ERGODIC RAMSEY THEORY – NOTES

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These notes are being written for the TCC module on Ergodic Ramsey Theory, which runs in the Fall of 2021. Ergodic Ramsey Theory is a relatively young subject of mathematics whose purpose is to apply techniques, methods and ideas from ergodic theory, and more the general theory of dynamical systems, to problems that arise in Ramsey theory, combinatorics, and number theory. The main interface between the dynamical and the combinatorial realms is provided by the Correspondence Principle of Furstenberg, first introduced in [9] to give a new ergodic theoretic proof of Szemerédi's theorem on arithmetic progressions.

The module will start by introducing some of the problems from Ramsey theory that we will consider, as well as the preliminary results from Ergodic theory, and then, after introducing the Furstenberg Correspondence Principle we will go over the ergodic theoretic proof of Szemerédi's theorem. The last half (or third) of the course focuses on more recent developments in ergodic Ramsey theory (still to be decided).

We will not follow any single textbook from beginning to end, but both Furstenberg's book [10] and Einsiedler-Ward's book [7] share the same spirit of introducing ergodic theory both as a theory on its own and as a tool to approach problems in combinatorics and number theory. Bergelson's survey [1] with the same title as these notes obviously shares a great deal of content. A more advanced text on this subject is the recent book of Host and Kra [17], which goes into much more depth. For an introductory text to general ergodic theory, Walters [27] is an excellent source, which can be complemented with Glasner's [12] or Cornfeld-Fomin-Sinai's [4]. For an introductory text to general Ramsey Theory, the book [14] by Graham, Rothschild and Spencer of that title is still one of the best sources.

1. Ramsey Theory

Ramsey theory is a branch of combinatorics which, roughly speaking, explores structures that persist when partitioned. Instead of trying to give a more precise description, we illustrate this principle with a few examples of results from Ramsey theory.

Theorem 1.1 (Schur, [24])**.** *Given a finite coloring of $\mathbb{N}$, one can always find $x, y \in \mathbb{N}$ with $x, y, x+y$ all having the same colour.*

To be clear, a finite coloring of $\mathbb{N}$ is a function $f : \mathbb{N} \to F$, where F is a finite set (whose elements are the "colours"). Two elements x, y of $\mathbb{N}$ have the same color if $f(x) = f(y)$.

In fact, it is not necessary to color all of $\mathbb{N}$ before one finds a ***monochromatic*** (i.e. with a single color) triple of the form $\{x, y, x+y\}$. Here's an alternative formulation of Schur's theorem.

Theorem 1.2 (Schur, again)**.** *For every $r \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that whenever the set $\{1, \dots, N\}$ is colored with r colors there is a monochromatic triple of the form $\{x, y, x+y\} \subset \{1, \dots, N\}$.*

The difference between Theorems 1.1 and 1.2 is that in the latter, N is chosen depending only on the number of colors r. To estimate the smallest N in terms of r is a difficult and interesting problem, but the purely qualitative Theorem 1.2 as formulated turns out to be equivalent to the apparently weaker Theorem 1.1 (and not in the uninteresting sense that any two true statements are tautological equivalent).

Exercise 1.3. Prove that Theorems 1.1 and 1.2 are equivalent. *[Hint: One implication is easy. For the other, suppose you have counterexamples to Theorem 1.2 for every N, then you can use them to find a counterexample to Theorem 1.1.]*

The most common way to solve the previous exercise is to use, explicitly or implicitly, the so-called ***compactness principle***, which in this case is simply the statement that the set of all colorings of $\mathbb{N}$

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into r colors is a compact set. The compactness principle allows one to formulate many Ramsey theoretic statements in an ***infinitary*** form, such as Theorem 1.1. This is the form that ergodic theory can handle, but it is useful to keep in mind that the statements are equivalent to their ***finitistic*** forms.

The next theorem was considered by Khinchine as one of "Three Pearls in Number Theory" [19].

Theorem 1.4 (Van der Waerden, [26])**.** *In any finite coloring of $\mathbb{N}$ there exist arbitrarily long monochromatic arithmetic progressions.*

In other words, for any $k \in \mathbb{N}$ there are $x, y \in \mathbb{N}$ such that the arithmetic progression $\{x, x+y, x+2y, \ldots, x+ky\}$ is monochromatic.

There is a natural finitistic form of van der Waerden's theorem.

Exercise 1.5. *Show that Theorem 1.4 is equivalent to the following statement:*

"For any $r, k \in \mathbb{N}$ there exists N such that for any coloring of the set $\{1, \ldots, N\}$ with r colors there exists a monochromatic arithmetic progression of the form $\{x, x+y, x+2y, \ldots, x+ky\} \subset \{1, \ldots, N\}$."

There is yet another equivalent formulation of van der Warden's theorem with a more geometric flavour.

Exercise 1.6. *Show that Theorem 1.4 is equivalent to the following statement:*

"For any finite coloring of $\mathbb{N}$ and for any finite set $F \subset \mathbb{N}$ there exists a monochromatic affine image of F, i.e. there exist $a, b \in \mathbb{N}$ such that the set $aF + b := \{ax + b : x \in F\}$ is monochromatic."

The next result was conjectured by Erdős and Turán as an attempt to better understand the true nature of van der Waerden's theorem. After some initial progress it was finally settled by Szemerédi in a remarkably involved combinatorial proof. In order to state it we need the notion of (upper) density.

Definition 1.7 (Upper density)**.** *Given a set $A \subset \mathbb{N}$ its* **upper density***, denoted $\bar{d}(A)$ is the quantity*

$$\bar{d}(A) = \limsup_{N\to\infty} \frac{1}{N}\big|A \cap \{1, \ldots, N\}\big|.$$

Replacing $\limsup$ *with* $\liminf$ *we obtain the analogous notion of* **lower density***.*

Here and elsewhere in these notes, when X is a finite set we denote by $|X|$ its cardinality.

Exercise 1.8. *Show that upper density is subadditive and shift invariant, i.e. if $A, B \subset \mathbb{N}$ and $n \in \mathbb{N}$ then $\bar{d}(A \cup B) \leq \bar{d}(A) + \bar{d}(B)$, and $\bar{d}(A - n) = \bar{d}(A)$, where $A - n := \{x \in \mathbb{N} : x + n \in A\}$.*

Theorem 1.9 (Szemerédi, [25])**.** *If $A \subset \mathbb{N}$ has positive upper density, then it contains arbitrarily long arithmetic progressions.*

Note that Szemerédi's theorem implies van der Waerden's theorem, since for any finite coloring of $\mathbb{N}$ one can use Exercise 1.8 to deduce that at least one of the colors has positive density.

Here is the finitistic form of Szemerédi's theorem.

Exercise 1.10. *Show that Theorem 1.9 is equivalent to the following statement:*

"For any $\delta > 0$ and $k \in \mathbb{N}$ there exists N such that any set $A \subset \{1, \ldots, N\}$ with $|A| > \delta N$ contains an arithmetic progression of the form $\{x, x+y, x+2y, \ldots, x+ky\}$."

Exercise 1.11. *Let $k \in \mathbb{N}$. Show that there exists $\delta < 1$ such that any set $A \subset \mathbb{N}$ with $\bar{d}(A) > \delta$ contains an arithmetic progression of the form $\{x, x+y, x+2y, \ldots, x+ky\}$.*

More than twenty years before Szemerédi's theorem was first proved, Roth obtained the special case corresponding to arithmetic progressions of length 3.

Theorem 1.12 (Roth, [22])**.** *Any set $A \subset \mathbb{N}$ with $\bar{d}(A) > 0$ contains a 3-term arithmetic progression.*

Roth's proof of Theorem 1.12 made use of Fourier Analysis, and would later inspire Gowers to obtain a full proof of Szemerédi's theorem [13] by developing what is now called "Higher order Fourier Analysis". Another Ramsey theoretic result that can be obtained using Fourier Analysis is the following.

Theorem 1.13 (Sárközy, [23])**.** *If $A \subset \mathbb{N}$ has $\bar{d}(A) > 0$, then there exist $x, y \in A$ whose difference is a perfect square.*

Theorem 1.13 is connected with the study of *sets of differences* of large sets. In this context, we think of a set $A \subset \mathbb{N}$ with positive upper density as a *large set*, and are interested in understand the structure of the set of differences $A - A := \{x - y : x, y \in A\}$. A related concept is that of intersective sets:

Definition 1.14. *A set $R \subset \mathbb{Z}$ is a called* **intersective** *if for every $A \subset \mathbb{N}$ with $\bar{d}(A) > 0$, the intersection $(A - A) \cap R$ is non-empty.*

Using this terminology, Theorem 1.13 states that the set of perfect squares is an intersective set.

Exercise 1.15. *Show that the following are intersective sets.*

- *Any set with lower density* 1.
- *The set $k\mathbb{N}$ of all multiples of k, for an arbitrary $k \in \mathbb{N}$.*
- $(*)$ *Any set of differences $I - I$ for any infinite set I (not necessarily with positive upper density).*

Exercise 1.16. *Show that the following are not intersective sets.*

- *The odd numbers.*
- *The set $\mathbb{N} \setminus (k\mathbb{N})$ of numbers not divisible by k, for an arbitrary $k \in \mathbb{N}$.*

Sárközy's theorem can be extended to more general polynomials than $p(x) = x^2$. The exact extent of this generalization was only fully understood after work of Furstenberg [9, 10] and of Kamae and Mendes-France [18].

Theorem 1.17. *Let $p \in \mathbb{Z}[x]$ be a polynomial with integer coefficients and no constant term. Then the set $R := \{p(n) : n \in \mathbb{N}\}$ is intersective if and only if it contains a multiple of any $k \in \mathbb{N}$ (in other words, if p has a root modulo k for every k).*

Notice that an easy sufficient condition on a polynomial to have a root modulo k for every k, is to satisfy $p(0) = 0$.

One can interpret Sárközy's theorem as stating that any set $A \subset \mathbb{N}$ with positive upper density contains a 2-term arithmetic progression whose common difference is a perfect square. From this angle it makes sense to ask about longer arithmetic progressions. The following powerful theorem of Bergelson and Leibman gives an affirmative answer.

Theorem 1.18 (Polynomial Szemerédi theorem, [2])**.** *Let $p_1, \dots, p_k \in \mathbb{Z}[x]$ satisfy $p_i(0) = 0$. Then any set $A \subset \mathbb{N}$ with $\bar{d}(A) > 0$ contains a "polynomial progression" of the form*

$$\big\{x, x + p_1(y), x + p_2(y), \dots, x + p_k(y)\big\}.$$

Observe that by taking $p_i(y) = iy$ one recovers Szemerédi's theorem from Theorem 1.18.

2. Ergodic theory background

In this section we collect some of the basic definitions and facts about ergodic theory that we will need later on.

Definition 2.1 (Measure preserving transformation)**.** *Given two probability spaces $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$, we say that a map[1] $T : X \to Y$* **preserves the measure** *or is a* **measure preserving transformation** *if for every $B \in \mathcal{B}$, the set $T^{-1}B := \{x \in X : Tx \in B\}$ is in $\mathcal{A}$ and satisfies $\mu(T^{-1}B) = \nu(B)$.*

A map between probability spaces induces a linear operator between the corresponding L^p spaces.

Exercise 2.2. *Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be probability spaces and let $T : X \to Y$ be a measurable map.*

- *Show that T preserves the measure if and only if for every $f \in L^2(Y)$, the function $f \circ T$ belongs to $L^2(X)$ and satisfies*
$$\int_X f \circ T \, \mathsf{d}\mu = \int_Y f \, \mathsf{d}\nu. \tag{2.1}$$
- *If both μ and ν are Radon measures, show that T preserves the measure if and only if (2.1) holds for every $f \in C(Y)$. [Hint: $C(Y)$ is dense in $L^2(Y)$.]*

[1] To be completely precise, T may be defined only on a full measure subset of X.

The basic object in ergodic theory is a ***measure preserving system*** (m.p.s. for short), which we now define.

Definition 2.3 (Measure preserving system)**.** *A measure preserving system is a quadruple* $(X, \mathcal{B}, \mu, T)$ *where* $(X, \mathcal{B}, \mu)$ *is a probability space and* $T : X \to X$ *is a measure preserving transformation.*

Example 2.4 (Circle rotation)**.** *Let* $X = [0, 1)$*, endowed with the Borel* σ*-algebra* $\mathcal{B}$ *and the Lebesgue measure* μ*. Given* $\alpha \in \mathbb{R}$ *we consider the map* $T = T_\alpha : X \to X$ *given by* $Tx = x + \alpha \bmod 1$*. The fact that* T *preserves the measure* μ *follows from the basic properties of Lebesgue measure.*

Alternatively, we can identify the space X *with the compact group* $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ *in the obvious way. The Lebesgue measure on* $[0, 1)$ *gets identified with the Haar measure on* $\mathbb{T}$*, and* T *becomes the map* $Tx = x + \tilde{\alpha}$ *(where* $\tilde{\alpha} = \alpha + \mathbb{Z} \in \mathbb{T}$*). This map clearly preserves the Haar measure.*

The reason to call this system a circle rotation is that the group $\mathbb{T}$ *is isometrically isomorphic to the circle* $S^1 \subset \mathbb{C}$*, viewed as a group under multiplication. The map* T *under this identification becomes the rotation* $T : z \mapsto \theta z$*, where* $\theta = e^{2\pi i\alpha} \in S^1$*.*

The above example can be extended to "rotations" on any compact group X, endowed with the Borel σ-algebra $\mathcal{B}$ and Haar measure μ. Taking any $\alpha \in X$, the map $T : x \mapsto x + \alpha$ preserves μ and hence $(X, \mathcal{B}, \mu, T)$ is a measure preserving system, called a ***group rotation*** or a ***Kronecker system***.

