq C|r|^{2+\delta} \quad \text{for some } \delta > 0 \text{ and } |r| \text{ sufficiently large.} \end{aligned} \right\} \quad (3.1)$$

The standard choice for F is

$$F(r) = \frac{1}{4}(1 - r^2)^2,$$

such that the nonlinearity in (1.1) becomes $F'(r) = -r(1 - r^2)$.

Next we give some geometric meaning to u_ε . We define the *normal direction* with respect to u_ε by

$$\nu_\varepsilon(t, x) := \begin{cases} \frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|}(t, x) & \text{if } |\nabla u_\varepsilon(t, x)| \neq 0, \\ \vec{e} & \text{else,} \end{cases} \quad (3.2)$$

where \vec{e} is an arbitrary fixed unit vector. We define the *diffuse surface area measures*

$$\mu_\varepsilon^t(\eta) := \int_U \eta \left(\frac{\varepsilon}{2} |\nabla u_\varepsilon(t, \cdot)|^2 + \frac{1}{\varepsilon} F(u_\varepsilon(t, \cdot)) \right) dx \quad (3.3)$$

for $\eta \in C_c^0(U)$. We denote the *diffuse mean curvature* by

$$w_\varepsilon := -\varepsilon \Delta u_\varepsilon + \frac{1}{\varepsilon} F'(u_\varepsilon). \quad (3.4)$$

For the initial data we assume that u_ε^0 is smooth and that

$$E_\varepsilon(u_\varepsilon^0) \leq \Lambda \quad (3.5)$$

holds for all $\varepsilon > 0$ and a fixed $\Lambda > 0$. Note that by [25, page 423] the boundary of every open set that verifies a density bound and that can be approximated in BV by smooth hypersurfaces can be approximated by phase fields with uniformly bounded diffuse surface area. On the other hand (3.5) implies by [35, 36] that the sequence u_ε^0 is compact in $L^1(U_T)$ and that every limit belongs to the space of phase indicator functions $BV(U, \{\pm 1\})$.

3.2. Stochastic Flows. Let us briefly introduce some notations for stochastic flows. We refer the reader to Kunita's book [29] Chapter 3 and Section 2,5 and 6 in Chapter 4 for further background.

Let (Ω, \mathcal{F}, P) be a probability space with a right continuous and complete filtration $\{\mathcal{F}_{s,t}\}_{0 \leq s \leq t \leq T}$. Let $(X(t, x), t \in [0, T], x \in U)$ be a continuous vector-field valued semimartingale with local characteristics $\tilde{A} := (\tilde{a}_{ij}(t, x, y))_{i,j=1,\dots,n}$ and $b := (b_i(t, x))_{i=1,\dots,n}$ on (Ω, \mathcal{F}, P) . This means that for every $x \in U$ the process $X(t, x)$ is a continuous \mathbb{R}^n valued semimartingale with finite variation process $\int_0^t b(s, x) ds$ and quadratic variation

$$\langle X_i(t, x), X_j(t, y) \rangle = \int_0^t \tilde{a}_{ij}(s, x, y) ds. \quad (3.6)$$

We assume that for every $(x, y) \in U \times U$ the function \tilde{a} is continuous in time and of class $C^{4,\alpha}$ in both space variables, and that $x \mapsto b^i(t, x)$ is of class $C^{3,\alpha}$ for some $\alpha > 0$. Finally we assume that \tilde{a} and b have compact support in $U \times U$ resp. in U .

Denote by $(\varphi_{s,t}, s < t)$ the Stratonovich-Flow associated to $-X$. This means that almost surely $(\varphi_{s,t}, s < t)$ is a two parameter family of diffeomorphisms of U fixing the boundary and verifying the flow property

$$\varphi_{s,t} \circ \varphi_{r,s} = \varphi_{r,t} \quad \text{for } r \leq s \leq t. \quad (3.7)$$

Furthermore, for every x and every $s \in [0, T)$ the process $(\varphi_{s,t}(x), t \geq s)$ is a solution of the stochastic differential equation

$$\begin{aligned} d\varphi_{s,t}(x) &= -X(\circ dt, \varphi_{s,t}(x)) \\ \varphi_{s,s}(x) &= x. \end{aligned} \quad (3.8)$$

Under the above regularity assumption for all $s \leq t$ the mapping $\varphi_{s,t}$ is a $C^{3,\beta}$ diffeomorphism of U , for all $\beta < \alpha$.

A particular example is that of a stochastic flow given by a usual Stratonovich-differential equation. If $X^k(t, \cdot), k = 0, \dots, N$ are smooth time dependent vectorfields on U and B_1, \dots, B_N are independent standard Brownian motions, then

$$X(t, x) = \sum_{k=1}^N \int_0^t X^k(s, x) \circ dB_k(s) + \int_0^t X^0(s, x) ds, \quad (3.9)$$

is a vectorfield valued Brownian motion as considered above. Its local characteristic is given by

$$\begin{aligned} \tilde{a}_{ij}(s, x, y) &= \sum_{k=1}^N X_i^k(s, x) X_j^k(s, y) \\ b_i(s, x) &= X_i^0(s, x). \end{aligned} \quad (3.10)$$

In this case the stochastic differential equation (3.8) reduces to the more familiar

$$\begin{aligned} d\varphi_{s,t}(x) &= X^0(t, \varphi_{s,t}(x))dt + \sum_{k=1}^N X^k(t, \varphi_{s,t}(x)) \circ dB_k(t) \\ \varphi_{s,s}(x) &= x. \end{aligned} \quad (3.11)$$

The advantage of Kunita's framework is that it allows for infinite sums in the noise part i.e. for noise fields of the form

$$X(t, x) = \sum_{i=1}^{\infty} \int_0^t X^i(s, x) \circ dB_i(s) + \int_0^t X^0(s, x) ds, \quad (3.12)$$

for vectorfields with the right summability properties. We prefer this approach as a restriction to finite dimensional noises is unnecessary and a severe restriction.

4. RESULTS

In this section we state our main results. For the proofs see the subsequent sections. We first address the question of existence and uniqueness of solutions for (1.4). There are some classical existence and uniqueness results for equations similar to (1.4), see for example [11, p. 212 ff.], [28], [21]. In those references either mild or weak variational solutions are constructed. Using the technique from [29] we obtain here Hölder-continuous strong solutions.

Theorem 4.1. *Let u_ε^0 , F , and X satisfy the assumptions (3.1), (3.5) and the smoothness conditions stated in Section 3.2. Then for every $\varepsilon > 0$ there exists a unique solution u_ε of*

$$u_\varepsilon(t, x) = u_\varepsilon^0(x) + \int_0^t \left(\Delta u_\varepsilon(s, x) - \frac{1}{\varepsilon^2} F'(u_\varepsilon(s, x)) \right) ds + \int_0^t \nabla u_\varepsilon(s, x) \cdot X(x, \circ ds) \quad (4.1)$$

$$\nabla u_\varepsilon \cdot \nu_U = 0 \quad \text{on } (0, T) \times \partial U. \quad (4.2)$$

The function $u_\varepsilon(t, \cdot)$ is a continuous $C^{3, \beta}(\bar{U})$ -valued semimartingale for any $0 < \beta < \alpha$. Furthermore, we have the following bound for the spatial derivatives:

$$\mathbb{E}[\sup_{x \in U} |\partial^\gamma u_\varepsilon(t, x)|^p] < \infty, \quad (4.3)$$

for any multi-index γ with $|\gamma| \leq 3$, every $p \geq 1$, and for any $t \in (0, T)$.

Our main result concerns the tightness of the solutions u_ε in the limit $\varepsilon \rightarrow 0$. In addition we show that limit points are concentrated on the space of phase indicator function of bounded variation. The key step in the proof of these results is a uniform bound on the diffuse surface area.

Theorem 4.2. *Let the assumptions of Theorem 4.1 be satisfied and let u_ε be the solution of (4.1)-(4.2) for $\varepsilon > 0$. Then the following statements hold:*

(1) *Uniform bounds on the energy: For every $T > 0$ and every $p \geq 1$ we have*

$$\sup_{\varepsilon > 0} \mathbb{E} \left[\sup_{0 \leq t \leq T} E_\varepsilon(u_\varepsilon(t))^p \right] < \infty. \quad (4.4)$$

(2) *Uniform bounds on the diffuse mean curvature: For every $T > 0$ and every $p \geq 1$ we have*

$$\sup_{\varepsilon > 0} \mathbb{E} \left[\left(\int_0^T \int_U \frac{1}{\varepsilon} w_\varepsilon(t, x)^2 dx dt \right)^p \right] < \infty. \quad (4.5)$$

(3) *Tightness of the sequence: Let \mathbb{Q}^ε be the distribution of the solution (1.4). Then the family \mathbb{Q}^ε is tight on $C([0, T], L^1(U))$. In particular, there exists a sequence $\varepsilon_i \downarrow 0$ such that the processes u_{ε_i} can jointly be realized on a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and converge $\tilde{\mathbb{P}}$ -almost-surely in $C([0, T], L^1(U))$ to a limiting process*

u. For almost all $t \in (0, T)$ we have $u(t, \cdot) \in BV(U, \{\pm 1\})$ almost surely and

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} \|u(t)\|_{BV(U)}^p \right] < \infty$$

holds for every $T > 0$ and every $p \geq 1$.

In most of the sequel we will use the Itô-form of (4.1), which is by [29, Section 6.2] given as

$$\begin{aligned} u_\varepsilon(t, x) = & u_\varepsilon(0, x) + \int_0^t \left(\Delta u_\varepsilon(s, x) - \frac{1}{\varepsilon^2} F'(u_\varepsilon(s, x)) \right) ds + \int_0^t \nabla u_\varepsilon(s, x) \cdot X(ds, x) \\ & + \frac{1}{2} \int_0^t \left(A(s, x) : D^2 u_\varepsilon(s, x) + c(s, x) \cdot \nabla u_\varepsilon(s, x) \right) ds. \end{aligned} \quad (4.6)$$

Here and below we use the notation $A : B = \sum_{i,j} A_{ij} B_{ij}$ for the Hilbert-Schmidt scalar product of two matrices.