Example 2.5 (Doubling map)**.** *Again take* $(X, \mathcal{B}, \mu)$ *to be the unit interval* $X = [0, 1]$ *equipped with its Borel* σ*-algebra and Lebesgue measure. Let* $T : X \to X$ *be the doubling map* $Tx = 2x \bmod 1$*.*

At first sight it may seem that the doubling map doubles the measure, but in fact it preserves the measure! For instance, given an interval $[a, b] \subset [0, 1]$*, the pre-image* $T^{-1}[a, b]$ *is the union of two intervals, each half the length of the original interval:*

$$T^{-1}\big([a, b]\big) = \left[\frac{a}{2}, \frac{b}{2}\right] \cup \left[\frac{a+1}{2}, \frac{b+1}{2}\right].$$

Exercise 2.6. *Show that the doubling map does indeed preserve the Lebesgue measure. [Hint: use Exercise 2.2]*

Here is the first theorem of ergodic theory.

Theorem 2.7 (Poicaré recurrence theorem)**.** *Let* $(X, \mathcal{B}, \mu, T)$ *be a measure preserving system and let* $A \in \mathcal{B}$ *with* $\mu(A) > 0$*. Then for some* $n \in \mathbb{N}$ *we have*

$$\mu(A \cap T^{-n} A) > 0. \tag{2.2}$$

Proof. The sets $A, T^{-1}A, T^{-2}A, \ldots$ all have the same (positive) measure, and all live in X which has measure 1. Therefore we must have $\mu(T^{-i}A \cap T^{-j}A) > 0$ for some $i > j$. Finally, letting $n = i - j$, observe that

$$\mu(A \cap T^{-n}A) = \mu\big(T^{-j}(A \cap T^{-n}A)\big) = \mu(T^{-i}A \cap T^{-j}A) > 0.$$

□

While Poicaré's recurrence theorem is a simple result, it has a lot of potential for extensions, which in turn reveal a lot about the structure of measure preserving systems. For instance, one may ask how small can we choose n? How large is the set of n for which (2.2) holds? How large can we make the measure of the intersection be?

In order to address some of these questions, we make the following definition.

Definition 2.8. *A set* R *of natural numbers is called a* **set of recurrence** *if for every measure preserving system* $(X, \mathcal{B}, \mu, T)$ *and every* $A \in \mathcal{B}$ *with* $\mu(A) > 0$ *there exists* $n \in R$ *such that* $\mu(A \cap T^{-n}A) > 0$*.*

With this notion we can reformulate Poicaré's recurrence theorem as stating that $\mathbb{N}$ is a set of recurrence.

Exercise 2.9. *Show that the set* $2\mathbb{N}$ *of even numbers is a set of recurrence but the set* $2\mathbb{N} - 1$ *of odd numbers is not.*

Here is a more sophisticated result, due to Furstenberg, which will be proved later in the course.

Theorem 2.10. *The set $Q := \{m^2 : m \in \mathbb{N}\}$ of perfect squares is a set of recurrence. In fact, for every m.p.s. $(X, \mathcal{B}, \mu, T)$, every $A \in \mathcal{B}$ and for every $\epsilon > 0$ there exists a perfect square $n = m^2 \in \mathbb{N}$ such that*

$$\mu(A \cap T^{-n} A) > \mu(A)^2 - \epsilon$$

It turns out that the notion of sets of recurrence coincides with the notion of intersective sets.

Proposition 2.11. *A set $R \subset \mathbb{N}$ is a set of recurrence if and only if it is intersective (see Definition 1.14).*

Proposition 2.11 provides the first connection we've encountered between combinatorics and Ramsey theory; to prove it we will need the Furstenberg Correspondence Principle.

Exercise 2.12. $(*)$ *Show that if $R \subset \mathbb{N}$ is a set of recurrence and is decomposed as $R = A \cup B$ then either A or B is a set of recurrence. [Hint: Proceed by contradiction and take the product system of the two presumed counter-examples.]*

2.1. **Ergodicity.** The word ergodic arises from Boltzman's "ergodic hypothesis" in termodynamics, which describes a system where, *over long periods of time, the time spent by a system in some region of the phase space of microstates with the same energy is proportional to the volume of this region*[2]. In the language of measure preserving systems, the ergodic hypothesis would imply that the proportion of time that the orbit of a point (i.e. the sequence $x, Tx, T^2x, \dots$) is in a set A, tends to $\mu(A)$. This is in fact the conclusion of the ergodic theorem, which will be discussed below.

However, there is an obvious obstruction to the ergodic hypothesis: suppose $(X_i, \mathcal{A}_i, \mu_i, T_i)$ is a measure preserving system for each $i = 1, 2$ with X_1 and X_2 disjoint. Now let $Y = X_1 \cup X_2$, let $\mathcal{B}$ be the σ-algebra generated by $\mathcal{A}_1 \cup \mathcal{A}_2$, let $\nu = \frac{1}{2}\mu_1 + \frac{1}{2}\mu_2$ and let $S : Y \to Y$ be the map that maps $x \in X_i$ to $T_i x$, for $i = 1, 2$. Then $(Y, \mathcal{B}, \nu, S)$ is a measure preserving system, but a point $x \in X_1$ (or, more precisely, its orbit) will never visit X_2, even though $\mu(X_2) = 1/2 > 0$. A system is ergodic when it avoids this behavior.

Definition 2.13. *A measure preserving system $(X, \mathcal{B}, \mu, T)$ is* **ergodic** *if every set $A \in \mathcal{B}$ satisfying $T^{-1}A = A$ is trivial in the sense that either $\mu(A) = 0$ or $\mu(A) = 1$.*

Proposition 2.14. *A measure preserving system $(X, \mathcal{B}, \mu, T)$ is ergodic if and only if every $f \in L^2$ which is invariant in the sense that $f \circ T = f$ a.e. is constant a.e.*

Proof. For every $A \in \mathcal{B}$ the indicator function 1_A is in L^2, and hence we obtain the "only if" implication.

For the converse implication, suppose the system is ergodic and $f \in L^2$ is invariant. Then for every $t \in \mathbb{R}$, the set $A_t := \{x \in X : f(x) > t\}$ is invariant and hence has either measure 0 or 1. Let $r = \inf\{t : \mu(A_t) = 0\}$. Then $\mu(A_r) = 0$ because $A_r = \bigcup_{n \geq 1} A_{r+1/n}$. On the other hand $\mu(A_t) = 1$ for every $t < r$ and hence $\mu(\{x : f(x) \geq r\}) = 1$. We conclude that $f = r$ a.e. $\square$

The ergodic theorems assert, roughly speaking, that ergodic systems satisfy the ergodic hypothesis. Given a measure preserving system $(X, \mathcal{B}, \mu, T)$, the set $I \subset L^2(X)$ consisting of (almost everywhere) T-invariant functions, i.e. $I := \{f \in L^2(X) : f \circ T = f\}$ is a closed subspace. Therefore we can consider the orthogonal projection $P_I : L^2(X) \to I$ defined so that $P_I f$ is the element of I which is closest to f. It is not hard to show that P_I is a linear operator, and that it satisfies $\langle f - P_I f, g \rangle = 0$ for every $g \in I$. Here and in these notes, the inner product in L^2 is defined by

$$\langle f, g \rangle = \int_X f(x)\overline{g(x)}\, \mathsf{d}\mu(x).$$

Theorem 2.15 (Birkhoff's pointwise ergodic theorem, L^2 version)**.** *Let $P_I : L^2(X) \to I$ denote the orthogonal projection onto the subspace of T-invariant functions. Then for every $f \in L^2$*

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f \circ T^n = P_I f \qquad \text{a.e.}. \tag{2.3}$$

If the system is ergodic, then I consists only of the constant functions and $P_I f = \int_X f \, \mathsf{d}\mu$ a.e. Therefore for ergodic systems we have the following corollary.

[2] https://en.wikipedia.org/wiki/Ergodic_hypothesis

Corollary 2.16. *Let $(X, \mathcal{B}, \mu, T)$ be an ergodic measure preserving system. Then for every $A \in \mathcal{B}$ and almost every $x \in X$,*

$$\lim_{N\to\infty} \frac{1}{N} \Big| \big\{ n \in \{1, \dots, N\} : T^n x \in A \big\} \Big| = \mu(A).$$

Proof. Apply Theorem 2.15 to the indicator function 1_A of A and observe that, for each $x \in X$,

$$\sum_{n=1}^{N} (1_A \circ T^n)(x) = \Big| \big\{ n \in \{1, \dots, N\} : T^n x \in A \big\} \Big|.$$

□

A different version of the ergodic theorem was obtained by von Neumann, usually called the mean ergodic theorem because it deals with convergence in L^2 (or more generally in L^p) instead of almost everywhere convergence. This version has the advantage that it holds even if one changes the averaging scheme from $\{1, \dots, N\}$ to any sequence of intervals $\{a_N, a_N + 1, \dots, a_N + N\}$. Moreover, the simpler proof of von Neumann's theorem can be easily modified to apply to measure preserving actions of any amenable group.

Theorem 2.17 (von Neumann's mean ergodic theorem, L^2 version)**.** *Let $P_I : L^2(X) \to I$ denote the orthogonal projection onto the subspace of T-invariant functions. Then for every $f \in L^2$*

$$\lim_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N} f \circ T^n = P_I f \qquad \text{in } L^2(X). \tag{2.4}$$

Remark 2.18.

$$\lim_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N} h_n = c$$

means that for every $\epsilon > 0$ there exists some K such that if $M, N \in \mathbb{N}$ satisfy $N - M > K$, then $\left| \frac{1}{N-M} \sum_{n=M}^{N} h_n - c \right| < \epsilon$. This mode of convergence is often used in ergodic theory and is called a **uniform Cesàro limit** *or a* **uniform Cesàro average***, as opposed to the kind of averages used in the pointwise ergodic theorem, called simply* **Cesàro averages***.*

Exercise 2.19. *Show that*

$$\lim_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N} h_n = c$$

is equivalent to

$$\forall\ (I_N)_{N\in\mathbb{N}} \qquad \lim_{N\to\infty} \frac{1}{|I_N|} \sum_{n\in I_N} h_n = c$$

where $(I_N)_{N\in\mathbb{N}}$ is a sequence of intervals $I_N = \{a_N + 1, a_N + 2, \dots, a_N + b_N\}$ whose lengths b_N tend to infinity.

Given a measure preserving system $(X, \mathcal{B}, \mu, T)$, the ***Koopman operator*** $\Phi_T : L^2(X) \to L^2(X)$ is the linear operator defined by the equation $\Phi_T f := f \circ T$. Since T is measure preserving, it follows that Φ_T is an isometry, i.e., $\langle \Phi_T f, \Phi_T g \rangle = \langle f, g \rangle$. Therefore Theorem 2.17 is a corollary of the following.

Theorem 2.20 (von Neumann's mean ergodic theorem, Hilbert space version)**.** *Let H be a Hilbert space, let $\Phi : H \to H$ be an isometry and let $I \subset H$ be the subspace of invariant vectors, i.e. $I = \{f \in H : \Phi f = f\}$. Let $P : H \to I$ be the orthogonal projection onto I. Then for every $f \in H$,*

$$\lim_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N} \Phi^n f = P f \qquad \text{in norm} \tag{2.5}$$

Proof. If $f \in I$ then (2.5) holds trivially (with both sides equal to f).

On the other hand, if $f = g - \Phi g$ for some $g \in H$, then for any $h \in I$ we have

$$\langle f, h\rangle = \langle g, h\rangle - \langle \Phi g, h\rangle = \langle g, h\rangle - \langle g, \Phi h\rangle = 0$$

hence f is orthogonal to I and so $Pf = 0$. Moreover we have that $\sum_{n=M}^{N} \Phi^n f = \Phi^M g - \Phi^{N+1} g$, which has norm at most $2\|g\|$, and so the limit in the left hand side of (2.5) is also 0.

Call J the subspace of the vectors of the form $g - \Phi g$. We claim that $H = I \oplus J$ and this concludes the proof. To prove the claim, letting $f \perp J$, we have:

$$\begin{aligned} \|f - \Phi f\| &= \|f\|^2 + \|\Phi f\|^2 - 2Re\langle f, \Phi f\rangle \\ &= 2\|f\|^2 - 2Re\langle f, \Phi f\rangle - 2Re\langle f, f - \Phi f\rangle = 2\|f\|^2 - 2Re\langle f, f\rangle = 0 \end{aligned}$$

so $f \in I$ and hence $I = J^\perp$ and this finishes the proof. ☐

Corollary 2.21. *A measure preserving system $(X, \mathcal{B}, \mu, T)$ is ergodic if and only if for every $A, B \in \mathcal{B}$,*

$$\lim_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N} \mu(T^{-n}A \cap B) = \mu(A)\mu(B). \tag{2.6}$$

Proof. If the system is not ergodic, then there exists $A \in \mathcal{B}$ with $\mu(A) \in (0,1)$ which is invariant. Therefore, taking $B = X \setminus A$, we see that $T^{-n}A \cap B = \emptyset$ for every n, contradicting (2.6).

Let $f = 1_A$ and $g = 1_B$. Observe that $1_{T^{-n}A} = f \circ T^n = \Phi_T^n f$. Therefore $\mu(T^{-n}A \cap B) = \int_X \Phi_T^n 1_A \cdot 1_B \,\mathsf{d}\mu = \langle \Phi_T^n 1_A, 1_B\rangle$. Since strong (or norm) convergence in L^2 implies weak convergence, it follows from(2.4) that

$$\lim_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N} \mu(T^{-n}A \cap B) = \langle P_I f, g\rangle.$$

Finally, in view of ergodicity, we have that $P_I f$ is the constant $\int_X f \,\mathsf{d}\mu = \mu(A)$, and (2.6) follows from the fact that $\int_X \mu(A) g \,\mathsf{d}\mu = \mu(A)\mu(B)$. ☐

Setting $A = B$ in Corollary 2.21 we see that, in ergodic system, one can improve Poincaré's recurrence theorem by finding $n \in \mathbb{N}$ such that $\mu(T^{-n}A \cap A)$ is arbitrarily close to $\mu^2(A)$. One can in fact obtain a stronger version of this fact, which also applies to non-ergodic systems.