The Itô-Stratonovich correction terms in (4.6) are given by the matrix $A = (a_{ij})_{i,j=1,\dots,n}$ and the vector field $c = (c_i)_{i=1,\dots,n}$,

$$a_{ij}(t, x) = \tilde{a}_{ij}(t, x, x), \quad (4.7)$$

$$c_j(t, x) = \partial_{y_i} \tilde{a}_{ij}(t, x, y)|_{y=x}, \quad (4.8)$$

where we sum here and in the following over repeated indices. Note that the extra term $A : D^2 u$ is of highest order, such that it changes the diffusion coefficient in (4.1). In particular, the stochastic-parabolicity condition (see e.g. [21, condition (1)]), which is often needed in the case of gradient dependend noise, is always satisfied.

5. EXISTENCE AND UNIQUENESS

In this section we prove Theorem 4.1 by reducing the existence of solutions of (1.4) to an existence statement for a deterministic reaction-diffusion equation with random coefficients. This technique is borrowed from [29].

Proof of Theorem 4.1. As above denote by $\varphi_{s,t}$ the stochastic flow generated by $-X$. For a function $u : U \rightarrow \mathbb{R}$ define the transformation $w_t(u)(x) = u(\varphi_{0,t}^{-1}(x))$. By the regularity of the stochastic flow it is clear that w_t maps $C^{3,\beta}(U)$ into itself. Denote by \mathcal{L} the nonlinear operator

$$\mathcal{L}(u) = \Delta u - \frac{1}{\varepsilon^2} F'(u)$$

and by \mathcal{L}_t^w the operator $w_t^{-1} \mathcal{L} w_t$. Then a direct computation shows that \mathcal{L}_t^w is given by

$$\mathcal{L}_t^w u(t, x) = \sum_{i,j=1}^n R_w^{i,j}(t, x) \frac{\partial^2}{\partial x^i \partial x^j} u(t, x) + \sum_{i=1}^n S_w^i(t, x) \frac{\partial}{\partial x^i} u(t, x) - \frac{1}{\varepsilon^2} F'(u(t, x)), \quad (5.1)$$

with coefficients

$$R_w^{i,j}(t, x) = \sum_{k,l} \partial_k \left(\varphi_{0,t}^{-1} \right)^i (\varphi_{0,t}(x)) \partial_l \left(\varphi_{0,t}^{-1} \right)^j (\varphi_{0,t}(x)) \quad (5.2)$$

$$S_w^i(t, x) = \sum_{k,l} \partial_k \partial_l \left(\varphi_{0,t}^{-1} \right)^i (\varphi_{0,t}(x)) \quad (5.3)$$

In particular, the coefficients are random, the $R_w^{i,j}$ are of class $C^{3,\beta}$ in space and continuous C^γ in time for every $\gamma < \frac{1}{2}$, and the S^i are of class $C^{2,\beta}$ in space and C^γ in time. Furthermore, note that $R^{i,j} = \delta^{i,j}$ and $S = 0$ close to the boundary. Similar to Lemma 6.2.3 in [29] it can be seen by another straightforward computation that a smooth semimartingale u is a solution to (4.1) if and only if $u' = w_t^{-1}u$ is a solution to

$$\frac{\partial}{\partial t} v(t, x) = \mathcal{L}_t^w v(t, x). \quad (5.4)$$

Existence and uniqueness of smooth solutions to reaction diffusion equation like (5.4) can be derived in a standard way: For example, apply Schaefer's Fixed Point Theorem [18, Theorem 9.2.4] in combination with Schauder-estimates [30, Theorem IV.5.3] for the linear part of (5.4) and a-priori estimates by the maximum principle. This yields existence of solutions that are $C^{2,\beta}$ in space and $C^{1,\beta/2}$ in time. Differentiation with respect to space and another application of [30, Theorem IV.5.3] proves $C^{4,\beta}$ -regularity in space. To derive (4.3) note that by (5.4) the derivatives of u up to order 3 can be bounded in terms of the derivatives of v and φ^{-1} . The bounds on v follow from the Schauder-estimates [30, Theorem IV.5.3] applied to the random coefficients $R^{i,j}$ and S^j . The bounds on these coefficients as well as on the derivatives of φ follow from [29, Theorem 6.1.10]. \square

6. TIGHTNESS

In this section we derive estimates for the diffuse surface area.

Proposition 6.1. *Let u_ε satisfy (1.4). Then for all $0 \leq t_0 < t_1$ and all $\varepsilon > 0$*

$$\begin{aligned} & E_\varepsilon(u_\varepsilon(t_1)) - E_\varepsilon(u_\varepsilon(t_0)) \\ &= - \int_{t_0}^{t_1} \int_U \frac{1}{\varepsilon} w_\varepsilon(t, x)^2 dx dt + \int_U \int_{t_0}^{t_1} w_\varepsilon(t, x) \nabla u_\varepsilon(t, x) \cdot X(dt, x) dx \\ &\quad - \frac{1}{2} \int_{t_0}^{t_1} \int_U w_\varepsilon(t, x) c(t, x) \cdot \nabla u_\varepsilon(t, x) dx dt \\ &\quad + \frac{1}{2} \int_{t_0}^{t_1} \int_U \varepsilon \nabla u_\varepsilon(t, x) \cdot \Psi(t, x) \nabla u_\varepsilon(t, x) + \psi(t, x) \frac{1}{\varepsilon} F(u_\varepsilon(t, x)) dx dt \end{aligned} \quad (6.1)$$

holds with

$$\begin{aligned} \Psi_{ij} &= \left(\partial_k \left[\partial_{x_l} \tilde{a}_{lk}(t; x, y) \Big|_{x=y} - \partial_{y_l} \tilde{a}_{lk}(t; x, y) \Big|_{x=y} \right] \right) \delta_{ij} \\ &\quad + \partial_{x_k} \partial_{y_k} \tilde{a}_{ij}(t; x, y) \Big|_{x=y} + 2 \partial_j \left[\partial_{x_k} \tilde{a}_{ik}(t; x, y) \Big|_{x=y} \right] - 2 \partial_k \left[\partial_{x_j} \tilde{a}_{ik}(t; x, y) \Big|_{x=y} \right], \end{aligned}$$

$$\psi = \left(\partial_j \left[\partial_{x_i} \tilde{a}_{ij}(t; x, y) \Big|_{x=y} - \partial_{y_i} \tilde{a}_{ij}(t; x, y) \Big|_{x=y} \right] \right). \quad (6.2)$$

Proof. Recalling the conventions (3.4), (4.7), and (4.8) we can rewrite (4.6) as

$$du_\varepsilon = -\frac{1}{\varepsilon} w_\varepsilon dt + \nabla u_\varepsilon \cdot X(dt) + \frac{1}{2} A : D^2 u_\varepsilon dt + \frac{1}{2} c \cdot \nabla u_\varepsilon dt. \quad (6.3)$$

Itô's formula then yields

$$\begin{aligned} dE_\varepsilon(u_\varepsilon) &= \int_U w_\varepsilon(t, x) du_\varepsilon(t, x) dx \\ &\quad + \frac{1}{2} \left[\int_U \varepsilon \operatorname{tr}(Q)(t, x, x) dx + \int_U \frac{1}{\varepsilon} F''(u_\varepsilon(t, x)) q(t, x, x) dx \right] dt. \end{aligned} \quad (6.4)$$

Here in the Itô correction term we have used the short hand notation

$$Q_{kl}(t, x, y) = \partial_{x_k} \partial_{y_l} \left[\partial_i u_\varepsilon(t, x) \tilde{a}_{ij}(t; x, y) \partial_j u_\varepsilon(t, y) \right] \quad (6.5)$$

$$q(t, x, y) = \partial_i u_\varepsilon(t, x) \tilde{a}_{ij}(t, x, y) \partial_j u_\varepsilon(t, y). \quad (6.6)$$

See [29, Theorem 3.1.3] for a proof that the cross variation of $\nabla(\nabla u_\varepsilon \cdot X)$ is indeed given by Q . Recall that we use the convention to sum over repeated indices.

When evaluating the right-hand side of (6.4) we obtain one ‘good’ term, which corresponds to minus the integral over the squared diffuse mean curvature in the purely deterministic case. Additional terms are due to the stochastic drift in (6.3) and the second order terms in the Itô formula that can be found in the second line of (6.4). The objective is to show that these extra terms can finally be controlled by the ‘good’ mean curvature term or by a Gronwall argument. Note that a priori this is not obvious at all, because on the right hand side of (6.4) there appear terms like

$$\int_U w_\varepsilon(t, x) A(t, x) : D^2 u_\varepsilon(t, x) dx dt \quad (6.7)$$

that are not controlled by the mean curvature term.

We now start modifying the Itô terms in the second line of (6.4). These calculations only involve the spatial integrals and we drop the time argument for simplicity. We start the calculation by treating the terms involving the potential F on the right hand side of (6.4).

$$\begin{aligned} &\frac{1}{\varepsilon} \int_U F''(u_\varepsilon(x)) \partial_i u_\varepsilon(x) a_{ij}(x) \partial_j u_\varepsilon(x) dx \\ &= -\frac{1}{\varepsilon} \int_U F'(u_\varepsilon(x)) \partial_i a_{ij}(x) \partial_j u_\varepsilon(x) dx - \frac{1}{\varepsilon} \int_U F'(u_\varepsilon(x)) a_{ij}(x) \partial_i \partial_j u_\varepsilon(x) dx \\ &= -\frac{1}{\varepsilon} \int_U F'(u_\varepsilon(x)) \left[\partial_{x_i} \tilde{a}_{ij}(x, y) \Big|_{x=y} - \partial_{y_i} \tilde{a}_{ij}(x, y) \Big|_{x=y} \right] \partial_j u_\varepsilon(x) dx \\ &\quad - \frac{2}{\varepsilon} \int_U F'(u_\varepsilon(x)) \partial_{y_i} \tilde{a}_{ij}(x, y) \Big|_{x=y} \partial_j u_\varepsilon(x) dx \\ &\quad - \frac{1}{\varepsilon} \int_U F'(u_\varepsilon(x)) a_{ij}(x) \partial_i \partial_j u_\varepsilon(x) dx \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\varepsilon} \int_U F(u_\varepsilon(x)) \partial_j \left[\partial_{x_i} \tilde{a}_{ij}(x, y) \Big|_{x=y} - \partial_{y_i} \tilde{a}_{ij}(x, y) \Big|_{x=y} \right] dx \\
&\quad - \frac{2}{\varepsilon} \int_U F'(u_\varepsilon(x)) \partial_{y_i} \tilde{a}_{ij}(x, y) \Big|_{x=y} \partial_j u_\varepsilon(x) dx \\
&\quad - \frac{1}{\varepsilon} \int_U F'(u_\varepsilon(x)) a_{ij}(x) \partial_i \partial_j u_\varepsilon(x) dx.
\end{aligned} \tag{6.8}$$

Here in the first step we have performed a partial integration with respect to the x_i coordinate. In the second step we have added and subtracted an extra term. Finally, in the last line, we have performed another partial integration with respect to x_j in the first term.