Definition 2.22. *A set $S \subset \mathbb{N}$ is called* **syndetic** *if it has bounded gaps. More precisely, S is syndetic if there exists $L \in \mathbb{N}$ such that every interval $\{n, n+1, \ldots, n+L-1\}$ of length L contains some element of S.*

Exercise 2.23. *Let (a_n) be a sequence of non-negative real numbers and let $a \in \mathbb{R}$. Show that if*

$$\lim_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N} a_n = a,$$

then for every $\epsilon > 0$ the set

$$\{n \in \mathbb{N} : a_n \geq a - \epsilon\}$$

is syndetic.

Theorem 2.24 (Khintchine's recurrence theorem)**.** *Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving system, let $A \in \mathcal{B}$ and let $\epsilon > 0$. Then there exists $n \in \mathbb{N}$ such that $\mu(A \cap T^{-n}A) > \mu^2(A) - \epsilon$, and moreover the set*

$$\left\{n \in \mathbb{N} : \mu(A \cap T^{-n}A) > \mu^2(A) - \epsilon\right\}$$

is syndetic.

Proof. Applying Theorem 2.17 to the indicator function 1_A of A we have

$$\lim_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N} \mu(T^{-n}A \cap A) = \int_X P_I 1_A \cdot 1_A \,\mathsf{d}\mu.$$

Since P_I is an orthogonal projection it follows that $\int_X P_I 1_A \cdot 1_A \, \mathsf{d}\mu = \|P_I 1_A\|^2$. We now use the Cauchy-Schwarz inequality to get

$$\|P_I 1_A\|^2 \geq \left(\int_X P_I 1_A \, \mathsf{d}\mu \right)^2 = \mu(A)^2.$$

□

3. Furstenberg's Correspondence Principle

The connection between Ramsey theory and ergodic theory hinges on the Furstenberg Correspondence Principle which we will soon formulate. Recall from Exercise 1.8 that the upper density satisfies $\bar{d}(A) = \bar{d}(A-1)$ for any set $A \subset \mathbb{N}$. Denote by T the successor map $T : \mathbb{N} \to \mathbb{N}$, $T : x \mapsto x+1$. Then $A - 1$ can be written as $T^{-1}A$. While $\bar{d}$ is not a probability measure (it is not even finitely additive), the tuple $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \bar{d}, T)$ looks a lot like a measure preserving system (by $\mathcal{P}(\mathbb{N})$ we denote the collection of all subsets of $\mathbb{N}$).

Furstenberg Correspondence Principle.

For many "arithmetic purposes", the tuple $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \bar{d}, T)$ behaves like a measure preserving system.

This is, of course a very vague statement, which is why it is a "principle" and not a "theorem". There are several incarnations of this principle as precise statements, but it is good to keep in mind the overarching principle, which can be adapted for different purposes.

Exercise 3.1. *Show that there are sets $A, B \subset \mathbb{N}$ with $\bar{d}(A) = \bar{d}(B) = 1$ but $A \cap B = \emptyset$.*

It is natural to wonder if the problem lies with the definition of upper density itself, and in particular with the lim sup. For instance, if one restricts attention to sets with ***natural density***, defined as the limit $d(A) := \lim_{N\to\infty} \frac{1}{N}|A \cap \{1, \ldots, N\}|$ only for those sets A for which it exists, could it have better properties than upper density? While it is true (and easy to check) that Unfortunately, doing so leads to problems of a different kind:

Exercise 3.2. *Show that there are sets $A, B \subset \mathbb{N}$ both having natural density but such that $A \cap B$ does not.*

The first instance of the Correspondence Principle was used by Furstenberg to give an ergodic theoretic proof of Szemerédi's theorem (Theorem 1.9), which states that if $A \subset \mathbb{N}$ has $\bar{d}(A) > 0$, then A contains an arithmetic progression of length k for any prescribed $k \in \mathbb{N}$. Note that

$$\exists x, n \in \mathbb{N} : \{x, x+n, \ldots, x+kn\} \subset A \qquad \Longleftrightarrow \qquad \exists n \in \mathbb{N} : A \cap (A-n) \cap \cdots \cap (A-kn) \neq \emptyset.$$

Using the shift T, we can write this as $\exists n \in \mathbb{N} : A \cap T^{-n}A \cap \cdots \cap T^{-kn}A \neq \emptyset$. If we subscribe to the Correspondence Principle, then Szemerédi's theorem becomes the statement that in a measure preserving system, whenever A has positive measure and $k \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that $A \cap T^{-n}A \cap \cdots \cap T^{-kn}A \neq \emptyset$. Since sets of measure 0 in a measure space might as well be empty, we have shown that Szemerédi's theorem is morally equivalent to the following.

Theorem 3.3 (Furstenberg's multiple recurrence theorem, [9])**.** *Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving system, let $A \in \mathcal{B}$ have $\mu(A) > 0$ and let $k \in \mathbb{N}$. Then there exists $n \in \mathbb{N}$ such that*

$$\mu\big(A \cap T^{-n}A \cap \cdots \cap T^{-kn}A\big) > 0.$$

It turns out that Theorem 3.3 is indeed equivalent to Szemerédi's theorem, which will be proved using a concrete instance of the Furstenberg Correspondence Principle. Note that for $k = 1$, Theorem 3.3 reduces to Poincaré's Recurrence theorem (Theorem 2.7), which has a fairly simple proof. On the other hand, the case $k = 1$ of Szemerédi's theorem states that the even more trivial fact that any set with positive upper density contains a 2-term arithmetic progression.

The proof of Theorem 3.3, which will occupy a few lectures, not only yields a proof of Szemerédi's theorem, but it reveals some deep structural results about *arbitrary* measure preserving systems.

Here is the version of the correspondence principle, formulated in [9], that we will use.

Theorem 3.4 (Correspondence Principle)**.** *Let $E \subset \mathbb{N}$. Then there exist a measure preserving system $(X, \mathcal{B}, \mu, T)$ and a set $A \in \mathcal{B}$ with $\mu(A) = \bar{d}(E)$ such that for any $n_1, \ldots, n_k \in \mathbb{N}$,*

$$\mu\big(A \cap T^{-n_1}A \cap \cdots \cap T^{-n_k}A\big) \leq \bar{d}\big(E \cap (E - n_1) \cap \cdots \cap (E - n_k)\big). \tag{3.1}$$

Proof. It turns out that $X, \mathcal{B}, T$ and A will not depend on E and only μ does. Take $X = \{0,1\}^{\mathbb{N}_0}$, with the product topology (where $\{0,1\}$ has the discrete topology) and the Borel σ-algebra $\mathcal{B}$. Let $T : X \to X$ be the left shift map $T : (x_n)_{n=0}^{\infty} = (x_{n+1})_{n=0}^{\infty}$ and let A be the cylinder set at 0, described as $A = \{(x_n) \in X : x_0 = 1\}$.

Then let $x \in X$ be the indicator function of E, so that $x_n = 1 \iff n \in E$. For each $N \in \mathbb{N}$, let $\mu_N = \frac{1}{N}\sum_{n=1}^{N} \delta_{T^n x}$ be the empirical measure (here, as usual, we denote by δ_y the Dirac measure (a.k.a the point mass) at y). Find a sequence $(N_k)_{k\in\mathbb{N}}$ such that $\bar{d}(E) = \lim_{k\to\infty} \frac{1}{N_k}|E \cap \{1, \dots, N_k\}|$. Since X is compact, so is the space of probability measures one X under the weak* topology[3]. Therefore, we may pass to a subsequence of (N_k) (which to simplify notation will still be denoted by (N_k)) so that the limit

$$\mu = \lim_{k\to\infty} \mu_{N_k}$$

exists. It is not hard to show that μ is T-invariant (see Exercise 3.5) so that $(X, \mathcal{B}, \mu, T)$ is indeed a measure preserving system. Note that

$$\delta_{T^n x}(A) = 1 \iff T^n x \in A \iff x_n = 1 \iff n \in E,$$

so $\mu_N(A) = \frac{1}{N}|E \cap \{1, \dots, N\}|$ and hence $\mu(A) = \bar{d}(E)$. Finally, for any $n_1, \dots, n_k \in \mathbb{N}$ we have $\delta_{T^n x}\big(A \cap T^{-n_1}A \cap \dots \cap T^{-n_k}A\big) = 1 \iff n \in E \cap (E - n_1) \cap \dots \cap (E - n_k)$, and hence

$$\begin{aligned} \mu\big(A \cap T^{-n_1}A \cap \dots \cap T^{-n_k}A\big) &= \lim_{k\to\infty} \frac{1}{N_k}\big|E \cap (E - n_1) \cap \dots \cap (E - n_k) \cap \{1, \dots, N_k\}\big| \\ &\le \bar{d}\big(E \cap (E - n_1) \cap \dots \cap (E - n_k)\big) \end{aligned}$$

□

Exercise 3.5. *Show that the measure μ constructed in the proof of the Furstenberg Correspondence Principle is T-invariant.*

3.1. **Applications of the Correspondence Principle.** The first application, as mentioned in the previous subsection, is to reduce Szemerédi's theorem to the multiple recurrence theorem, which, while a deep result, is purely about ergodic theory. Indeed, given $E \subset \mathbb{N}$ with $\bar{d}(E) > 0$, applying the correspondence principle in the form of Theorem 3.4 yields a measure preserving system $(X, \mathcal{B}, \mu, T)$ and a set $A \in \mathcal{B}$ with $\mu(A) > 0$ and satisfying (3.1). Then, using Theorem 3.3, one can find for any $k \in \mathbb{N}$ a number $n \in \mathbb{N}$ such that $\mu(A \cap T^{-n}A \cap \dots \cap T^{-kn}A) > 0$, which in view of (3.1) implies that $\bar{d}\big(E \cap (E - n) \cap \dots \cap (E - kn)\big) > 0$. Now any $x \in E \cap (E - n) \cap \dots \cap (E - kn)$ gives rise to an arithmetic progression $\{x, x + n, \cdots, x + kn\}$ contained in E.

As also mentioned above, it turns out that the converse direction is also true, i.e., taking Szemerédi's theorem as a blackbox, one can easily prove Theorem 3.3 (see Exercise 3.6).

The next application of the correspondence principle is a proof of Proposition 2.11:

Proof of Proposition 2.11. Suppose first that R is a set of recurrence and let $E \subset \mathbb{N}$ with $\bar{d}(E) > 0$. We need to find $n \in R \cap (E - E)$. Applying Theorem 3.4 we get a m.p.s. $(X, \mathcal{B}, \mu, T)$ and a set $A \in \mathcal{B}$ with $\mu(A) > 0$ satisfying (3.1). Since R is a set of recurrence, there exists $n \in R$ with $\mu(A \cap T^{-n}A) > 0$, which in view of (3.1) implies that $\bar{d}(E \cap (E - n)) > 0$. In particular $E \cap (E - n)$ is non-empty, and if x belongs to it, then $x, x + n \in E$, whence $n = (x + n) - x \in (E - E) \cap R$. We conclude that R is an intersective set.

Next suppose that R is intersective. Let $(X, \mathcal{B}, \mu, T)$ be a m.p.s. and let $A \in \mathcal{B}$ with $\mu(A) > 0$. For each $x \in X$ let $E_x = \{n \in \mathbb{N} : T^n x \in A\}$. The upper density of E_x is

$$\bar{d}(E_x) = \limsup_{N\to\infty} \frac{1}{N}\big|\{n \in \{1, \dots, N\} : T^n x \in A\}\big| = \limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} 1_A(T^n x).$$

Using Fatou's lemma, we can now estimate the average upper density of E_x:

$$\int_X \bar{d}(E_x)\,\mathsf{d}\mu = \int_X \limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} 1_{T^{-n}A}\,\mathsf{d}\mu \ge \limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \int_X 1_{T^{-n}A}\,\mathsf{d}\mu = \mu(A)$$

[3]This follows from combining the Riesz representation theorem for measures with the Banach-Alaoglu theorem and the trivial fact that the constant function 1 has compact support.

Therefore the set $B := \{x \in X : \bar{d}(E_x) > \mu(A)/2\}$ has positive measure, and for each $x \in B$ we can use the fact that R is intersective to find $a_x, b_x \in E_x$ with $a_x - b_x \in R$. Since there only countably many choices for the pairs (a_x, b_x), there exists a pair $(a, b) \in \mathbb{N}^2$ and a positive measure subset $C \subset B$ such that for every $x \in C$ we have $\{a, b\} \subset E_x$ and $n := a - b \in R$. Therefore $C \subset T^{-a}A \cap T^{-b}A = T^{-b}(T^{-n}A \cap A)$ which implies that $\mu(T^{-n}A \cap A) > 0$, and hence that R is a set of recurrence. $\square$

The method used for the second half of the proof can be adapted to show that Szemerédi's theorem implies Furstenberg's Multiple Recurrence theorem.

Exercise 3.6. *Adapting the proof of Proposition 2.11, show that Theorem 1.9 implies Theorem 3.3.*

Exercise 3.7. *Show that, in the proof of Proposition 2.11, the function $x \mapsto \bar{d}(E_x)$ is measurable and hence we can in fact consider its integral.*

Exercise 3.8. *Show that, in the proof of Proposition 2.11, for μ-a.e. $x \in X$ the set E_x has a natural density, i.e., show that the limit $\lim_{N\to\infty} \frac{1}{N}|E_x \cap \{1, \dots, N\}|$ exists.*

Recall Khintchine's theorem (Theorem 2.24). Applying the correspondence principle we obtain the following combinatorial corollary.

Corollary 3.9. *Let $E \subset \mathbb{N}$ have $\bar{d}(E) > 0$. Then the set $E - E$ is syndetic.*
In fact, given sets $E_1, E_2, \dots, E_k \subset \mathbb{N}$ with $\bar{d}(E_i) > 0$ for all i, the intersection $(E_1 - E_1) \cap \cdots \cap (E_k - E_k)$ is syndetic.

Proof. We prove only the second statement, which naturally implies the first one. Let $E_1, \dots, E_k \subset \mathbb{N}$ have all positive upper density. Apply Theorem 3.4 to each of them to get measure preserving systems $(X_i, \mathcal{B}_i, \mu_i, T_i)$ and sets $A_i \in \mathcal{B}_i$ for each $i = 1, \dots, k$ satisfying $\mu_i(A_i) = \bar{d}(E_i) > 0$. Then let $X = \prod_{i=1}^k X_i$, $\mathcal{B} = \bigotimes_{i=1}^k \mathcal{B}_i$, $\mu = \bigotimes_{i=1}^k \mu_i$ and $T : X \to X$ be the map $T(x_1, \dots, x_k) = (T_1x_1, \dots, T_kx_k)$. Let $A = \prod_{i=1}^k A_i \subset X$ and note that $\mu(A) = \mu_1(A_1) \times \cdots \times \mu_k(A_k)$.

In view of Theorem 2.24, the set $R := \{n \in \mathbb{N} : \mu(A \cap T^{-n}A) > 0\}$ is syndetic. Noting that $A \cap T^{-n}A = \prod_{i=1}^k (A_i \cap T_i^{-n}A_i)$ it follows that whenever $n \in R$, for each $i = 1, \dots, k$ we have $\mu_i(A_i \cap T_i^{-n}A_i) > 0$. Using (3.1) it follows that $\bar{d}(E_i \cap (E_i - n)) > 0$ and in particular that $n \in E_i - E_i$ for each i. We conclude that $R \subset (E_1 - E_1) \cap \cdots \cap (E_k - E_k)$ and hence that this intersection is syndetic. $\square$

As an application of this circle of ideas, here is a proof of Schur's theorem (Theorem 1.1) essentially first discovered by Bergelson.

Proof of Theorem 1.1. Let $\mathbb{N} = C_1 \cup \cdots \cup C_r$ be a finite partition (i.e. coloring) of $\mathbb{N}$. After reordering the C_i's if needed we can find $s \in \{1, \dots, r\}$ such that $\bar{d}(C_i) > 0$ for every $i = 1, \dots, s$ and $\bar{d}(C_i) = 0$ for each $i > s$. It follows that the (possibly empty) intersection $E := \bigcup_{i>s} C_i$ has 0 density and in particular is not a syndetic set.

Using Corollary 3.9 it follows that the intersection $(C_1 - C_1) \cap \cdots \cap (C_s - C_s)$ is syndetic, and hence is not contained in E. Therefore there exists $x \in (C_1 - C_1) \cap \cdots \cap (C_s - C_s) \cap (\mathbb{N} \setminus E)$. Say $x \in C_j$; we then have that $j \leq s$, so $x \in C_j - C_j$ as well. Let $z, y \in C_j$ be such that $z - y = x$. It follows that $\{x, y, x + y\} \subset C_j$. $\square$

4. Polynomial Recurrence

In this section we prove a polynomial recurrence theorem, which in view of the Correspondence Principle implies Sàrközy's theorem (Theorem 2.10). To prove it we introduce an important tool in Ergodic Ramsey Theory – the van der Corput trick.

Another idea that is briefly explored in this section is that of a dichotomy between "structure" and "randomness", albeit in a very embryonic form. In this context, structure is captured by periodic functions, and randomness (or "mixing") is captured by the notion of total ergodicity. This kind of dichotomy will become more clear (and useful) in the following sections.

4.1. **The van der Corput trick.** If the Correspondence Principle is the soul of Ergodic Ramsey Theory, its beating heart is the so-called van der Corput trick. There are many variations of this technique (the interested reader may read the expository article [3]), catered for specific applications throughout Ergodic Ramsey Theory.

The original lemma due to van der Corput [5] is concerned with uniform distribution in the unit interval.

Definition 4.1. *A sequence $(x_n)_{n=1}^\infty$ taking values in $[0,1]$ is said to be* **uniformly distributed** *or* **equidistributed** *if for every interval $(a,b)\subset[0,1]$,*

$$\lim_{N\to\infty}\frac{1}{N}\Big|\big\{n\in[1,N]:x_n\in(a,b)\big\}\Big|=b-a. \tag{4.1}$$

Due to the fact that there are uncountably many intervals (a,b) inside $[0,1]$, it is not clear that uniformly distributed sequences even exist. However, we have the following criterion by Weil [28] (for a proof, see [20, Theorems 1.1.1 and 1.2.1]).

Lemma 4.2 (Weyl criterion)**.** *Let $(x_n)_{n=1}^\infty$ be a sequence taking values in $[0,1]$. The following are equivalent.*

(1) *$(x_n)_{n=1}^\infty$ is uniformly distributed.*
(2) *The sequence of measures $\mu_N=\frac{1}{N}\sum_{n=1}^N\delta_{x_n}$ converges in the weak* topology to the Lebesgue measure.*
(3) *For every continuous function $f\in C[0,1]$,*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^N f(x_n)=\int_0^1 f(t)\,\mathsf{d}t.$$

(4)

$$\forall\ h\in\mathbb{N}\qquad \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^N e^{2\pi i h x_n}=0. \tag{4.2}$$

Example 4.3. *Let $\alpha\in\mathbb{R}\setminus\mathbb{Q}$. Then the sequence $x_n=n\alpha \bmod 1$ is uniformly distributed. Indeed, for every $h,N\in\mathbb{N}$ we have*

$$\frac{1}{N}\sum_{n=1}^N e^{2\pi i h x_n}=\frac{1}{N}\sum_{n=1}^N\left(e^{2\pi i h\alpha}\right)^n=\frac{1}{N}\cdot\frac{e^{2\pi i h\alpha(N+1)}-e^{2\pi i h\alpha}}{e^{2\pi i h\alpha}-1}$$

and the last expression converges to 0 as $N\to\infty$.

Exercise 4.4. *Show that the sequence $x_n=\sqrt{n} \bmod 1$ is uniformly distributed.*

Exercise 4.5. *Show that the sequence $x_n=\log n \bmod 1$ is* **not** *uniformly distributed.*

Here is the original version of the van der Corput trick.

Lemma 4.6. *Let $(x_n)_{n=1}^\infty$ be a sequence taking values in $\mathbb{R}$. If for every $m\in\mathbb{N}$ the sequence $n\mapsto x_{n+m}-x_n \bmod 1$ is uniformly distributed, then also the sequence $n\mapsto x_n \bmod 1$ is uniformly distributed.*

We will prove a more general result below. As a corollary of Lemma 4.6 we obtain Weyl's equidistribution theorem.

Corollary 4.7. *Let $f\in\mathbb{R}[t]$ be a polynomial with real coefficients. If at least one of the coefficients of f, other than the constant term, is irrational, then $f(n) \bmod 1$ is uniformly distributed.*

Proof. We proceed by induction on the degree $d=d(f)$ of the largest degree term of f with an irrational coefficient. If $d=1$, then the sequence $f(n) \bmod 1$ is the sum of a periodic sequence (say of period p) and the sequence $n\mapsto n\alpha \bmod 1$ where α is the irrational coefficient of degree 1. Since $p\alpha$ is still irrational, one can adapt the argument in Example 4.3 to show that $f(n) \bmod 1$ is indeed uniformly distributed when $d=1$.

Next suppose that $d>1$. For each $m\in\mathbb{N}$, the sequence $g_m:n\mapsto f(n+m)-f(n)$ is itself a polynomial with $d(g_m)=d(f)-1$ by induction, $g_m \bmod 1$ is uniformly distributed, and in view of Lemma 4.6, so is $f(n) \bmod 1$. $\square$

The most useful versions of the van der Corput trick for Ergodic Ramsey Theory deal with sequences of vectors in a Hilbert space; here is a simple formulation that will be useful later.

Lemma 4.8. *Let H be a Hilbert space and let $(x_n)_{n=1}^{\infty}$ be a bounded sequence taking values in H. If for every $d \in \mathbb{N}$,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \langle x_{n+d}, x_n \rangle = 0 \tag{4.3}$$

then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} x_n = 0.$$

There is also a version for uniform Cesàro averages, which can be proved in the same way (see Exercise 4.9 below).

Proof of Lemma 4.8. For any $\epsilon > 0$ and any $D \in \mathbb{N}$, if $N \in \mathbb{N}$ is large enough we have

$$\left\| \frac{1}{N} \sum_{n=1}^{N} x_n - \frac{1}{D} \sum_{d=1}^{D} \frac{1}{N} \sum_{n=1}^{N} x_{n+d} \right\| < \frac{\epsilon}{2}$$

Hence it suffices to show that, if D is large enough,

$$\limsup_{N\to\infty} \left\| \frac{1}{D} \sum_{d=1}^{D} \frac{1}{N} \sum_{n=1}^{N} x_{n+d} \right\| < \frac{\epsilon}{2}$$

Using the Cauchy-Schwarz inequality we have

$$\begin{aligned}
\limsup_{N\to\infty} \left\| \frac{1}{N} \sum_{n=1}^{N} \frac{1}{D} \sum_{d=1}^{D} x_{n+d} \right\|^2 &\leq \limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \left\| \frac{1}{D} \sum_{d=1}^{D} x_{n+d} \right\|^2 \\
&= \limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \frac{1}{D^2} \sum_{d_1,d_2=1}^{D} \langle x_{n+d_1}, x_{n+d_2} \rangle \\
&\leq \frac{1}{D^2} \sum_{d_1,d_2=1}^{D} \limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \langle x_{n+d_1}, x_{n+d_2} \rangle \qquad (4.4)
\end{aligned}$$

Note that, for $d_1 \neq d_2$, it follows from (4.3) that $\frac{1}{N}\sum_{n=1}^{N} \langle x_{n+d_1}, x_{n+d_2} \rangle \to 0$ as $N \to \infty$. We conclude that the quantity in (4.4) is bounded by $\frac{D}{D^2} = \frac{1}{D}$ which is arbitrarily small for large enough D. $\square$

Exercise 4.9. *Adapt the proof of Lemma 4.8 to the following version for uniform Cesàro averages (see Remark 2.18): Let H be a Hilbert space and let $(x_n)_{n=1}^{\infty}$ be a bounded sequence taking values in H. If for every $d \in \mathbb{N}$,*

$$\lim_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N} \langle x_{n+d}, x_n \rangle = 0$$

then

$$\lim_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N} x_n = 0.$$

Exercise 4.10. $(*)$
Let $p \in \mathbb{R}[x]$ have at least one irrational coefficient (other than the constant term) and let $U \subset [0,1]$ be open and non-empty. Is it true that the set $\{n \in \mathbb{N} : p(n) \bmod 1 \in U\}$ is syndetic? [Hint: Use Exercise 4.9 to obtain versions of Lemma 4.6 and Corollary 4.7 for uniform Cesàro averages and then use a similar argument as for Exercise 2.23.]

4.2. **Totally ergodicity.**

Definition 4.11. *A measure preserving system* $(X, \mathcal{B}, \mu, T)$ *is* **totally ergodic** *if for every* $n \in \mathbb{N}$*, the measure preserving system* $(X, \mathcal{B}, \mu, T^n)$ *is ergodic.*

A convenient notation we will often use from now on is the following: given a m.p.s. $(X, \mathcal{B}, \mu, T)$ and a function $f \in L^2(X)$, we denote by Tf the composition $f \circ T$ (another way to think about this is, as an abuse of language, to denote by T the associated Koopman operator).

Example 4.12. *Recall the circle rotation* $(X, \mathcal{B}, \mu, T)$ *described in Example 2.4, where* $X = [0,1]$*,* $\mathcal{B}$ *is the Borel* σ*-algebra,* μ *is the Lebesgue measure and* $T : x \mapsto x + \alpha \bmod 1$*. This system is totally ergodic if and only if* α *is irrational. Indeed, if* α *is rational, say* $\alpha = p/q$*, then* $q\alpha$ *is an integer and hence* T^q *is the identity map on* $[0,1]$*, which is trivially not ergodic.*

On the other hand, if α *is irrational, then the system is ergodic. To see this we use the ergodic theorem. Then we need to show that for every* $f \in L^2$ *the average*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^n f$$

is a constant function. But this is easy to check for functions $t \mapsto e(nt)$ *with* $n \in \mathbb{Z}$*, and finite linear combinations of functions of this kind form a dense subset of* L^2*.*

Finally, for every $n \in \mathbb{N}$*, the measure preserving system* $(X, \mathcal{B}, \mu, T^n)$ *is the circle rotation by* $n\alpha$*; since* $n\alpha$ *is also irrational when* α *is, the system* $(X, \mathcal{B}, \mu, T)$ *is totally ergodic in this case.*

When a system $(X, \mathcal{B}, \mu, T)$ is totally ergodic, we obtain from the ergodic theorem the following corollary.