In order to treat the terms involving Q we start by evaluating the partial derivatives in (6.5). We get

$$Q_{kl} = T_{kl}^1 + T_{kl}^2 + T_{kl}^3 + R_{kl}, \tag{6.9}$$

where

$$\begin{aligned}
T_{kl}^1(x, y) &= \partial_k \partial_i u_\varepsilon(x) \tilde{a}_{ij}(x, y) \partial_l \partial_j u_\varepsilon(y), \\
T_{kl}^2(x, y) &= \partial_i u_\varepsilon(x) \partial_{x_k} \tilde{a}_{ij}(x, y) \partial_l \partial_j u_\varepsilon(y), \\
T_{kl}^3(x, y) &= \partial_k \partial_i u_\varepsilon(x) \partial_{y_l} \tilde{a}_{ij}(x, y) \partial_j u_\varepsilon(y), \\
R_{kl}(x, y) &= \partial_i u_\varepsilon(x) \partial_{x_k} \partial_{y_l} \tilde{a}_{ij}(x, y) \partial_j u_\varepsilon(y).
\end{aligned} \tag{6.10}$$

We have chosen to denote the last term with a different character R to indicate that it will remain unchanged throughout the rest of the calculation.

We start our calculation with the term in (6.4) that involves T^1 . By first performing first a partial integration in x_j and then another partial integration in x_k in the first term we get

$$\begin{aligned}
&\int_U \varepsilon \partial_k \partial_i u_\varepsilon(x) a_{ij}(x) \partial_k \partial_j u_\varepsilon(x) dx \\
&= - \int_U \varepsilon \partial_j \partial_k \partial_i u_\varepsilon(x) a_{ij}(x) \partial_k u_\varepsilon(x) dx - \int_U \varepsilon \partial_k \partial_i u_\varepsilon(x) \partial_j a_{ij}(x) \partial_k u_\varepsilon(x) dx \\
&= \int_U \varepsilon \partial_j \partial_i u_\varepsilon(x) a_{ij}(x) \Delta u_\varepsilon(x) dx + \int_U \varepsilon \partial_j \partial_i u_\varepsilon(x) \partial_k a_{ij}(x) \partial_k u_\varepsilon(x) dx \\
&\quad - \int_U \varepsilon \partial_k \partial_i u_\varepsilon(x) \partial_j a_{ij}(x) \partial_k u_\varepsilon(x) dx.
\end{aligned} \tag{6.11}$$

The first term on the right hand side of (6.11) has already the desired form. It cancels exactly with the corresponding part of the ‘bad’ term mentioned above in (6.7). Let us treat the two extra terms on the right hand side of (6.11) separately. For the last term we get

$$\begin{aligned}
&- \int_U \varepsilon \partial_k \partial_i u_\varepsilon(x) \partial_j a_{ij}(x) \partial_k u_\varepsilon(x) dx \\
&= - \int_U \varepsilon \partial_k \partial_i u_\varepsilon(x) \left[\partial_{y_j} \tilde{a}_{ij}(x, y) \Big|_{x=y} - \partial_{x_j} \tilde{a}_{ij}(x, y) \Big|_{x=y} \right] \partial_k u_\varepsilon(x) dx
\end{aligned}$$

$$\begin{aligned}
& - 2 \int_U \varepsilon \partial_k \partial_i u_\varepsilon(x) \partial_{x_j} \tilde{a}_{ij}(x, y) \big|_{x=y} \partial_k u_\varepsilon(x) dx \\
& = \int_U \varepsilon \frac{1}{2} |\nabla u_\varepsilon(x)|^2 \partial_i [\partial_{y_j} \tilde{a}_{ij}(x, y) \big|_{x=y} - \partial_{x_j} \tilde{a}_{ij}(x, y) \big|_{x=y}] dx \\
& \quad + 2 \int_U \varepsilon \partial_i u_\varepsilon(x) \partial_{x_j} \tilde{a}_{ij}(x, y) \big|_{x=y} \Delta u_\varepsilon(x) dx \\
& \quad + 2 \int_U \varepsilon \partial_i u_\varepsilon(x) \partial_k [\partial_{x_j} \tilde{a}_{ij}(x, y) \big|_{x=y}] \partial_k u_\varepsilon(x) dx. \tag{6.12}
\end{aligned}$$

Here in the second equality for the first term we have used the fact that $\partial_k \partial_i u_\varepsilon \partial_k u_\varepsilon = \frac{1}{2} \partial_i |\nabla u_\varepsilon|^2$ to perform an integration by part. For the second term we have performed another integration by part in x_k .

For the second term on the right hand side of (6.11) we add the terms involving T^3 and T^4 to obtain

$$\begin{aligned}
& \int_U \varepsilon \partial_j \partial_i u_\varepsilon(x) \partial_k a_{ij}(x) \partial_k u_\varepsilon(x) dx + \int \varepsilon (T_{k,k}^2(x, x) + T_{k,k}^3(x, x)) dx \\
& = \int_U \varepsilon \partial_j \partial_i u_\varepsilon(x) \partial_{x_k} \tilde{a}_{ij}(x, y) \big|_{x=y} \partial_k u_\varepsilon(x) dx + \int_U \varepsilon \partial_i u_\varepsilon(x) \partial_{x_k} \tilde{a}_{ij}(x, y) \big|_{x=y} \partial_k \partial_j u_\varepsilon(x) dx \\
& + \int_U \varepsilon \partial_j \partial_i u_\varepsilon(x) \partial_{y_k} \tilde{a}_{ij}(x, y) \big|_{x=y} \partial_k u_\varepsilon(x) dx + \int_U \partial_k \partial_i u_\varepsilon(x) \partial_{y_k} \tilde{a}_{ij}(x, y) \big|_{x=y} \partial_j u_\varepsilon(x) dx = \\
& = - \int_U \varepsilon \partial_i u_\varepsilon(x) \partial_j [\partial_{x_k} \tilde{a}_{ij}(x, y) \big|_{x=y}] \partial_k u_\varepsilon(x) dx \\
& \quad - \int_U \varepsilon \partial_j u_\varepsilon(x) \partial_i [\partial_{y_k} \tilde{a}_{ij}(x, y) \big|_{x=y}] \partial_k u_\varepsilon(x) \\
& = -2 \int_U \varepsilon \partial_i u_\varepsilon(x) \partial_j [\partial_{x_k} \tilde{a}_{ij}(x, y) \big|_{x=y}] \partial_k u_\varepsilon(x) dx. \tag{6.13}
\end{aligned}$$

Here for the second equality we have performed an integration by part in x_j in the second line and another integration by part in x_i in the third line. In the last equality we have used the identity $\tilde{a}_{ij}(x, y) = \tilde{a}_{ji}(y, x)$ which follows immediately from the definition (3.6).

So finally, summarising the calculations, we get by collecting and regrouping the terms from (6.4), (6.8), (6.10), (6.11), (6.12), and (6.13)

$$\begin{aligned}
dE_\varepsilon(u_\varepsilon(t)) & = \int_U w_\varepsilon(t, x) \left[-\frac{1}{\varepsilon} w_\varepsilon(t, x) dt + \nabla u_\varepsilon(t, x) \cdot X(dt) \right. \\
& \quad \left. + \frac{1}{2} A(t, x) : D^2 u_\varepsilon(t, x) dt + \frac{1}{2} c(t, x) \cdot \nabla u_\varepsilon(t, x) dt \right] dx \\
& - \frac{1}{2} \int_U \frac{1}{\varepsilon} (F'(u_\varepsilon(t, x)) - \Delta u_\varepsilon(t, x)) a_{ij}(t, x) \partial_i \partial_j u_\varepsilon(t, x) dx dt \\
& + \frac{1}{2} \int_U \frac{1}{\varepsilon} F(u_\varepsilon(t, x)) \partial_j [\partial_{x_i} \tilde{a}_{ij}(t, x, y) \big|_{x=y} - \partial_{y_i} \tilde{a}_{ij}(t, x, y) \big|_{x=y}] dx dt \\
& + \frac{1}{2} \int_U \varepsilon \frac{1}{2} |\nabla u_\varepsilon(t, x)|^2 \partial_i [\partial_{y_j} \tilde{a}_{ij}(t, x, y) \big|_{x=y} - \partial_{x_j} \tilde{a}_{ij}(t, x, y) \big|_{x=y}] dx dt
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{\varepsilon} \int_U F'(u_\varepsilon(t, x)) \partial_{y_i} \tilde{a}_{ij}(t; x, y)|_{x=y} \partial_j u_\varepsilon(t, x) dx dt \\
& + \int_U \varepsilon \partial_i u_\varepsilon(t, x) \partial_{x_j} \tilde{a}_{ij}(t; x, y)|_{x=y} \Delta u_\varepsilon(t, x) dx \\
& + \frac{1}{2} \int_U \varepsilon \partial_i u_\varepsilon(t, x) \partial_{x_k} \partial_{y_k} \tilde{a}_{ij}(t; x, y)|_{x=y} \partial_j u_\varepsilon(t, x) dx dt \\
& + \int_U \varepsilon \partial_i u_\varepsilon(t, x) \partial_k [\partial_{x_j} \tilde{a}_{ij}(t; x, y)|_{x=y}] \partial_k u_\varepsilon(t, x) dx dt \\
& - \int_U \varepsilon \partial_i u_\varepsilon(t, x) \partial_j [\partial_{x_k} \tilde{a}_{ij}(t; x, y)|_{x=y}] \partial_k u_\varepsilon(t, x) dx dt.
\end{aligned} \tag{6.14}$$