Corollary 4.13. *Let* $(X, \mathcal{B}, \mu, T)$ *be a measure preserving system. Then it is totally ergodic if and only if for every* $f \in L^2(X)$ *and every* $q, r \in \mathbb{N}$*,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^{qn+r} f = \int_X f \, \mathsf{d}\mu. \qquad \textit{in } L^2(X) \tag{4.5}$$

Proof. If the system is not totally ergodic, then there exists $q \in \mathbb{N}$ and a non-constant $f \in L^2(X)$ such that $T^q f = f$. Thus (4.5) implies that the system is totally ergodic.

To prove the converse direction, let $(X, \mathcal{B}, \mu, T)$ be totally ergodic and let $f \in L^2(X)$ and $q, r \in \mathbb{N}$ be arbitrary. Applying the ergodic theorem (Theorem 2.17) to the (ergodic) system $(X, \mathcal{B}, \mu, T^q)$ we conclude that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^{qn+r} f = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} (T^q)^n (T^r f) = \int_X T^r f \, \mathsf{d}\mu = \int_X f \, \mathsf{d}\mu.$$

□

Remark 4.14. *A measure preserving system* $(X, \mathcal{B}, \mu, T)$ *is called* **invertible** *if* T *is invertible a.e. and the inverse is measurable and measure preserving. In this situation we can allow* q *and* r *in Corollary 4.13 to be negative, but if the system is not invertible, then the expression* $T^n f$ *does not make sense for a negative value of* n*.*

Nevertheless, Corollary 4.13 still makes sense when $r < 0$*, even if the system is not invertible. Indeed, in this case the expression* $qn + r$ *is positive for all but finitely many values of* n*, and since we take an average over* $\mathbb{N}$ *we can just ignore those finitely many values.*

One could interpret the expression T^{qn+r} appearing in (4.5) as $T^{p(n)}$ where p is a linear polynomial. The following theorem reveals the power of the van der Corput trick, which allows one to upgrade Corollary 4.13 to general polynomials.

Theorem 4.15. *Let* $(X, \mathcal{B}, \mu, T)$ *be a totally ergodic system and let* $p \in \mathbb{Z}[x]$ *be such that either the system is invertible or the polynomial has a positive leading coefficient. Then for every* $f \in L^2(X)$*,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^{p(n)} f = \int_X f \, \mathsf{d}\mu. \qquad \textit{in } L^2(X) \tag{4.6}$$

Proof. We proceed by induction on the degree of p. If p is linear, then the result follows from Corollary 4.13, so assume that p has degree at least 2. Eq. (4.6) holds for f if and only if it holds for $f-c$ where c is a constant; therefore, after subtracting $\int_X f\,\mathsf{d}\mu$ from f we can assume that $\int_X f\,\mathsf{d}\mu=0$. Letting $x_n=T^{p(n)}f$, we need to show that $\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^N x_n=0$, and to this end we will invoke the van der Corput lemma (Lemma 4.8). Fixing $d\in\mathbb{N}$ we can compute

$$\langle x_{n+d},x_n\rangle=\int_X T^{p(n+d)}f\cdot T^{p(n)}\overline{f}\,\mathsf{d}\mu=\int_X T^{p(n+d)-p(n)}f\cdot\overline{f}\,\mathsf{d}\mu=\Big\langle T^{p(n+d)-p(n)}f,f\Big\rangle.$$

Since $n\mapsto p(n+d)-p(n)$ is a polynomial of degree smaller than the degree of p, we can use the induction hypothesis (together with the fact that convergence in $L^2(X)$ implies convergence in the weak topology) to conclude

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^N\langle x_{n+d},x_n\rangle=\left\langle\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^N T^{p(n+d)-p(n)}f,f\right\rangle=0.$$

This establishes the hypothesis (4.3) of the van der Corput lemma, so we conclude that $\displaystyle\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^N x_n=0$, as desired. □

Remark 4.16. *Both Corollary 4.13 and Theorem 4.15 have versions for uniform Cesàro averages, which can be proved in the exact same way. The choice to present the regular Cesàro versions was made with the hope that the main ideas became more transparent.*

4.3. **Total ergodicity and finite factors.**

This subsection is not necessary to the proof of Sárközy's theorem, but it leads to some important ideas that will appear in later sections.

Here is another example of an ergodic system that is not totally ergodic.

Example 4.17. *Let $X=\{0,1\}$, $\mathcal{B}$ the discrete σ-algebra, μ the normalized counting measure and $T:x\mapsto x+1\bmod 2$. In other words $(X,\mathcal{B},\mu,T)$ is a transposition on 2 points. Then this system is ergodic, since the only sets with measure in $(0,1)$ are the singletons $\{0\}$ and $\{1\}$, and neither of them is invariant. However, the system is not totally ergodic, since T^2 is the identity map and leaves both singletons (which have positive measure) invariant.*

While Example 4.17 seems rather trivial, it turns out that finite systems are in some sense the only obstruction to total ergodicity. To better capture this, we need the notion of factor maps.

Definition 4.18 (Factor map). *Let $(X,\mathcal{A},\mu,T)$ and $(Y,\mathcal{B},\nu,S)$ be m.p.s. and let $\phi:X\to Y$. Then ϕ is a* **factor map** *if it is surjective, preserves the measure (i.e. $\mu(\phi^{-1}B)=\nu(B)$ for every $B\in\mathcal{B}$) and intertwines T and S, in the sense that $S\circ\phi=\phi\circ T$.*

More generally, one can allow ϕ to be a surjective map between full measure sets $X_0\in\mathcal{A}$ and $Y_0\in\mathcal{B}$ such that $T^{-1}X_0=X_0$ and $S^{-1}Y_0=Y_0$, and the relation $S\circ\phi=\phi\circ T$ only needs to hold in X_0.

We say that the system $(Y,\mathcal{B},\nu,S)$ is a ***factor*** of $(X,\mathcal{A},\mu,T)$ if there is a factor map $\phi:X\to Y$. We will also say that, in this case, $(X,\mathcal{A},\mu,T)$ is an ***extension*** of $(Y,\mathcal{B},\nu,S)$.

Theorem 4.19. *Let $(X,\mathcal{A},\mu,T)$ be a measure preserving system. Then it is totally ergodic if and only if it does not allow for any non-trivial finite factor.*

Proof. Let $(Y,\mathcal{B},\nu,S)$ be a non-trivial finite system and suppose that there is a factor map $\pi:X\to Y$. Let $y\in Y$ be such that $\nu(\{y\})\in(0,1)$ and let $A=\pi^{-1}(\{y\})$. Then $\mu(A)=\nu(\{y\})\in(0,1)$. Let $k=|Y|!$. Then S^k acts trivially on Y, and in particular $S^{-k}\{y\}=\{y\}$. Therefore $T^{-k}A=A$ and we conclude that $(X,\mathcal{A},\mu,T^k)$ is not ergodic.

To prove the converse direction, suppose that $(X,\mathcal{A},\mu,T)$ is not totally ergodic. Let $n\in\mathbb{N}$ be such that T^n is not ergodic and let $A\in\mathcal{A}$ be such that $\mu(A)\in(0,1)$ and $T^{-n}A=A$. It follows that the σ-algebra $\mathcal{B}$ generated by the sets $A,T^{-1}A,\dots,T^{-(n-1)}A$ is invariant under T, finite and non-trivial. Let Y be the (finite) set of atoms of $\mathcal{B}$, and let $\pi:X\to Y$ be the containment map (i.e. $\pi(x)$ is the atom of $\mathcal{B}$ that contains x; more explicitly $\pi(x)=\bigcap_{\substack{B\in\mathcal{B}\\ x\in B}}B$). It is easy to check that π is indeed a factor map. □

Exercise 4.20. *Finish the proof of Theorem 4.19 by explicitly describing the measure preserving system structure of Y and showing that π is indeed a factor map.*

Exercise 4.21. *Let $(X,\mathcal{A},\mu,T)$ be a measure preserving system and let $(Y,\mathcal{B},\nu,S)$ be a factor. Prove that:*

- *If $(X,\mathcal{A},\mu,T)$ is ergodic, then so is $(Y,\mathcal{B},\nu,S)$.*
- *If $(X,\mathcal{A},\mu,T)$ is totally ergodic, then so is $(Y,\mathcal{B},\nu,S)$.*

4.4. **Proof of Sárközy's theorem.** Let $(X,\mathcal{B},\mu,T)$ be a measure preserving system. Consider the following subspaces of $L^2(X)$:

$$H_{rat} := \overline{\{f \in L^2(X) : T^k f = f \text{ for some } k \in \mathbb{N}\}};\ H_{te} := \left\{ f \in L^2(X) : \forall\, k \in \mathbb{N},\ \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} T^{kn} f = 0 \right\}$$

Exercise 4.22. *Show that if $f,g \in H_{rat}$ are bounded, then their product $f\cdot g$ is also in H_{rat}. Can you find an example showing that the same is not tru for H_{te}?*

Exercise 4.23. $(*)$
Show that the collection $\{A \in \mathcal{B} : 1_A \in H_{rat}\}$ is a σ-algebra.

Observe that in a totally ergodic system the space H_{rat} consists only of constant functions, while the space H_{te} contains every function with 0 integral. The following proposition generalizes this observation.

Proposition 4.24. *For any measure preserving system $(X,\mathcal{B},\mu,T)$, the spaces H_{rat} and H_{te} are orthogonal and $L^2(X) = H_{rat} \oplus H_{te}$.*

Proof. Let $f \in L^2(X)$ be such that $T^k f = f$ for some $k \in \mathbb{N}$ and let $g \in H_{te}$. Then $\langle f,g\rangle = \langle T^k f, T^k g\rangle = \langle f, T^k g\rangle$. Iterating this observation we deduce that $\langle f,g\rangle = \langle f, T^{kn} g\rangle$ for every $n \in \mathbb{N}$. Averaging over n we then deduce

$$\langle f,g\rangle = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \langle f, T^{kn} g\rangle = \left\langle f,\ \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} T^{kn} g \right\rangle = 0,$$

showing that H_{rat} and H_{te} are orthogonal.

Now suppose that $f \in L^2(X)$ is orthogonal to H_{rat}, we need to show that $f \in H_{te}$. But for every $k \in \mathbb{N}$, the space H_{rat} contains the invariant subspace I_k for the system $(X,\mathcal{B},\mu,T^k)$. It follows that f is orthogonal to I_k for every k, and in view of the mean ergodic theorem, $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} T^{kn} f = 0$, so that indeed $f \in H_{te}$. □

We are now ready to prove Sárközy's theorem (Theorem 1.13). Using the correspondence principle, or more precisely, using Proposition 2.11, our task is reduced to establishing polynomial recurrence, formulated in Theorem 2.10. The proof we provided for the Poincaré recurrence theorem (Theorem 2.7) does not extend far beyond the scope of Theorem 2.7. However, we saw a different proof of Poincaré's recurrence when proving the stronger Khintchine's recurrence (Theorem 2.24) using the ergodic theorem. Our proof of Theorem 2.10 follows this second strategy, replacing the ergodic theorem with the "polynomial ergodic theorem" for totally ergodic systems that we obtained in Eq. (4.6).

We will in fact establish a stronger version of Theorem 2.10.

Definition 4.25. *A polynomial $p \in \mathbb{Z}[x]$ is called* **divisible** *or* **intersective** *if for every $k \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $p(n)$ is a multiple of k.*

If $p(0) = 0$ or, more generally, p has an integer root, then it is divisible. However there are polynomials, such as $p(x) = (x^2-3)(x^2-5)(x^2-15)$ which have no integer root but are divisible. It is easy to see that if p is not divisible, then there exists a finite system where recurrence does not occur at times of the form $p(n)$. In other words, if p is not divisible, then the set $\{p(n) : n \in \mathbb{N}\}$ is not a set of recurrence. The converse of this observation is the content of the following theorem, which significantly extends Theorem 2.10.

Theorem 4.26. *Let $(X,\mathcal{B},\mu,T)$ be a measure preserving system, let $A \in \mathcal{B}$, let $\epsilon > 0$ and let $p \in \mathbb{Z}[x]$ be a divisible polynomial with a positive leading coefficient. Then there exists $n \in \mathbb{N}$ such that $\mu(A \cap T^{-p(n)}A) > \mu^2(A) - \epsilon$.*

Proof. Decompose $1_A = f + g$ with $f \in H_{rat}$ and $g \in H_{te}$. Since H_{rat} contains the constant functions, using the Cauchy-Schwarz inequality we have $\langle 1_A, f\rangle = \|f\|^2 \geq \langle f, 1\rangle^2 = \mu(A)^2$. Find $h \in H_{rat}$ such that $T^k h = h$ for some $k \in \mathbb{N}$, and such that $\|f - h\| < \epsilon/2$. In particular it follows that $\langle 1_A, h\rangle > \mu(A)^2 - \epsilon/2$.

Using divisibility of p, find $a \in \mathbb{N}$ such that $p(a) \equiv 0 \bmod k$ and consider the polynomial $q(n) = p(a+kn)$. Then $T^{q(n)}h = h$ for all $n \in \mathbb{N}$. As in the proof of Theorem 4.15, an application of the van der Corput trick implies that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} T^{q(n)} g = 0.$$

Finally, we have

$$\begin{aligned}
\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \mu(A \cap T^{-q(n)}A) &= \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \left\langle 1_A, h + T^{q(n)}(f-h) + T^{q(n)}g \right\rangle \\
&= \left\langle 1_A, h + \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} T^{q(n)}(f-h) + T^{q(n)}g \right\rangle \\
&\geq \langle 1_A, h\rangle - \epsilon/2 \quad \geq \quad \mu(A)^2 - \epsilon.
\end{aligned}$$

□

Exercise 4.27. *Adapt the proof of Theorem 4.26 to obtain that, under the same conditions, if additionally $\mu(A) > 0$, then*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \mu(A \cap T^{-p(n)}A) > 0.$$

Exercise 4.28. $(*)$ *Using Exercise 4.9 in the proof of Theorem 4.26, show that for any set $E \subset \mathbb{N}$ with $\bar{d}(E) > 0$, the set $\{n \in \mathbb{N} : n^2 \in E - E\}$ is syndetic.*

5. Proof of Roth's Theorem

In this section we prove Roth's theorem (Theorem 1.12). In view of the Furstenberg correspondence principle (Theorem 3.4) it suffices to prove the following triple recurrence theorem.