As mentioned above the terms in the third line of (6.14) cancel with the first term in the second line. Also note that using the symmetry $\tilde{a}_{ij}(x, y) = \tilde{a}_{ji}(y, x)$ once more we can see that the terms in square brackets in the fourth and fifth lines are identical, so that these two terms can be written as

$$\mu_\varepsilon^t \left(\partial_j \left[\partial_{x_i} \tilde{a}_{ij}(t; x, y)|_{x=y} - \partial_{y_i} \tilde{a}_{ij}(t; x, y)|_{x=y} \right] \right). \tag{6.15}$$

Finally, using the symmetry of \tilde{a} once more and recalling the definition (4.8), we see that $\partial_{x_j} \tilde{a}_{ij} = \partial_{y_i} \tilde{a}_{ij} = c_j$. Hence, the terms in the sixth and seventh lines yield -2 times the last term in the second line and the sign in this Itô-Stratonovich correction term changes. Putting everything together we get the identity

$$\begin{aligned}
& E_\varepsilon(u_\varepsilon(t_1)) - E_\varepsilon(u_\varepsilon(t_0)) \\
& = - \int_{t_0}^{t_1} \int_U \frac{1}{\varepsilon} w_\varepsilon(t, x)^2 dx dt + \int_U \int_{t_0}^{t_1} w_\varepsilon(t, x) \nabla u_\varepsilon(t, x) \cdot X(dt, x) dx \\
& - \frac{1}{2} \int_{t_0}^{t_1} \int_U c(t, x) \cdot \nabla u_\varepsilon(t, x) w_\varepsilon(t, x) dx dt \\
& + \frac{1}{2} \int_{t_0}^{t_1} \mu_\varepsilon^t \left(\partial_j \left[\partial_{x_i} \tilde{a}_{ij}(t; x, y)|_{x=y} - \partial_{y_i} \tilde{a}_{ij}(t; x, y)|_{x=y} \right] \right) dt \\
& + \frac{1}{2} \int_{t_0}^{t_1} \int_U \varepsilon \partial_i u_\varepsilon(t, x) \partial_{x_k} \partial_{y_k} \tilde{a}_{ij}(t; x, y)|_{x=y} \partial_j u_\varepsilon(t, x) dx dt \\
& + \int_U \varepsilon \partial_i u_\varepsilon(t, x) \partial_k [\partial_{x_j} \tilde{a}_{ij}(t; x, y)|_{x=y}] \partial_k u_\varepsilon(t, x) dx dt \\
& - \int_U \varepsilon \partial_i u_\varepsilon(t, x) \partial_j [\partial_{x_k} \tilde{a}_{ij}(t; x, y)|_{x=y}] \partial_k u_\varepsilon(t, x) dx dt.
\end{aligned} \tag{6.16}$$

This equality is equivalent to (6.1). \square

Remark 6.2. In the special case that the Brownian vector field $X(\cdot, \circ dt)$ is given by (3.9), the equation (6.16) yields

$$\begin{aligned}
& E_\varepsilon(u_\varepsilon(t_1)) - E_\varepsilon(u_\varepsilon(t_0)) \\
& = - \int_{t_0}^{t_1} \int_U \frac{1}{\varepsilon} w_\varepsilon^2 dx dt + \int_U \int_{t_0}^{t_1} w_\varepsilon \nabla u_\varepsilon \cdot X^k dB_k(t) dx
\end{aligned}$$

$$\begin{aligned}
& + \int_{t_0}^{t_1} \int_U w_\varepsilon \nabla u_\varepsilon \cdot X^0 dx dt \\
& - \frac{1}{2} \int_{t_0}^{t_1} \int_U DX^k X^k \cdot \nabla u_\varepsilon w_\varepsilon dx dt \\
& + \frac{1}{2} \int_{t_0}^{t_1} \mu_\varepsilon^t \left((\nabla \cdot X^k)^2 - \text{tr}(DX^k DX^k) \right) dt \\
& + \int_{t_0}^{t_1} \int_U \frac{\varepsilon}{2} |DX^T \nabla u_\varepsilon|^2 + \varepsilon \nabla u_\varepsilon \cdot DX^k DX^k \nabla u_\varepsilon dx dt \\
& - \int_{t_0}^{t_1} \int_U \varepsilon (\nabla u_\varepsilon \cdot DX^k \nabla u_\varepsilon) \nabla \cdot X^k dx dt.
\end{aligned} \tag{6.17}$$

We next remark that the first *inner* variation of E_ε at u_ε in direction of a vector field Y is given by

$$\delta E_\varepsilon(u_\varepsilon)(Y) = \int_U -w_\varepsilon \nabla u_\varepsilon \cdot Y dx,$$

and that we obtain from [31] for the second inner variation of E_ε at u_ε in direction of a vector field Y

$$\begin{aligned}
\delta^2 E_\varepsilon(u_\varepsilon)(Y, Y) &= \left(\frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} F'(u_\varepsilon) \right) \left((\nabla \cdot Y)^2 - \text{tr}(DY DY) \right) + \\
&+ \frac{\varepsilon}{2} |DX^T \nabla u_\varepsilon|^2 + \varepsilon \nabla u_\varepsilon \cdot DY DY \nabla u_\varepsilon - (\nabla u_\varepsilon \cdot DY \nabla u_\varepsilon) \nabla \cdot Y.
\end{aligned}$$

Therefore, we can rewrite (6.17) as

$$\begin{aligned}
E_\varepsilon(u_\varepsilon(t_1)) - E_\varepsilon(u_\varepsilon(t_0)) &= - \int_{t_0}^{t_1} \int_U \frac{1}{\varepsilon} w_\varepsilon^2 dx dt \\
&+ \int_{t_0}^{t_1} \delta E_\varepsilon(u_\varepsilon(t)) (-X^k) \circ dB_k(t) \\
&+ \int_{t_0}^{t_1} \delta E_\varepsilon(u_\varepsilon(t)) (-X^0) dt \\
&+ \frac{1}{2} \int_{t_0}^{t_1} \delta^2 E_\varepsilon(u_\varepsilon(t)) (X^k, X^k) dt,
\end{aligned}$$

which reflects the fact that the perturbation $\nabla u_\varepsilon \cdot X^k \circ dB_k$ to the Allen-Cahn equation in (1.4) corresponds to an inner variation of u_ε in direction of $-X^k \circ dB_k(t)$.

With Proposition 6.1 in hand we are now ready to derive the desired moment estimates (4.4) and (4.5). We give the proof only for $p \geq 2$. The general case then follows from Young's inequality.

Proposition 6.3. *For every $p \geq 2$ and all $T > 0$ we have*

$$\sup_{\varepsilon > 0} \mathbb{E} \left[\sup_{0 \leq t \leq T} E_\varepsilon(u_\varepsilon(t))^p \right] < \infty \quad \text{and} \tag{6.18}$$

$$\sup_{\varepsilon > 0} \mathbb{E} \left[\left(\int_0^T \int_U \frac{1}{\varepsilon} w_\varepsilon^2(t, x) dx dt \right)^p \right] < \infty. \quad (6.19)$$

Proof. We start by observing that due to the positivity of both summands we can write

$$\begin{aligned} & \mathbb{E} \left[\sup_{0 \leq s \leq t} E_\varepsilon(u_\varepsilon(s))^p \right] + \mathbb{E} \left[\left(\int_0^t \int_U \frac{1}{\varepsilon} w_\varepsilon^2(s, x) dx ds \right)^p \right] \\ & \leq 2^{p-1} \mathbb{E} \left[\left(\sup_{0 \leq s \leq t} E_\varepsilon(u_\varepsilon(s)) + \int_0^t \int_U \frac{1}{\varepsilon} w_\varepsilon^2(s, x) dx ds \right)^p \right]. \end{aligned} \quad (6.20)$$

According to (6.1) we can write for any $t > 0$

$$E_\varepsilon(u_\varepsilon(t)) + \int_0^t \int_U \frac{1}{\varepsilon} w_\varepsilon^2(t, x) dx dt = E_\varepsilon(u_\varepsilon^0) + M_\varepsilon(t) + H_\varepsilon(t), \quad (6.21)$$

where

$$M_\varepsilon(t) = \int_U \int_0^t w_\varepsilon(t, x) \nabla u_\varepsilon(t, x) \cdot \tilde{X}(dt, x) dx, \quad (6.22)$$

and

$$\begin{aligned} H_\varepsilon(t) &= \int_0^t \int_U w_\varepsilon(s, x) \tilde{b}(s, x) \cdot \nabla u_\varepsilon(s, x) dx ds \\ &+ \frac{1}{2} \int_0^{t_1} \int_U \varepsilon \nabla u_\varepsilon(s, x) \cdot \Psi(s, x) \nabla u_\varepsilon(s, x) + \psi(t, x) \frac{1}{\varepsilon} F(u_\varepsilon(s, x)) dx ds. \end{aligned} \quad (6.23)$$

Here we have used the notation

$$\tilde{X}(t, x) = X(t, x) - \int_0^t b(s, t) ds \quad \text{and} \quad \tilde{b}(t, x) = b(t, x) - \frac{1}{2} c(t, x). \quad (6.24)$$

Then M_ε is a local martingale and H_ε is a process of bounded variation. The quadratic variation of M_ε is given by

$$d\langle M_\varepsilon \rangle_t = \int_U \int_U w_\varepsilon(t, x) \nabla u_\varepsilon(t, x) \tilde{A}(t, x, y) w_\varepsilon(t, y) \nabla u_\varepsilon(t, y) dx dy dt. \quad (6.25)$$