Theorem 5.1 (Dynamical Roth theorem)**.** *Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving system and let $A \in \mathcal{B}$ with $\mu(A) > 0$. Then there exists $n \in \mathbb{N}$ such that $\mu(A \cap T^{-n}A \cap T^{-2n}A) > 0$.*

There are a few steps in the proof of Theorem 5.1. First we make some simplifying assumptions, then we deal with the special case of weak mixing systems. Finally, drawing upon the Jacobs-de Leeuw-Glicksberg Decomposition, we prove the general case.

5.1. **Simplifying assumptions in multiple recurrence.** Recall from Remark 4.14 the notion of invertible system. In the proof of the correspondence principle (Theorem 3.4) that we presented, the system constructed is not in general invertible; however it is possible to make the system invertible by suitably modifying the proof.

Exercise 5.2. *Show that in Theorem 3.4 one can obtain an invertible system. In other words, show that for any $E \subset \mathbb{N}$ there exist an* **invertible** *measure preserving system $(X, \mathcal{B}, \mu, T)$ and a set $A \in \mathcal{B}$ with $\mu(A) = \bar{d}(E)$ such that for any $n_1, \dots, n_k \in \mathbb{N}$,*

$$\mu\big(A \cap T^{-n_1}A \cap \cdots \cap T^{-n_k}A\big) \leq \bar{d}\big(E \cap (E - n_1) \cap \cdots \cap (E - n_k)\big).$$

[Hint: Think of E as a subset of $\mathbb{Z}$, and replace everywhere in the proof $\mathbb{N}$ with $\mathbb{Z}$.]

Another way in which the conclusion of Theorem 3.4 can be improved is by noting that in the system $(X, \mathcal{B}, \mu, T)$ constructed in the proof, X is a compact metric space, $\mathcal{B}$ is the Borel σ-algebra on X and T is continuous. Putting these observations together we conclude that Roth's theorem Theorem 1.12 follows from an apparently weaker version of Theorem 5.1 where the system is invertible, X is a compact metric space, $\mathcal{B}$ is the Borel σ-algebra and T is continuous. However, since Theorem 1.12 also implies Theorem 5.1 (cf. Exercise 3.6) we have that Theorem 5.1 is in fact equivalent to this apparently weaker version.

The next simplification is to further assume that the system is ergodic. This is possible by using the so-called ergodic decomposition theorem.

Theorem 5.3 (Ergodic Decomposition)**.** *Let X be a compact metric space, let $\mathcal{B}$ be the Borel σ-algebra on X and let $T : X \to X$ be continuous. Let μ be a T-invariant Borel probability measure. Then there exists a probability space $(Y, \mathcal{D}, \nu)$ and, for each $y \in Y$, a T-invariant probability measure μ_y on $(X, \mathcal{B})$ satisfying*

(0) *The map $y \mapsto \mu_y$ is measurable, i.e. every integral below makes sense.*
(1) *For ν-a.e. $y \in Y$ the measure μ_y is ergodic (i.e. the system $(X, \mathcal{B}, \mu_y, T)$ is ergodic).*
(2) *$\mu = \int_Y \mu_y \, \mathsf{d}\nu(y)$ (i.e. $\forall f \in C(X)$, $\int_X f \, \mathsf{d}\mu = \int_Y \int_X f \, \mathsf{d}\mu_y \, \mathsf{d}\nu(y)$).*

The proof of Theorem 5.3 will be omitted but it can be found in many standard texts on ergodic theory, eg: [7, Theorem 4.8]. Here are some examples that illustrate this theorem.

Example 5.4. *Let $X = \{1, 2, 3\}$ be given the discrete topology and discrete σ-algebra $\mathcal{B}$ and let μ be the uniform measure (more precisely, $\mu(\{1\}) = \mu(\{2\}) = \mu(\{3\}) = 1/3$). Let $T(1) = 2$, $T(2) = 1$ and $T(3) = 3$. The set $A = \{1, 2\}$ is invariant under T and $0 < \mu(A) < 1$, hence the system $(X, \mathcal{B}, \mu, T)$ is not ergodic.*

However, if we restrict μ to A and renormalize it, we obtain a probability measure which makes the system ergodic. More precisely, let $\nu(\{1\}) = \nu(\{2\}) = 1/2$ and $\nu(\{3\}) = 0$. Then ν is and ergodic measure, in other words, the system $(X, \mathcal{B}, \nu, T)$ is ergodic.

Also, if ν_3 is the point mass at 3 (so that $\nu_3(\{1\}) = \nu_3(\{2\}) = 0$ and $\nu_3(\{3\}) = 1$), then the system $(X, \mathcal{B}, \nu_3, T)$ is also ergodic (one can also think of ν_3 as the normalized restriction of μ to the invariant set $\{3\}$).

Finally, observe that we can write μ as the convex combination $\mu = \frac{2}{3}\nu + \frac{1}{3}\nu_3$ of the ergodic measures ν and ν_3. If we let $\nu_1 = \nu_2 = \nu$, then we can write informally $\mu = \int_X \nu_y d\mu(y)$.

Example 5.5. *Let $X = \mathbb{T}^2$ with the usual topology and let μ be the Lebesgue measure. Let $T(x, y) = (x+y, y)$. Any set of the form $\mathbb{T} \times B$, where $B \subset \mathbb{T}$ is a Borel set, is invariant under T and hence the measure preserving system (X, μ, T) is not ergodic.*

Let λ denote the Lebesgue/Haar measure on $\mathbb{T}$. For each $y \in \mathbb{T}$, let $\mu_y = \lambda \otimes \delta_y$ (we are using the standard notation δ_y for a Dirac point mass, and $\otimes$ for the product of two measures). It is not hard to see that μ_y is T-invariant. Moreover, μ_y is ergodic exactly when y is irrational (this can be proved with some Fourier analysis).

Since the set of irrational y have full measure on $\mathbb{T}$, the ergodic decomposition of μ can be described by $\mu = \int_{\mathbb{T}} \mu_y \, \mathsf{d}\lambda(y)$.

Exercise 5.6. *Using Theorem 5.3 and the simplifications made at the beginning of this subsection, show that in Theorem 5.1 we can assume that the system is ergodic (in other words, show that if we Theorem 5.1 holds for ergodic systems then it holds for any measure preserving system).*

5.2. **Mixing and weak-mixing.** As we saw in Corollary 2.21, a measure preserving system is ergodic if and only if any two sets became asymptotically independent on average. For certain systems, this asymptotic independence occurs even without averaging, and we call this property ***mixing***.

Definition 5.7. *A measure preserving system $(X\ B, \mu, T)$ is* **mixing** *or* **strong-mixing** *if for every $A, B \in \mathcal{B}$,*

$$\lim_{n\to\infty} \mu(T^{-n}A \cap B) = \mu(A)\mu(B).$$

Proposition 5.8. *Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving system. Then the following are equivalent.*

- *The system is mixing.*
- *For every $f, g \in L^2(X)$, $\lim_{N\to\infty} \int_X T^n f \cdot g \, \mathsf{d}\mu = \int_X f \, \mathsf{d}\mu \int_X g \, \mathsf{d}\mu$.*
- *For every $f \in L^2(X)$ with $\int_X f \, \mathsf{d}\mu = 0$, the orbit $T^n f$ converges to 0 in the weak topology.*

Proof. The equivalence between the first two follows from the fact that the set of finite linear combinations of indicator functions is dense in L^2. The equivalence between the last two is immediate, after replacing f with $\tilde{f} := f - \int_X f \, \mathsf{d}\mu$ and noticing that $\int_X \tilde{f} \, \mathsf{d}\mu = 0$. □

It should be clear that every mixing system is ergodic, but the opposite is not true. There is also a notion of higher order mixing.

Definition 5.9. *A measure preserving system* $(X\ B,\mu,T)$ *is* **mixing of order** k *if for every* $A_1,\dots,A_k\in\mathcal{B}$ *and every sequences* $\left(n_i^{(1)}\right)_{i=1}^\infty,\dots,\left(n_i^{(k)}\right)_{i=1}^\infty$ *with* $\lim_{i\to\infty} n_i^{(r)}-n_i^{(s)}=\infty$ *for every* $1\le r,s\le k$

$$\lim_{i\to\infty}\mu\Big(T^{-n_i^{(1)}}A_1\cap T^{-n_i^{(2)}}A_2\cap\cdots\cap T^{-n_i^{(k)}}A_k\Big)=\mu(A_1)\mu(A_2)\cdots\mu(A_k).$$

Notice that mixing of order 2 is the same a strong-mixing. It is clear that k-mixing implies $k-1$-mixing; it is in fact a major open problem in ergodic theory whether the converse holds, even for $k=3$.

A weaker notion of mixing is ***weak-mixing***.

Definition 5.10. *Let* $\mathbf{X}=(X,\mathcal{B},\mu,T)$ *be a measure preserving system and let* $\mathbf{X}\times\mathbf{X}$ *be the self product system. The system* $\mathbf{X}$ *is* **weak-mixing** *or* **weakly mixing** *if and only if* $\mathbf{X}\times\mathbf{X}$ *is ergodic.*

The following theorem states several equivalent properties to weak-mixing.

Theorem 5.11. *Let* $\mathbf{X}=(X,\mathcal{B},\mu,T)$ *be a measure preserving system. Then the following are equivalent*

(1) $\mathbf{X}$ *is weak mixing.*
(2) *For every ergodic m.p.s.* $\boldsymbol{Y}$*, the product* $\boldsymbol{X}\times\boldsymbol{Y}$ *is ergodic.*
(3) *For any two sets* $A,B\in\mathcal{B}$ *we have* $\displaystyle\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^N\big|\mu(A\cap T^{-n}B)-\mu(A)\mu(B)\big|=0$
(4) *For any* $f,g\in L^2$ *we have* $\displaystyle\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^N\left|\int_X T^nf\cdot g\,\mathsf{d}\mu-\int_X f\,\mathsf{d}\mu\int_X g\,\mathsf{d}\mu\right|=0$
(5) *For any* $A,B\in\mathcal{B}$ *there exists a subset* $E\subset\mathbb{N}$ *with upper density* $\bar{d}(E)=0$ *such that*
$$\lim_{\substack{n\to\infty\\ n\notin E}}\mu(A\cap T^{-n}B)=\mu(A)\mu(B).$$

Condition (3) explains why it is called weak mixing, and makes it clear that every mixing system is weak mixing, and that every weak mixing system is ergodic. Not every weak-mixing system is strong-mixing, but examples are not easy to come by. On the other hand, it is easy to show, using directly the definition, that irrational circle rotations are ergodic but not weakly mixing.

Condition (2) implies that if $\mathbf{X}$ is weak mixing, then $\mathbf{X}\times\mathbf{X}\times\mathbf{X}\times\mathbf{X}$ is ergodic, and hence $\mathbf{X}\times\mathbf{X}$ is weak mixing. Therefore any self product $\mathbf{X}\times\mathbf{X}$ is weak mixing if and only if it is ergodic.

Proof of Theorem 5.11. The proof was not given in class, but we provide it here for completeness.

(1)⇒(4) Replacing f with $f-\int_X f\,\mathsf{d}\mu$ we can assume that $\int_X f\,\mathsf{d}\mu=0$. Using the Cauchy-Schwartz inequality we have

$$\limsup_{N\to\infty}\left(\frac{1}{N}\sum_{n=1}^N\left|\int_X T^nf\cdot g\,\mathsf{d}\mu\right|\right)^2\le\limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^N\left|\int_X T^nf\cdot g\,\mathsf{d}\mu\right|^2.$$

Using the hypothesis that $\mathbf{X}\times\mathbf{X}$ is ergodic, and applying the von Neumann's Ergodic Theorem (Theorem 2.17) to the functions $f\otimes\overline{f}\in L^2(X\times X)$ and $g\otimes\overline{g}\in L^2(X\times X)$ we obtain

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^N\int_{X\times X}(f\otimes\overline{f})\circ(T\times T)^n\cdot g\otimes\overline{g}\,\mathsf{d}(\mu\otimes\mu)=\int_{X\times X}f\otimes\overline{f}\,\mathsf{d}(\mu\otimes\mu)\int_{X\times X}g\otimes\overline{g}\,\mathsf{d}(\mu\otimes\mu).$$

Observe that $\int_{X\times X}f\otimes\overline{f}\,\mathsf{d}(\mu\otimes\mu)=\left|\int_X f\,\mathsf{d}\mu\right|^2=0$, so the previous equation can be rewritten as

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^N\left|\int_X T^nf\cdot g\,\mathsf{d}\mu\right|^2=0,$$

finishing the proof.

(4)⇒(3) This is immediate by letting $f=1_A$ and $g=1_B$.