Then recalling (3.5) we can write for any $\varepsilon > 0$ and any $p \geq 1$ and any $t > 0$ that

$$\begin{aligned} & \mathbb{E} \left[\left(\sup_{0 \leq s \leq t} E_\varepsilon(u_\varepsilon(s)) + \int_0^t \int_U \frac{1}{\varepsilon} w_\varepsilon^2(s, x) dx ds \right)^p \right] \\ & \leq 3^{p-1} \Lambda^p + 3^{p-1} \mathbb{E} \left[\sup_{0 \leq s \leq t} |M_\varepsilon(s)|^p \right] + 3^{p-1} \mathbb{E} \left[\sup_{0 \leq s \leq t} |H_\varepsilon(s)|^p \right]. \end{aligned} \quad (6.26)$$

For the martingale term we get using the Burkholder-Davis-Gundy inequality [26, Theorem 3.28 on page 166] that

$$\mathbb{E} \left[\sup_{0 \leq s \leq t} |M_\varepsilon(s)|^p \right] \leq C \mathbb{E} \left[\langle M_\varepsilon \rangle_t^{\frac{p}{2}} \right], \quad (6.27)$$

for a constant C depending only on p . In order to bound this expression we use

$$w_\varepsilon \nabla u_\varepsilon = -\varepsilon \Delta u_\varepsilon \nabla u_\varepsilon + \frac{1}{\varepsilon} F'(u_\varepsilon) \nabla u_\varepsilon =$$

$$= -\nabla \cdot (\varepsilon \nabla u_\varepsilon \otimes \nabla u_\varepsilon) + \nabla \left(\frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} F(u_\varepsilon) \right); \quad (6.28)$$

a partial integration yields

$$\begin{aligned} & \int_U \tilde{A}(s, x, y) w_\varepsilon(s, y) \nabla u_\varepsilon(s, y) dy \\ &= \int_U \nabla_y \tilde{A}(s, x, y) (\varepsilon \nabla u_\varepsilon(s, y) \otimes \nabla u_\varepsilon(s, y)) dy \\ & \quad - \int_U \nabla_y \tilde{A}(s, x, y) \left(\frac{\varepsilon}{2} |\nabla u_\varepsilon(s, y)|^2 + \frac{1}{\varepsilon} F(u_\varepsilon(s, y)) \right) dy. \end{aligned} \quad (6.29)$$

Thus repeating the same partial integration in the x -variable we can conclude that

$$\begin{aligned} & \left| \int_U \int_U w_\varepsilon(s, x) \nabla u_\varepsilon(s, x) \cdot \tilde{A}(s, x, y) w_\varepsilon(s, y) \nabla u_\varepsilon(s, y) dx dy \right| \\ & \leq 4 \|\tilde{A}\|_{C^0([0, T]; C^2(\bar{U} \times \bar{U}))} E_\varepsilon(u_\varepsilon(s))^2. \end{aligned} \quad (6.30)$$

Hence for any $t \leq T$ the right hand side of (6.27) can be bounded by

$$\begin{aligned} \mathbb{E} \left[\langle M_\varepsilon \rangle_t^{\frac{p}{2}} \right] & \leq 4 \|\tilde{A}\|_{C^0([0, T]; C^2(\bar{U} \times \bar{U}))} \mathbb{E} \left[\left(\int_0^t E_\varepsilon(u_\varepsilon(s))^2 ds \right)^{\frac{p}{2}} \right] \\ & \leq 4 \|\tilde{A}\|_{C^0([0, T]; C^2(\bar{U} \times \bar{U}))} T^{\frac{p-2}{2}} \int_0^t \mathbb{E} [E_\varepsilon(u_\varepsilon(s))^p] ds. \end{aligned} \quad (6.31)$$

Here in the second line we have used Hölder inequality and Fubini theorem. For the term H_ε we write

$$\begin{aligned} & \mathbb{E} \left[\sup_{0 \leq s \leq t} |H_\varepsilon(s)|^p \right] \\ & \leq 2^{p-1} \mathbb{E} \left[\left| \int_0^t \int_U w_\varepsilon(r, x) \nabla u_\varepsilon(r, x) \cdot \tilde{b}(r, x) dx dr \right|^p \right] \\ & \quad + 2^{p-1} \mathbb{E} \left[\left| \int_0^t \int_U \frac{\varepsilon}{2} \nabla u_\varepsilon(r, x) \cdot \Psi(r, x) \nabla u_\varepsilon(r, x) + \psi(r, x) \frac{1}{2\varepsilon} F(u_\varepsilon(r, x)) dx dr \right|^p \right]. \end{aligned} \quad (6.32)$$

Using Young's inequality for any $\delta > 0$ the first term on the right hand side of (6.32) can be estimated by

$$\begin{aligned} & \mathbb{E} \left[\left| \int_0^t \int_U w_\varepsilon \nabla u_\varepsilon \cdot \tilde{b} dx ds \right|^p \right] \\ & \leq \|\tilde{b}\|_{C^0([0, T] \times U)}^p \mathbb{E} \left[\left| \int_0^t \int_U \frac{\delta}{2\varepsilon} w_\varepsilon^2 + \frac{\varepsilon}{2\delta} |\nabla u_\varepsilon|^2 dx ds \right|^p \right] \\ & \leq \delta^p \frac{1}{2} \|\tilde{b}\|_{C^0([0, T] \times U)}^p \mathbb{E} \left[\left| \int_0^t \int_U \frac{1}{\varepsilon} w_\varepsilon^2 dx ds \right|^p \right] \\ & \quad + \frac{1}{\delta^p} 2^{p-1} \|\tilde{b}\|_{C^0([0, T] \times U)}^p T^{p-1} \int_0^t \mathbb{E} \left[\left(\int_U \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 dx \right)^p \right] ds. \end{aligned} \quad (6.33)$$

For the second term on the right hand side of (6.32) we write

$$\begin{aligned} & \mathbb{E} \left[\left| \int_s^t \int_U \frac{\varepsilon}{2} \nabla u_\varepsilon(r, x) \cdot \Psi(r, x) \nabla u_\varepsilon(r, x) + \psi(r, x) \frac{1}{2\varepsilon} F(u_\varepsilon(r, x)) dx dr \right|^p \right] \\ & \leq (\|\psi\|_{C^0([0, T] \times U)} + \|\Psi\|_{C^0([0, T] \times U)})^p T^{p-1} \int_0^t \mathbb{E} [E_\varepsilon(u_\varepsilon(s))^p] ds. \end{aligned} \quad (6.34)$$

So finally, (6.20) and (6.26)–(6.34) we obtain

$$\begin{aligned} & \mathbb{E} \left[\sup_{0 \leq s \leq t} E_\varepsilon(u_\varepsilon(s))^p \right] + \mathbb{E} \left[\left(\int_0^t \int_U \frac{1}{\varepsilon} w_\varepsilon^2(s, x) dx ds \right)^p \right] \\ & \leq C_1 \left(1 + \frac{1}{\delta^p} \right) \int_0^t \mathbb{E} [E_\varepsilon(u_\varepsilon(s))^p] ds + C_2 \delta^p \mathbb{E} \left[\left(\int_0^t \int_U \frac{1}{\varepsilon} w_\varepsilon^2(s, x) dx ds \right)^p \right], \end{aligned} \quad (6.35)$$

for positive constants $C_1 = C_1(p, T, \|b\|_{C^0([0, T] \times U)}, \|\tilde{A}\|_{C^0([0, T]; C^2(U))})$ and $C_2 = C_2(p, T, \|\tilde{b}\|_{C^0([0, T] \times U)})$. Hence choosing $\delta = \left(\frac{1}{C_2}\right)^{\frac{1}{p}}$ and applying Gronwall's Lemma we obtain (6.18). Then applying the same estimate for $\delta = \frac{1}{2} \left(\frac{1}{C_2}\right)^{\frac{1}{p}}$ and using (6.18) we also obtain (6.19). \square

As in [34] it is convenient to consider instead of u_ε the transformation $G(u_\varepsilon)$ given by the function $G(s) := \int_0^s \sqrt{2F(r)} dr$. Note that G is smooth, increasing and that $G(0) = 0$. We will need the following bound on the increments of $G(u_\varepsilon(t))$:

Lemma 6.4. *For every smooth test function $\varphi \in C_b^\infty(U)$ and every $p \in \mathbb{N}$ we have for all $0 \leq s < t \leq T$*

$$\mathbb{E} \left[\left| \int_U G(u_\varepsilon(t, x)) \varphi(x) dx - \int_U G(u_\varepsilon(s, x)) \varphi(x) dx \right|^{2p} \right] \leq C |t - s|^p, \quad (6.36)$$

where $C = C(p, \|\tilde{A}\|_{C^0([0, T] \times U \times U)}, \|A\|_{C^0([0, T]; C^1(U))}, \|b, c\|_{C^0([0, T] \times U)}, \|\varphi\|_{C^1(U)})$.

Proof. Noting that $G'(u_\varepsilon) = \sqrt{2F(u_\varepsilon)}$, Itô's formula and (4.6) imply that

$$\begin{aligned} & \int_U G(u_\varepsilon(t, x)) \varphi(x) dx - \int_U G(u_\varepsilon(s, x)) \varphi(x) dx \\ & = \int_U \left[\int_s^t \varphi(x) \sqrt{2F(u_\varepsilon(r, x))} \left(\Delta u_\varepsilon(r, x) - \frac{1}{\varepsilon^2} F'(u_\varepsilon(r, x)) \right) dr \right. \\ & \quad + \int_s^t \nabla u_\varepsilon(r, x) \cdot X(dr, x) \\ & \quad + \int_s^t \frac{1}{2} A(r, x) : D^2 u_\varepsilon(r, x) dr + \frac{1}{2} c(r, x) \cdot \nabla u_\varepsilon(r, x) dr \Big] dx \\ & \quad + \frac{1}{2} \int_s^t \int_U G''(u_\varepsilon(r, x)) \varphi(x) \nabla u_\varepsilon(r, x) \cdot A(r, x) \nabla u_\varepsilon(r, x) dx dr. \end{aligned} \quad (6.37)$$

Thus for $p \in \mathbb{N}$ one can write

$$\mathbb{E} \left[\left| \int_U G(u_\varepsilon(t)) \varphi dx - \int G(u_\varepsilon(s)) \varphi dx \right|^{2p} \right] \leq 4^{p-1} (I_1 + I_2 + I_3 + I_4). \quad (6.38)$$

Let us bound the individual terms:

$$\begin{aligned} I_1 &= \mathbb{E} \left[\left| \int_s^t \int_U \varphi(x) \sqrt{2F(u_\varepsilon(r, x))} \left[\Delta u_\varepsilon(r, x) - \frac{1}{\varepsilon^2} F'(u_\varepsilon(r, x)) \right] dx dr \right|^{2p} \right] \leq \\ &\leq \|\varphi\|_{C^0(U)}^{2p} \mathbb{E} \left[\left(\int_s^t \int_U \frac{1}{\varepsilon} F(u_\varepsilon(r, x)) dx dr \right)^p \left(\int_s^t \int_U \frac{1}{\varepsilon} w_\varepsilon(r, x)^2 dx dr \right)^p \right] \\ &\leq \|\varphi\|_{C^0(U)}^{2p} \mathbb{E} \left[\left(\int_s^t E_\varepsilon(u_\varepsilon(r, \cdot)) dr \right)^{2p} \right]^{1/2} \mathbb{E} \left[\left(\int_s^t \int_U \frac{1}{\varepsilon} w_\varepsilon(r, x)^2 dx dr \right)^{2p} \right]^{1/2} \\ &\leq \|\varphi\|_{C^0(U)}^{2p} (t-s)^p \mathbb{E} \left[\left(\sup_{s \leq r \leq t} E_\varepsilon(u_\varepsilon(r, \cdot)) \right)^{2p} \right]^{1/2} \mathbb{E} \left[\left(\int_s^t \int_U \frac{1}{\varepsilon} w_\varepsilon(r, x)^2 dx dr \right)^{2p} \right]^{1/2} \\ &\leq C(t-s)^p. \end{aligned} \quad (6.39)$$

Here in the second and third line we have used Cauchy-Schwarz inequality. In the last line we have used (6.18) and (6.19). The second term can be bounded using Youngs inequality:

$$\begin{aligned} I_2 &= \mathbb{E} \left[\left| \int_s^t \int_U \varphi \sqrt{2F(u_\varepsilon(r, x))} \left[\nabla u_\varepsilon(r, x) \cdot \left(\frac{1}{2} c(r, x) + b(r, x) \right) \right] dx dr \right|^{2p} \right] \\ &\leq \|\varphi\|_{C^0(U)}^{2p} \left\| \frac{1}{2} c + b \right\|_{C^0([0, T] \times U)}^{2p} \mathbb{E} \left[\left| \int_s^t E_\varepsilon(u_\varepsilon(r, \cdot)) dr \right|^{2p} \right] \\ &\leq C(t-s)^{2p}. \end{aligned} \quad (6.40)$$

Here we have used the inequalities $|\nabla u_\varepsilon \sqrt{2F(u_\varepsilon)}| \leq \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} F(u_\varepsilon)$ and (6.18) in the last line. For the martingale term we get using Burkholder-Davis-Gundy inequality in the second line and then Youngs inequality in the third line:

$$\begin{aligned} I_3 &= \mathbb{E} \left[\left| \int_s^t \int_U \varphi(x) \sqrt{2F(u_\varepsilon(r, x))} \left[\nabla u_\varepsilon(r, x) \cdot (X(dr, x) - b(r, x)dr) \right] dx \right|^{2p} \right] \\ &\leq \mathbb{E} \left[\left| \int_s^t \int_U \varphi(x) \sqrt{2F(u_\varepsilon(r, x))} \nabla u_\varepsilon(r, x) \cdot \right. \right. \\ &\quad \left. \left. \cdot \tilde{A}(r, x, y) \nabla u_\varepsilon(r, y) \sqrt{2F(u_\varepsilon(r, y))} \varphi(y) dx dy dr \right|^p \right] \\ &\leq \|\varphi\|_{C^0(U)}^{2p} \|\tilde{A}\|_{C^0([0, T] \times U)}^p \mathbb{E} \left[\left(\int_s^t \left(\int_U \sqrt{2F(u_\varepsilon(r, x))} |\nabla u_\varepsilon(r, x)| dx \right)^2 dr \right)^p \right] \\ &\leq \|\varphi\|_{C^0(U)}^{2p} \|\tilde{A}\|_{C^0([0, T] \times U)}^p \mathbb{E} \left[\left| \int_s^t E_\varepsilon(u_\varepsilon(r, \cdot))^2 dr \right|^p \right] \\ &\leq C(t-s)^p. \end{aligned} \quad (6.41)$$

For the fourth term we write:

$$I_4 = \frac{1}{2} \mathbb{E} \left[\left| \int_s^t \int_U \varphi(x) \sqrt{2F(u_\varepsilon(r, x))} \left(A(r, x) : D^2 u_\varepsilon(r, x) \right) + \varphi(x) G''(u_\varepsilon(r, x)) \nabla u_\varepsilon(r, x) \cdot A(r, x) \nabla u_\varepsilon(r, x) dx dr \right|^{2p} \right]. \quad (6.42)$$

After a partial integration the second summand can be written as:

$$\begin{aligned} & \int_U \varphi(x) G''(u_\varepsilon(r, x)) \nabla u_\varepsilon(x) \cdot A(r, x) \nabla u_\varepsilon(x) dx = \\ &= \int_U \varphi(x) \nabla \left(\sqrt{2F(u_\varepsilon(r, x))} \right) \cdot A(r, x) \nabla u_\varepsilon(x) dx \\ &= - \int_U \nabla \varphi(x) \cdot A(r, x) \nabla u_\varepsilon(x) \sqrt{2F(u_\varepsilon(r, x))} dx \\ &\quad - \int_U \varphi(x) \sqrt{2F(u_\varepsilon(r, x))} (\nabla \cdot A(r, x)) \nabla u_\varepsilon(r, x) dx \\ &\quad - \int_U \varphi(x) \sqrt{2F(u_\varepsilon(r, x))} A(r, x) : D^2 u_\varepsilon(r, x) dx. \end{aligned} \quad (6.43)$$

Noting that the terms involving $D^2 u$ in (6.42) and (6.43) cancel it remains to bound:

$$\begin{aligned} & \mathbb{E} \left[\left| \int_s^t \int_U \nabla \varphi(x) \cdot A(r, x) \nabla u_\varepsilon(x) \sqrt{2F(u_\varepsilon(r, x))} dx + \int_s^t \int_U \varphi(x) \sqrt{2F(u_\varepsilon(r, x))} (\nabla \cdot A(r, x)) \nabla u_\varepsilon(r, x) dx ds \right|^{2p} \right] \\ & \leq \left(2 \|A\|_{C^0([0, T], C^1(U))} \|\varphi\|_{C^1(U)} \right)^{2p} \mathbb{E} \left[\left| \int_s^t E_\varepsilon(u_\varepsilon(r)) dr \right|^{2p} \right] \\ & \leq C(t-s)^{2p}. \end{aligned} \quad (6.44)$$

This finishes the proof. \square

Now we are ready to prove our main theorem.

Proof of Theorem 4.2. As a first step we will show that the distributions of $G(u_\varepsilon)$ are tight on $C([0, T], L^1(U))$. To this end it suffices to show the following two conditions (see e.g. [12, Theorem 3.6.4, page 54] together with [5, Theorem 8.3, page 56]. Note that conditions (8.3) and (8.4) in [5] are implied by our stronger assumption (ii)):

- (i) (*Compact Containment*) For every $\delta > 0$ there exists a compact set $K^\delta \subseteq L^1(U)$ such that for all $\varepsilon > 0$

$$\mathbb{P} \left[G(u_\varepsilon(t)) \in K^\delta \text{ for all } 0 \leq t \leq T \right] \geq 1 - \delta \quad (6.45)$$

- (ii) (*Weak tightness*) For every smooth testfunction $\varphi \in C^\infty(U)$ there exist positive α, β, C such that for all $0 \leq s < t \leq T$

$$\mathbb{E} \left[\left| \int_U G(u_\varepsilon(t, x)) \varphi(x) dx - \int_U G(u_\varepsilon(s, x)) \varphi(x) dx \right|^\alpha \right] \leq C |t - s|^{1+\beta}. \quad (6.46)$$

To prove the first statement note that

$$\begin{aligned} \int_U |\nabla G(u_\varepsilon(t, x))| dx &= \int_U |\sqrt{2F(u_\varepsilon(t, x))}| |\nabla u_\varepsilon(t, x)| dx \\ &\leq \int_U \frac{\varepsilon}{2} |\nabla u_\varepsilon(t, x)|^2 + \frac{1}{\varepsilon} F(u_\varepsilon(t, x)) dx. \end{aligned} \quad (6.47)$$

Furthermore, (3.1) implies $G(r) \leq C(1 + F(r))$ such that

$$\int_U |G(u_\varepsilon(t, x))| dx \leq C \int_U F(u_\varepsilon(t, x)) dx + C|U|. \quad (6.48)$$

Thus applying Chebyshev's inequality equation (6.18) yields that

$$\mathbb{P} \left[\sup_{0 \leq t \leq T} \|G(u_\varepsilon(t))\|_{W^{1,1}(U)} \geq \lambda \right] \rightarrow 0, \quad (6.49)$$

for $\lambda \rightarrow \infty$ which together with Rellich's Theorem implies condition (i).

The second condition (ii) follows from (6.36). The tightness of $G(u_\varepsilon)$ is thus proved.