(3)⇒(5) Fix $m \in \mathbb{N}$ and set $A_m := \{n \in \mathbb{N} : |\mu(A \cap T^{-n}B) - \mu(A)\mu(B)| > 1/m\}$. Observe that

$$\frac{1}{N}\sum_{n=1}^{N} |\mu(A \cap T^{-n}B) - \mu(A)\mu(B)| \geq \frac{1}{m}\frac{|A_m \cap [1,N]|}{N}$$

Taking the limit as $N \to \infty$ we conclude that $\bar{d}(A_m) = 0$ for all $m \in \mathbb{N}$. For each $m \in \mathbb{N}$ let $N_m \in \mathbb{N}$ be such that for all $N > N_m$ we have $|A_m \cap [1,N]| \leq N/m$ and make

$$E = \bigcup_{m=1}^{\infty} \left(A_m \cap [N_m + 1, N_{m+1}]\right)$$

Now observe that $A_k \subset A_{k+1}$ for all $k \in \mathbb{N}$, hence for each $N \in \mathbb{N}$, choosing m such that $N \in [N_m + 1, N_{m+1}]$ we have $E \cap [1,N] \subset A_m \cap [1,N]$ and hence $|E \cap [1,N]| \leq N/m$. Taking $N \to \infty$ (note that also $m \to \infty$ because all A_m have 0 density) we conclude that $\bar{d}(E) = 0$.

Finally, for each $m \in \mathbb{N}$, let $N > N_m$, then if $N \notin E$ we also have $N \notin A_m$ and so $|\mu(A \cap T^{-n}B) - \mu(A)\mu(B)| < 1/m$ concluding the proof.

In the case when $\mathcal{B}$ is separable, let $\{B_n\}_{n=1}^{\infty}$ be a countable dense family. For each $m = (m_1, m_2) \in \mathbb{N}^2$ let $E_m \subset \mathbb{N}$ be such that $\bar{d}(E_m) = 0$ and $\lim_{n\to\infty} \mu(T^{-n}B_{m_1} \cap B_{m_2}) \to \mu(B_{m_1})\mu(B_{m_2})$ for $n \notin E_m$. As above we construct a set E of 0 density such that for all $m \in \mathbb{N}^2$ there exists $N = N(m) \in \mathbb{N}$ such that $E_m \setminus [1,N] \subset E$.

It is not hard to check that this set E satisfies the conditions, we omit the details.

(5)⇒(3) Assuming (5), for every ϵ the set $\left\{n \in \mathbb{N} : \left|\mu(A \cap T^{-n}B) - \mu(A)\mu(B)\right| > \epsilon\right\}$ has density 0. On the other hand $\left|\mu(A \cap T^{-n}B) - \mu(A)\mu(B)\right| \leq 1$ for every $n \in \mathbb{N}$, and hence

$$\limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \left|\mu(A \cap T^{-n}B) - \mu(A)\mu(B)\right| \leq \epsilon.$$

Since ϵ is arbitrary we conclude that (3) holds.

(3)⇒(4) Condition (3) is the special case of (4) when f and g are indicator functions. It is not hard to see that if (4) holds for pairs (f_1, g) and (f_2, g), then it holds for the pair $(af_1 + bf_2, g)$. Since every L^2 function is approximated by finite linear combinations of indicator functions, we deduce that (4) holds whenever g is an indicator function. But similarly, if (4) holds for (f, g_1) and (f, g_2), it holds for $(f, ag_1 + bg_2)$, and hence the same argument shows that it must hold for any $f, g \in L^2$.

(4)⇒(2) Let $\mathbf{Y} = (Y, \mathcal{A}, S, \nu)$. In order to show that $\mathbf{X} \times \mathbf{Y}$ is ergodic, we will show that for any $f, g \in L^2(X \times Y)$,

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \int_{X\times Y} (T \times S)^n f \cdot g \, \mathsf{d}(\mu \otimes \nu) = \int_{X\times Y} f \, \mathsf{d}(\mu \otimes \nu) \int_{X\times Y} g \, \mathsf{d}(\mu \otimes \nu). \tag{5.1}$$

Since finite linear combinations of tensor functions of the form $(f_1 \otimes f_2)(x,y) = f_1(x)f_2(y)$ form a dense subset of $L^2(X \times Y)$, it suffices to establish (5.1) when both f and g are tensor functions. Let $f(x,y) = f_1(x)f_2(y) \in L^2(X \times Y)$ and $g(x,y) = g_1(x)g_2(y) \in L^2(X \times Y)$ be arbitrary tensor functions. Then (5.1) can be written as

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \int_X T^n f_1 \cdot g_1 \, \mathsf{d}\mu \int_Y S^n f_2 \cdot g_2 \, \mathsf{d}\nu = \int_X f_1 \, \mathsf{d}\mu \int_Y f_2 \, \mathsf{d}\nu \int_X g_1 \, \mathsf{d}\mu \int_Y g_2 \, \mathsf{d}\nu. \tag{5.2}$$

Since (5.2) is linear in f_2 we can, splitting $f_2 = \int_Y f_2 \, \mathsf{d}\nu + \left(f_2 - \int_Y f_2 \, \mathsf{d}\nu\right)$, separate the proof of (5.2) in two cases: when f_2 is a constant and when $\int_Y f_2 \, \mathsf{d}\nu = 0$. For the first case, since $\mathbf{Y}$ is ergodic, it follows that f_2 is a constant, and hence the left hand side of (5.2) is

$$\int_Y f_2 \, \mathsf{d}\nu \int_Y g_2 \, \mathsf{d}\nu \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \int_X T^n f_1 \cdot g_1 \, \mathsf{d}\mu.$$

But now, using (4), it is clear that (5.2) holds in this case.

Next we establish (5.2) in the case that $\int_Y f_2 \, \mathsf{d}\nu = 0$. Applying Cauchy-Schwarz with f_2, g_2 and using (4) we get

$$\begin{aligned}\left|\frac{1}{N}\sum_{n=1}^{N}\int_X T^n f_1 \cdot g_1 \, \mathsf{d}\mu \int_Y S^n f_2 \cdot g_2 \, \mathsf{d}\nu\right| &\leq \frac{1}{N}\sum_{n=1}^{N}\left|\int_X T^n f_1 \cdot g_1 \, \mathsf{d}\mu \int_Y S^n f_2 \cdot g_2 \, \mathsf{d}\nu\right| \\ &\leq \|f_2\|.\|g_2\| \frac{1}{N}\sum_{n=1}^{N}\left|\int_X T^n f_1 \cdot g_1 \, \mathsf{d}\mu\right|\end{aligned}$$

Using (4) we conclude that this quantity converges to 0 as $N \to \infty$, establishing (5.2).

(2)⇒(1) It suffices to show that if (2) holds, then $\mathbf{X}$ is ergodic. To see this assume that $\mathbf{X}$ is not ergodic and let $A \in \mathcal{B}$ be an invariant set such that $0 < \mu(A) < 1$. Let $\mathbf{Y} = (Y, S)$ be the (ergodic) one point system. Then $A \times Y$ is invariant for $T \times S$ and so $\mathbf{X} \times \mathbf{Y}$ wouldn't also be ergodic.

□

Remark 5.12. *Conditions (3) and (4) can be formulated using uniform Cesàro averages, and the proof presented holds in that case as well. Therefore we obtain two other equivalent properties to weak mixing.*

Exercise 5.13. *Show that the doubling map $x \mapsto 2x \mod 1$ on $[0, 1)$ with respect to the Lebesgue measure is a weak-mixing system.*

We already saw that every weak mixing system is ergodic. It turns out that it must in fact be totally ergodic.

Theorem 5.14. *Let $k \in \mathbb{N}$. A system $(X, \mathcal{B}, \mu, T)$ is weak mixing if and only if the system $(X, \mathcal{B}, \mu, T^k)$ is weak mixing.*

Proof. First suppose that $(X, \mathcal{B}, \mu, T)$ is weak mixing. To show that $(X, \mathcal{B}, \mu, T^k)$ is weak mixing we will use Condition (5) from Theorem 5.11. Let $A, B \in \mathcal{B}$ and let $E \subset \mathbb{N}$ be the set with 0 density satisfying

$$\lim_{\substack{n\to\infty \\ n\notin E}} \mu(A \cap T^{-n}B) = \mu(A)\mu(B).$$

Let $\tilde{E} := \{m \in \mathbb{N} : mk \in E\}$. It is clear that $\bar{d}(\tilde{E}) = 0$ and that

$$\lim_{\substack{m\to\infty \\ m\notin \tilde{E}}} \mu\Big(A \cap \left(T^k\right)^{-m} B\Big) = \lim_{\substack{m\to\infty \\ m\notin \tilde{E}}} \mu\big(A \cap T^{-mk}B\big) = \mu(A)\mu(B).$$

To prove the converse, suppose that $(X, \mathcal{B}, \mu, T^k)$ is weak mixing. To show that $(X, \mathcal{B}, \mu, T)$ is weak mixing we will use Condition (1) from Theorem 5.11. Indeed, if $(X, \mathcal{B}, \mu, T) \times (X, \mathcal{B}, \mu, T)$ were not ergodic, there would exist a $T \times T$ invariant set $A \subset X \times X$ with $(\mu \otimes \mu)(A) \in (0, 1)$. But A would also be invariant under $T^k \times T^k = (T \times T)^k$, and hence $(X, \mathcal{B}, \mu, T^k) \times (X, \mathcal{B}, \mu, T^k)$ would not be ergodic, contradicting the assumption. □

As we will see later, weak mixing systems enjoy very good multiple recurrence properties. For the purposes of proving Roth's theorem however, one can isolate the exact property needed for individual functions.

Definition 5.15. *Let $(X, \mathcal{B}, \mu, T)$ be a m.p.s. and let $f \in L^2(X)$. We say that f is a* **weak-mixing** *function if for every $g \in L^2$,*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \left|\langle T^k f, g\rangle\right| = 0.$$

The set of all weak-mixing functions is denoted by H_{wm}.

Notice that, in view of Theorem 5.11, a system is weak-mixing if and only if every function f with 0 integral is a weak-mixing function.

Exercise 5.16. *Show that H_{wm} is a closed T-invariant subspace of L^2.*

The following theorem is the first in the class of "multiple ergodic theorems" – extensions of the ergodic theorem involving products of functions composed with different powers of T. We will make use of the convenient notation $\underset{n}{UC\text{-}\lim}\ a_n$ to denote $\lim_{N-M\to\infty}\frac{1}{N-M}\sum_{n=M}^{N}a_n$.

Lemma 5.17. *Let $\mathbf{X}=(X,\mathcal{B},\mu,T)$ be an ergodic measure preserving system and let $f,g\in L^\infty(X)$. If either f or g (or both) is weak mixing, then*

$$\underset{n}{UC\text{-}\lim}\ T^n f\cdot T^{2n}g=0\qquad \text{in norm.}$$

To prove Lemma 5.17 we need a version of the van der Corput trick slightly stronger than Lemma 4.8.

Lemma 5.18. *Let H be a Hilbert space and let $(x_n)_{n=1}^\infty$ be a bounded sequence taking values in H. If*

$$\lim_{D\to\infty}\frac{1}{D}\sum_{d=1}^{D}\underset{n}{UC\text{-}\lim}\ \langle x_{n+d},x_n\rangle=0 \tag{5.3}$$

then

$$\underset{n}{UC\text{-}\lim}\ x_n=0.$$

Remark 5.19. *As before, this version of the van der Corput trick also holds with regular Cesàro averages, as opposed to the uniform Cesàro averages used in Lemma 5.18.*

Exercise 5.20. $(*)$ *Let $(X\,B,\mu,T)$ be a measure preserving system and let $f\in L^2$ be such that*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left|\langle T^k f,f\rangle\right|=0.$$

Prove that f is weak mixing. [Hint: Using Lemma 5.18 with $u_n=\langle T^n f,g\rangle T^n f$.]

Proof of Lemma 5.17. With the goal of using Lemma 5.18, let $u_n=T^n f\cdot T^{2n}g$. We have

$$\langle u_{n+h},u_n\rangle=\int_X T^{n+h}f\cdot T^{2n+2h}g\cdot T^n\overline{f}\cdot T^{2n}\overline{g}\,\mathsf{d}\mu=\int_X\left(T^h f\cdot\overline{f}\right)\cdot T^n\left(T^{2h}g\cdot\overline{g}\right)\mathsf{d}\mu.$$

Using ergodicity and Theorem 2.17, taking a uniform Cesàro average in n we get

$$\underset{n}{UC\text{-}\lim}\ \langle u_{n+h},u_n\rangle=\int_X T^h f\cdot\overline{f}\,\mathsf{d}\mu\int_X T^{2h}g\cdot\overline{g}\,\mathsf{d}\mu.$$

Since both sequences $h\mapsto\int_X T^h f\cdot\overline{f}\,\mathsf{d}\mu$ and $h\mapsto\int_X T^{2h}g\cdot\overline{g}\,\mathsf{d}\mu$ are bounded (by Cauchy-Schwarz inequality) and the one associated with a weak mixing function is smaller than ϵ in a set of full density (for each $\epsilon>0$) it follows that

$$\limsup_{M\to\infty}\frac{1}{M}\sum_{h=1}^{M}\left|\underset{n}{UC\text{-}\lim}\ \langle u_{n+h},u_n\rangle\right|<\epsilon$$

for every $\epsilon>0$. This of course means that the limit is 0 and the conclusion follows from Lemma 5.18. $\square$

5.3. **Finishing the proof.** The complementary notion to weak mixing functions is that of compact functions:

Definition 5.21. *Let $(X,\mathcal{B},\mu,T)$ be a m.p.s. and let $f\in L^2(X)$. We say that f is a* **compact** *or* **almost periodic** *function if the orbit closure $\overline{\{T^n f:n\in\mathbb{N}\}}\subset L^2$ is compact as a subset of L^2 with the strong topology.*

The set of all compact functions is denoted by H_c.

Exercise 5.22. *Show that in the system $(X,\mathcal{B},\mu,T)$ where $X=[0,1)$, μ is the Lebesgue measure and $T:x\mapsto x+\alpha \bmod 1$ for some irrational α, every $f\in L^2$ is compact.*

Exercise 5.23. *Show that if f is compact, then for every $\epsilon>0$ the set $\{n\in\mathbb{N}:\|T^n f-f\|<\epsilon\}$ is syndetic.*

The Jacobs-de Leeuw-Glicksberg decomposition allows us to decompose a L^2 function from an arbitrary system into the sum of a compact function and a weak mixing function.