As a second step we prove that the distributions of u_ε are tight in $C([0, T], L^1(U))$. Denote by G^{-1} the inverse function of G . Then similar to [35, page 139] we observe that the operator $v \mapsto G^{-1} \circ v$ is continuous from $L^1(U)$ to itself. In fact assume $v_i \rightarrow v$ as $i \rightarrow \infty$ in $L^1(U)$. The growth condition in (3.1) implies that G^{-1} is uniformly continuous on \mathbb{R} . This implies convergence in measure and convergence pointwise in U for the sequence $G^{-1} \circ v_i$. Furthermore, using the growth condition once more one can see that $|G^{-1} \circ v_i| \leq C(|v_i| + 1)$ which then implies by Vitali's Convergence Theorem that $G^{-1}(v_i) \rightarrow G^{-1}(v)$ in $L^1(U)$. Thus using the following Lemma 6.5 we can conclude that the distributions of u_ε are tight on $C([0, T], L^1(U))$ as well.

In particular, there exists a decreasing sequence $\varepsilon_k \downarrow 0$ such that the distributions of u_ε converge weakly to a limiting measure on $C([0, T], L^1(U))$. We may now use Skorohod's observation that we can find a subsequence $\varepsilon_k \downarrow 0$ such that the random functions u_{ε_k} can be realized on a single probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$. On this space the u_{ε_k} converge almost surely in $C^0([0, T]; L^1(U))$ towards u (see [24, page 9]). By (6.18) and Fatou's Lemma we can for almost all $\omega \in \Omega, t \in (0, T)$ select a subsequence $\varepsilon'_k \rightarrow 0$ such that $\sup_{\varepsilon'_k} E_{\varepsilon'_k}(u_{\varepsilon'_k}(t)) < \infty$. Thus using the Gamma convergence of the functionals E_ε we can conclude that $u_{\varepsilon'_k}(t)$ converges for a subsequence strongly in $L^1(U)$ to a limit $v \in BV(U; \{-1, 1\})$ and that

$$\|v\| \leq \liminf_{\varepsilon'_k \rightarrow 0} E_{\varepsilon'_k}(u_{\varepsilon'_k}(t)) \quad (6.50)$$

But since $u_{\varepsilon_k} \rightarrow u$ almost surely in $C([0, T], L^1(U))$ we have $v = u(t, \cdot)$. This proves that $u(t, \cdot) \in BV(U; \{-1, 1\})$; (6.18), (6.50) yield

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} \|u(t)\|_{BV(U)}^p \right] < \infty.$$

which concludes the proof. \square

Lemma 6.5. *Let $((X_t^\varepsilon, t \in [0, T]), \varepsilon > 0)$ be a family of stochastic processes taking values in a separable metric space E . Let \tilde{E} be another separable metric space and $F: E \rightarrow \tilde{E}$ a continuous function. Suppose the family of distributions of X_t^ε is tight on $C([0, T], E)$. Then the family of distributions of $F(X_t^\varepsilon)$ is tight on $C([0, T], \tilde{E})$.*

Proof. According to [17, Thm. 7.2 and Rem. 7.3 on page 128] the distributions of X_t^ε are tight if and only if

- (1) For all $\eta > 0$ there exists a compact set $\Gamma_\eta \subseteq E$ such that for all ε

$$\mathbb{P}\left[X_t^\varepsilon \in \Gamma_\eta, \quad \forall 0 \leq t \leq T\right] \geq 1 - \eta \quad (6.51)$$

- (2) For all $\eta > 0$ there exists $\delta > 0$ such that

$$\sup_t \mathbb{P}\left[\omega(X_t^\varepsilon, \delta) \geq \eta\right] \leq \eta, \quad (6.52)$$

where the modulus of continuity is defined as $\omega(X_t, \delta) = \sup_{t-\delta \leq s \leq t+\delta} d(X_t, X_s)$.

It is clear from the continuity of F that the $F(X_t^\varepsilon)$ satisfy the property corresponding to (6.51) if X_t does.

To see that $F(X_t)$ also satisfies (6.52) fix $\eta > 0$ and the set $\Gamma_{\eta/2}$ such that

$$\mathbb{P}\left[X_t^\varepsilon \in \Gamma_{\eta/2}, \quad \forall 0 \leq t \leq T\right] \geq 1 - \eta/2.$$

As $\Gamma_{\eta/2}$ is compact F is uniformly continuous when restricted to $\Gamma_{\eta/2}$. Thus one can choose η' such that $d(x, y) \leq \eta'$ implies $d(F(x), F(y)) \leq \eta$ for all $x, y \in \Gamma_{\eta/2}$. Thus choosing δ small enough such that $\sup_t \mathbb{P}\left[\omega(X_t, \delta) \geq \eta'\right] \leq \eta/2$ one obtains

$$\begin{aligned} \sup_t \mathbb{P}\left[\omega(F(X_t), \delta) \geq \eta\right] &\leq \sup_t \mathbb{P}\left[X_t \notin \Gamma_{\eta/2}\right] + \sup_t \mathbb{P}\left[\omega(X_t, \delta) \geq \eta'\right] \\ &\leq \frac{\eta}{2} + \frac{\eta}{2}. \end{aligned} \quad (6.53)$$

This finishes the proof. \square

7. FORMAL SHARP INTERFACE LIMIT

In this section we discuss the sharp interface limit $\varepsilon \rightarrow 0$ of the stochastic Allen-Cahn equation (1.4). We will only provide formal arguments and do not intend to give a rigorous justification of a stochastically perturbed mean curvature flow and of the convergence of (1.4) to such a stochastic evolution. Such an analysis is quite delicate and beyond the aim of this paper.

Our approach is based on a formal sharp interface limit in a localized version of Proposition (6.1) (see (7.4) below). For the (deterministic) Allen-Cahn equation such an approach has been successfully used to prove convergence to mean curvature flow (see [25, 38]).

We start with the formulation of a stochastically driven mean curvature flow that formally corresponds to our stochastic Allen-Cahn equation. In the case of a Brownian vector field of the form (3.9) this could be formulated as a stochastic evolution of hypersurfaces $(\Gamma_t)_{t \in (0, T)}$ that is (locally) given by a parametrization

$\phi : M \times (0, T) \rightarrow \mathbb{R}^n$, $M \subset \mathbb{R}^n$ a smooth $(n - 1)$ -dimensional reference manifold, such that

$$d\phi(t, x) = \vec{H}(\phi(t, x)) dt - X^k(t, \phi(t, x)) \circ dB^k(t) - X^0(t, \phi(t, x)) \circ dt, \quad (7.1)$$

or in Itô formulation

$$\begin{aligned} d\phi(t, x) &= \vec{H}(\phi(t, x)) dt - X^k(t, \phi(t, x)) dB^k(t) - X^0(t, \phi(t, x)) dt \\ &\quad + \frac{1}{2} DX^k(t, \phi(t, x)) X^k(t, \phi(t, x)) dt. \end{aligned} \quad (7.2)$$

As explained above we make a connection between (1.4) and (7.1) by formally passing to the limit in an equality for the the time-derivative of a *localized* energy and to derive a Brakke-type (in)equality in the limit.

As above in (3.3) we consider for any $\eta \in C^2(\overline{U})$ and $u \in W^{1,2}(U)$

$$\mu_\varepsilon^t(\eta) := \int_U \left(\frac{\varepsilon}{2} |\nabla u_\varepsilon(t)|^2(x) + \frac{1}{\varepsilon} F(u_\varepsilon(t)(x)) \right) \eta(x) dx. \quad (7.3)$$

We then obtain by similar calculations as in the proof of Proposition 6.1 for a solution u_ε of (1.4)

$$\begin{aligned} \mu_\varepsilon^{t_1}(\eta) - \mu_\varepsilon^{t_0}(\eta) &= - \int_{t_0}^{t_1} \int_U \eta \frac{1}{\varepsilon} w_\varepsilon^2 dx dt + \int_U \int_{t_0}^{t_1} \eta w_\varepsilon \nabla u_\varepsilon \cdot X(dt) dx \\ &\quad - \frac{1}{2} \int_{t_0}^{t_1} \int_U \eta c \cdot w_\varepsilon \nabla u_\varepsilon dx dt \\ &\quad + \frac{1}{2} \int_{t_0}^{t_1} \mu_\varepsilon^t \left(\eta \partial_j [\partial_{x_i} \tilde{a}_{ij} - \partial_{y_i} \tilde{a}_{ij}] \right) dt \\ &\quad + \frac{1}{2} \int_{t_0}^{t_1} \int_U \eta \varepsilon \partial_i u_\varepsilon \partial_{x_k} \partial_{y_k} \tilde{a}_{ij} \partial_j u_\varepsilon dx dt \\ &\quad + \int_{t_0}^{t_1} \int_U \eta \varepsilon \partial_i u_\varepsilon \partial_k [\partial_{x_j} \tilde{a}_{ij}] \partial_k u_\varepsilon dx dt \\ &\quad - \int_{t_0}^{t_1} \int_U \eta \varepsilon \partial_i u_\varepsilon \partial_j [\partial_{x_k} \tilde{a}_{ij}] \partial_k u_\varepsilon dx dt \\ &\quad + \int_{t_0}^{t_1} \int_U \nabla \eta \cdot w_\varepsilon \nabla u_\varepsilon dx dt - \int_U \int_{t_0}^{t_1} \varepsilon (\nabla \eta \cdot \nabla u_\varepsilon) \nabla u_\varepsilon \cdot X(dt) dx \\ &\quad + \int_{t_0}^{t_1} \mu_\varepsilon^t \left(\partial_j \eta \partial_{x_i} \tilde{a}_{ij} + \frac{1}{2} A : D^2 \eta \right) dt \\ &\quad + \int_{t_0}^{t_1} \int_U \frac{\varepsilon}{2} \nabla \eta \cdot \nabla u_\varepsilon (c \cdot \nabla u_\varepsilon) - \varepsilon \partial_j \eta \partial_i u_\varepsilon \partial_{x_k} \tilde{a}_{ij} \partial_k u_\varepsilon dx dt. \end{aligned} \quad (7.4)$$

For the special choice (3.9) of Brownian vector fields this equality yields

$$\mu_\varepsilon^{t_1}(\eta) - \mu_\varepsilon^{t_0}(\eta)$$

$$\begin{aligned}
&= - \int_{t_0}^{t_1} \int_U \eta \frac{1}{\varepsilon} w_\varepsilon^2 dx dt + \int_U \int_{t_0}^{t_1} \eta w_\varepsilon \nabla u_\varepsilon \cdot (X^k dB_k(t) + X_0 dt) dx \\
&\quad - \frac{1}{2} \int_{t_0}^{t_1} \int_U \eta DX^k X^k \cdot \nabla u_\varepsilon w_\varepsilon dx dt + \\
&\quad + \frac{1}{2} \int_{t_0}^{t_1} \mu_\varepsilon^t \left(\eta (\nabla \cdot X^k)^2 - \eta \operatorname{tr}(DX^k DX^k) \right) dt \\
&\quad + \int_{t_0}^{t_1} \int_U \frac{\varepsilon}{2} \eta |DX^T \nabla u_\varepsilon|^2 + \eta \varepsilon \nabla u_\varepsilon \cdot DX^k DX^k \nabla u_\varepsilon dx dt \\
&\quad - \int_{t_0}^{t_1} \int_U \eta \varepsilon (\nabla \cdot X^k) \nabla u_\varepsilon \cdot DX^k \nabla u_\varepsilon dx dt \\
&\quad + \int_{t_0}^{t_1} \int_U \nabla \eta \cdot w_\varepsilon \nabla u_\varepsilon dx dt \\
&\quad - \int_U \int_{t_0}^{t_1} \varepsilon (\nabla \eta \cdot \nabla u_\varepsilon) \nabla u_\varepsilon \cdot (X^k dB_k(t) + X^0 dt) dx \\
&\quad + \int_{t_0}^{t_1} \mu_\varepsilon^t \left(\nabla \eta \cdot X^k \nabla \cdot X^k + \frac{1}{2} X^k \cdot D^2 \eta X^k \right) dt \\
&\quad + \int_{t_0}^{t_1} \int_U \frac{\varepsilon}{2} \nabla \eta \cdot \nabla u_\varepsilon (DX^k X^k \cdot \nabla u_\varepsilon) - \varepsilon \nabla \eta \cdot X^k \nabla u_\varepsilon \cdot DX^k \nabla u_\varepsilon dx dt. \tag{7.5}
\end{aligned}$$

In an analogous way we consider the derivative of the localized surface area energy

$$\mathcal{A}^\eta(\Gamma) = \int_\Gamma \eta(x) d\mathcal{H}^{n-1}(x), \tag{7.6}$$

for $\eta \in C^2(\overline{U})$ and a smooth hypersurface $\Gamma \subset \mathbb{R}^n$. For the first and second variation of the localized energy (7.6) in the direction of a vector field $Y \in C_c^2(\mathbb{R}^n, \mathbb{R}^n)$ we obtain by using [39]

$$\delta \mathcal{A}^\eta(\Gamma)(Y) = \int_\Gamma \left(\nabla \eta^\perp \cdot Y - \eta \vec{H} \cdot Y \right) d\mathcal{H}^{n-1} \tag{7.7}$$

$$\begin{aligned}
\delta^2 \mathcal{A}^\eta(\Gamma)(Y, Y) &= \int_\Gamma \eta \left((\operatorname{div}_{\tan} Y)^2 + \sum_{i=1}^{n-1} |D_{\tau_i}^\perp Y|^2 - \sum_{i,j=1}^{n-1} (\tau_i \cdot D_{\tau_j} Y)(\tau_j \cdot D_{\tau_i} Y) \right) d\mathcal{H}^{n-1} \\
&\quad + \int_\Gamma \left(2(\nabla \eta \cdot Y)(\operatorname{div}_{\tan} Y) + Y \cdot D^2 \eta Y \right) d\mathcal{H}^{n-1}, \tag{7.8}
\end{aligned}$$

where for $x \in \Gamma$ we have chosen an arbitrary orthonormal basis $\tau_i(x), \dots, \tau_{n-1}(x)$ of $T_x \Gamma$ and where \perp denotes the normal part of the corresponding vectors.

A formal application of Itô formula yields therefore for the formal stochastic evolution (7.1)

$$d\mathcal{A}^\eta(\Gamma_t) = \int_{\Gamma_t} -\eta \vec{H} \cdot \left(\vec{H} dt - X^k dB^k(t) - X^0 dt + \frac{1}{2} DX^k X^k dt \right) d\mathcal{H}^{n-1}$$

$$\begin{aligned}
& + \int_{\Gamma_t} \nabla \eta^\perp \cdot \left(\vec{H} dt - X^k dB^k(t) - X^0 dt + \frac{1}{2} DX^k X^k dt \right) d\mathcal{H}^{n-1} \\
& + \frac{1}{2} \int_{\Gamma_t} \eta \left((\nabla_{\tan} \cdot X^k)^2 + |D_{\tau_i}^\perp X^k|^2 - \sum_{i,j} (\tau_i D_{\tau_j} X^k)(\tau_j D_{\tau_i} X^k) \right) d\mathcal{H}^{n-1} dt \\
& + \int_{\Gamma_t} (\nabla_{\tan} \cdot X^k)(\nabla \eta \cdot X^k) d\mathcal{H}^{n-1} dt \\
& + \frac{1}{2} \int_{\Gamma_t} X^k \cdot D^2 \eta X^k d\mathcal{H}^{n-1} dt. \tag{7.9}
\end{aligned}$$

This has to be compared with the sharp interface limit of (7.5). We assume that the limit of a sequence $(u_\varepsilon)_{\varepsilon>0}$ is given by a smooth evolution $(\Gamma_t)_{t \in (0,T)}$ as above, that for all $t \in (0, T)$ the following convergence properties hold

$$\begin{aligned}
\mu_\varepsilon^t, \varepsilon |\nabla u_\varepsilon(t, \cdot)|^2 \mathcal{L}^n & \rightarrow c_0 \mathcal{H}^{n-1} \llcorner \Gamma_t, \\
\varepsilon \nu_\varepsilon(t, \cdot) \otimes \nu_\varepsilon(t, \cdot) \mathcal{L}^n & \rightarrow c_0 \nu(t, \cdot) \otimes \nu(t, \cdot) \mathcal{H}^{n-1} \llcorner \Gamma_t, \\
\frac{w_\varepsilon \nu_\varepsilon}{\varepsilon |\nabla u_\varepsilon|}(t, \cdot) \mathcal{L}^n & \rightarrow c_0 \vec{H}(t, \cdot) \mathcal{H}^{n-1} \llcorner \Gamma_t, \\
\frac{1}{\varepsilon} w_\varepsilon^2(t, \cdot) \mathcal{L}^n & \rightarrow c_0 H^2(t, \cdot) \mathcal{H}^{n-1} \llcorner \Gamma_t,
\end{aligned}$$

where to $x \in \Gamma_t$ we have associated a unit normal vector $\nu(t, x)$. We refer for example to [38] for a precise formulation and rigorous justification of such convergence properties under specific assumptions.

We then obtain from (7.5)

$$\begin{aligned}
& \frac{1}{c_0} \lim_{\varepsilon \rightarrow 0} d\mu_\varepsilon^t(\eta) \\
& = - \int_{\Gamma_t} \eta |\vec{H}|^2 d\mathcal{H}^{n-1} dt + \int_{\Gamma(t)} \eta \vec{H} \cdot (X^k dB_k(t) + X^0 dt) d\mathcal{H}^{n-1} \\
& \quad - \frac{1}{2} \int_{\Gamma(t)} \eta DX^k X^k \cdot \vec{H} d\mathcal{H}^{n-1} dt \\
& \quad + \frac{1}{2} \int_{\Gamma(t)} \eta (\nabla \cdot X^k)^2 - \eta \operatorname{tr}(DX^k DX^k) d\mathcal{H}^{n-1} dt \\
& \quad + \int_{\Gamma(t)} \eta |DX^T \nu|^2 + \frac{1}{2} \eta \nu \cdot DX^k DX^k \nu d\mathcal{H}^{n-1} dt \\
& \quad - \int_{\Gamma(t)} \eta (\nabla \cdot X^k) \nu \cdot DX^k \nu d\mathcal{H}^{n-1} dt \\
& \quad + \int_{\Gamma(t)} \nabla \eta \cdot \vec{H} d\mathcal{H}^{n-1} dt \\
& \quad - \int_{\Gamma(t)} (\nabla \eta \cdot \nu) \nu \cdot X^k d\mathcal{H}^{n-1} dB_k(t) - \int_{\Gamma(t)} (\nabla \eta \cdot \nu) \nu \cdot X^0 d\mathcal{H}^{n-1} dt
\end{aligned}$$

$$\begin{aligned}
& + \int_{\Gamma(t)} (\nabla \eta \cdot X^k \nabla \cdot X^k + \frac{1}{2} X^k \cdot D^2 \eta X^k) d\mathcal{H}^{n-1} dt \\
& + \int_{\Gamma(t)} \left(\frac{1}{2} \nabla \eta \cdot \nu (DX^k X^k \cdot \nu) - \nabla \eta \cdot X^k \nu \cdot DX^k \nu \right) d\mathcal{H}^{n-1} dt. \tag{7.10}
\end{aligned}$$

Comparing this equation with (7.9) and using the calculations from [31] we obtain

$$\frac{1}{c_0} \lim_{\varepsilon \rightarrow 0} d\mu_\varepsilon^t(\eta) = d\mathcal{A}^\eta(\Gamma_t) + \int_{\Gamma(t)} \eta (\nu \cdot X^k \nu)^2 d\mathcal{H}^{n-1} dt. \tag{7.11}$$

We therefore obtain in the formal sharp interface limit of the localized energy equality one additional positive term compared to the corresponding energy derivative for (7.1). However, this is consistent with recent work by Le [31] that compares the formal sharp interface limit of the second inner variation of E_ε to the second variation of the area functional and identifies exactly the same extra contribution.

In this respect the discrepancy should not be interpreted as an indication that (1.4) fails to approximate the stochastic mean curvature flow (7.1), but rather as consistent and a consequence of using an energy based proof. In contrast to the deterministic case, by Itô's formula the second inner variation of E_ε enters in the energy derivative and necessarily produces this extra term.

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