Theorem 5.24 (Jacobs-de Leeuw-Glicksberg)**.** *In any measure preserving system $(X, \mathcal{B}, \mu, T)$, the sets H_c and H_{wm} are closed invariant subspaces of $L^2(X)$, are orthogonal and $L^2(X) = H_c \oplus H_{wm}$.*

The proof of Theorem 5.24 is omitted but can be found in several places (eg. [8, Chapter 16] is dedicated to the decomposition in far greater generality).

Exercise 5.25. $(*)$ *Show directly from the definition that H_c is a closed T-invariant subspace of L^2.*

Exercise 5.26. *Let $X = \mathbb{T}^2$ have the Borel σ-algebra and the Haar measure and let $T : (x, y) \mapsto (x+\alpha, y+x)$ for some fixed irrational α.*

(1) *Show that every function of the form $f(x, y) = e^{2\pi i n x}$ with $n \in \mathbb{Z}$ is compact.*
(2) *Show that every function of the form $f(x, y) = e^{2\pi i(nx+my)}$, with $(n, m) \in \mathbb{Z}^2$ and $m \neq 0$, is weak mixing.*
(3) *Show that the conclusion of Theorem 5.24 holds in this system (without using the theorem) by explicitly describing H_c and H_{wm}.*

It turns out that more is true about H_c.

Lemma 5.27. *Let $f, g \in L^\infty(X) \cap H_c$. Then $fg \in H_c$.*

The proof of this lemma is left as an exercise. Iterating this lemma, it follows that the space $L^\infty(X) \cap H_c$ is closed under composition with polynomials, and hence, in view of the Stone-Weierstrass theorem, under composition with continuous functions. In particular, if $f \in L^\infty(X) \cap H_c$ is real valued, then for any constant $c \in \mathbb{R}$ both $\min(f, c)$ and $\max(f, c)$ are in H_c.

Corollary 5.28. *Let $(X, \mathcal{B}, \mu, T)$ be a m.p.s. and suppose $f \in L^2(X)$ is real valued and satisfies $f(x) \in [0, 1]$ for all $x \in X$. Let $f = f_c + f_{wm}$ be the decomposition of f arising from Theorem 5.24. Then $f_c(x) \in [0, 1]$ for (almost) all $x \in X$.*

Proof. Let $g = \min(f_c, 1)$. By the discussion above, $g \in H_c$. Since f takes values in $[0, 1]$ it follows that $\|f - g\| \leq \|f - f_c\|$. Since, according to Theorem 5.24, f_c is the orthogonal projection of f onto H_c, we conclude that $f_c = g$ and hence $f_c(x) \leq 1$ for almost all $x \in X$. A similar argument shows that $f_c(x) \geq 0$ for almost all $x \in X$. $\square$

We are now ready to prove Theorem 5.1 when the system is ergodic. In fact, we shall prove the following stronger statement.

Theorem 5.29. *Let $\mathbf{X} = (X, \mathcal{B}, \mu, T)$ be ergodic and let $A \in \mathcal{B}$ have $\mu(A) > 0$. Then*

$$UC\text{-}\lim_n \mu(A \cap T^{-n}A \cap T^{-2n}A) > 0. \tag{5.4}$$

Proof. Use Theorem 5.24 to decompose $1_A = f_c + f_w$ into $f_c \in H_c$ and $f_w \in H_w$. In view of Corollary 5.28, f_c takes values in $[0, 1]$. Moreover, since $1 \in H_c$ and hence $1 \perp f_w$, we deduce that $\int_X f_c \, \mathsf{d}\mu = \langle f_c, 1\rangle = \langle 1_A, 1\rangle = \mu(A) > 0$. Therefore f_c is not a.e. 0 and so we can use Exercise 5.23 with $\epsilon = \int_X f_c^3 \, \mathsf{d}\mu/2$ (say) to find a syndetic set $S \subset \mathbb{N}$ such that for any $n \in S$, $\|T^n f_c - f_c\| < \epsilon$. Since T preserves the measure, it follows that for $n \in S$ we also have $\|T^{2n} f_c - f_c\| < 2\epsilon$ and hence, using Jensen's inequality,

$$\int_X f_c \cdot T^n f_c \cdot T^{2n} f_c \, \mathsf{d}\mu > \int_X f_c^3 \, \mathsf{d}\mu - \epsilon > 0.$$

Using Exercise 2.23 we deduce that

$$UC\text{-}\lim_n \int_X f_c \cdot T^n f_c \cdot T^{2n} f_c \, \mathsf{d}\mu > 0.$$

Next, using Lemma 5.27 it follows that $T^n f_c \cdot T^{2n} f_c \in H_c$ and therefore it is orthogonal to H_{wm}. In particular, for every $n \in \mathbb{N}$, $\big(T^n f_c \cdot T^{2n} f_c\big) \perp f_w$ and hence

$$UC\text{-}\lim_n \int_X 1_A \cdot T^n f_c \cdot T^{2n} f_c \, \mathsf{d}\mu > 0. \tag{5.5}$$

Next we use Lemma 5.17 3 times to deduce that

$$UC\text{-}\lim_n \int_X 1_A \cdot T^n f_w \cdot T^{2n} f_c \, \mathsf{d}\mu = 0. \tag{5.6}$$

$$UC\text{-}\lim_n \int_X 1_A \cdot T^n f_c \cdot T^{2n} f_w \, \mathsf{d}\mu = 0. \tag{5.7}$$

$$UC\text{-}\lim_n \int_X 1_A \cdot T^n f_w \cdot T^{2n} f_w \, \mathsf{d}\mu = 0. \tag{5.8}$$

Finally, adding (5.5), (5.6), (5.7) and (5.8) we obtain (5.4). $\square$

6. Proof of Szemerédi's theorem

In this section we sketch the proof of Szemerédi's theorem and formulate the main steps. Recall that we already showed how the combinatorial statement Theorem 1.9 is equivalent to the multiple recurrence theorem Theorem 3.3. As was the case with other multiple recurrence theorems, we actually prove a stronger statement involving averages.

Theorem 6.1. *Let $(X, \mathcal{B}, \mu, T)$ be an ergodic system and let $A \in \mathcal{B}$ with $\mu(A) > 0$. Then for every $k \in \mathbb{N}$,*

$$\liminf_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N} \mu(A \cap T^{-n}A \cap T^{-2n}A \cap \cdots \cap T^{-kn}A) > 0. \tag{6.1}$$

It turns out that the liminf in (6.1) is an actual limit, but this was not proved until 2005 [16], almost 30 years after Theorem 6.1 was first established. Observe that in Theorem 6.1 we assume that the system is ergodic; but as was explained in the previous section this is just for convenience and one can still obtain Theorem 3.3.

6.1. Kronecker factor.

Let X be a compact metrizable abelian group, with the Borel σ-algebra and Haar measure and let $\alpha \in X$. Then the map $T : X \to X$ given by $Tx = x + \alpha$ is a measure preserving transformation. This transformation is ergodic precisely when α generates a dense subgroup of X. Systems of this form are called ***Kronecker systems***.

Theorem 6.2. *An ergodic system $(X, \mathcal{B}, \mu, T)$ is (isomorphic[4] to) a Kronecker system if and only if every $f \in L^2$ is compact (in the sense of Definition 5.21).*

Recall that a measure preserving system $\mathbf{Y} = (Y, \mathcal{B}, \nu, S)$ is a factor of another system $\mathbf{X} = (X, \mathcal{A}, \mu, T)$ if there exists a measurable map $\phi : X \to Y$ pushing μ to ν (i.e. satisfying $\mu(\phi^{-1}B) = \nu(B)$ for every $B \in \mathcal{B}$) and intertwining T and S, in the sense that $S \circ \phi = \phi \circ T$. In this situation one can embed $L^2(Y)$ into $L^2(X)$ by taking $f \in L^2(Y)$ to $f \circ \pi \in L^2(X)$; noting that this map is an isometric operator. We will then assume simply that $L^2(Y) \subset L^2(X)$. Let H_c denote the set of all compact functions in $L^2(X)$. If $\mathbf{Y}$ is a Kronecker system, then every function in $L^2(Y)$ is compact, and hence $L^2(Y) \subset H_c$. Conversely, if $\mathbf{Y}$ is a factor of $\mathbf{X}$ with $L^2(Y) \subset H_c$, then $\mathbf{Y}$ is a Kronecker system.

It turns out that every ergodic measure preserving system $\mathbf{X} = (X, \mathcal{A}, \mu, T)$ has a factor $\mathbf{Y} = (Y, \mathcal{B}, \nu, S)$ such that $L^2(Y) = H_c$; such factor is called the ***Kronecker factor*** of $\mathbf{X}$.

Theorem 6.3 (Kronecker factor)**.** *Let $\mathbf{X} = (X, \mathcal{B}, \mu, T)$ be an ergodic system and let H_c denote the space of all compact functions in $L^2(X)$. Then there exists a factor map $\pi : \mathbf{X} \to \mathbf{Y}$ such that*

- $\mathbf{Y}$ *is a Kronecker system.*
- $L^2(Y) = H_c$.
- $\mathbf{Y}$ *is the maximal Kronecker factor of $\mathbf{X}$ in the sense that if $\mathbf{Z}$ is a Kronecker system and there is a factor map $\phi : \mathbf{X} \to \mathbf{Z}$, then there exists a factor map $\psi : \mathbf{Y} \to \mathbf{Z}$ such that $\phi = \psi \circ \pi$.*
- $\mathbf{Y}$ *is unique up to isomorphism, and can be described as $\mathbf{Y} = (X, \mathcal{D}, \mu, T)$, where $\mathcal{D} = \{D \in \mathcal{B} : 1_D \in H_c\}$.*

It is possible that the Kronecker factor of a system is trivial (i.e., isomorphic to the identity transformation). This occurs precisely when H_c consists only of constant functions, which in view of Theorem 5.24 is equivalent to the system being weak mixing.

[4]Two systems $\mathbf{X}$ and $\mathbf{Y}$ are isomorphic if there exists a bijective factor map $\pi : \mathbf{X} \to \mathbf{Y}$ whose inverse is also a factor map.

Remark 6.4. *In view of Theorem 6.3, one can re-interpret the Jacobs-de Leeuw-Glicksberg in terms of conditional expectations. Indeed, given $f \in L^2(X)$ and writing $f = f_c + f_{wm}$ with $f_c \in H_c$ and $f_{wm} \in H_{wm}$, we know that f_c is the orthogonal projection of f onto the space H_c, which now we know is the same space as the space $L^2(X, \mathcal{D}, \mu)$ (where $\mathcal{D}$ is the σ-algebra described in the last item of Theorem 6.3) of L^2 functions that are measurable with respect to $\mathcal{D}$. It follows that $f_c = \mathbb{E}[f \mid \mathcal{D}]$. This is often denoted by $\mathbb{E}[f \mid \mathbf{Y}]$, and we can then re-formulate Theorem 5.24 as stating that for any ergodic system $\mathbf{X}$ with Kronecker factor $\mathbf{Y}$ and any $f \in L^2(X)$, the difference $f - \mathbb{E}[f \mid \mathbf{Y}]$ is a weak mixing function.*

6.2. **Special cases of multiple recurrence.** We start by proving Theorem 6.1 when the system is weak mixing.

Theorem 6.5. *Let $\mathbf{X} = (X, \mathcal{B}, \mu, T)$ be a weak mixing system, let $k \in \mathbb{N}$ and let $f_1, \dots, f_k \in L^\infty(X)$. Then*

$$UC\text{-}\lim_n \prod_{i=1}^k T^{ni} f_i = \prod_{i=1}^k \int_X f_i \, \mathsf{d}\mu \qquad \text{in } L^2(X). \tag{6.2}$$

Replacing each f_i with 1_A, a direct corollary of Theorem 6.5 is that Theorem 6.1 holds whenever the system is weak mixing.

The main idea in the proof of Theorem 6.5 is to use the van der Corput trick.

Proof of Theorem 6.5. We proceed by induction on k. The case $k = 1$ follows from the mean ergodic theorem (Theorem 2.17). Assume now that $k > 1$ and the result has been established for $k - 1$. Splitting f_k as the sum of a constant and function with 0 integral, we reduce the proof to those two cases. If f_k is a constant, then (6.2) follows immediately by induction.

Assume next that $\int_X f_k \, \mathsf{d}\mu = 0$. Since the right hand side of (6.2) is 0, we will use the van der Corput trick. Let $u_n := \prod_{i=1}^k T^{ni} f_i$. We have

$$\langle u_{n+h}, u_n \rangle = \int_X \prod_{i=1}^k T^{(n+h)i} f_i \cdot \overline{T^{ni} f_i} \, \mathsf{d}\mu = \int_X \prod_{i=1}^k T^{ni}\big(T^{hi} f_i \cdot \overline{f_i}\big) \, \mathsf{d}\mu = \int_X T^h f_1 \cdot \overline{f_1} \prod_{i=2}^k T^{n(i-1)}\big(T^{hi} f_i \cdot \overline{f_i}\big) \, \mathsf{d}\mu,$$

where the last equality follows from the fact that T^n preserves the measure. After using induction hypothesis on the $k-1$ functions $\{T^h f_2 \cdot \overline{f_2}, \dots, T^{h(k-1)} f_k \cdot \overline{f_k}\}$ and taking averages we get

$$\begin{aligned} UC\text{-}\lim_n \langle u_{n+h}, u_n \rangle &= UC\text{-}\lim_n \int_X T^h f_1 \cdot \overline{f_1} \prod_{i=2}^k T^{n(i-1)}\big(T^{hi} f_i \cdot \overline{f_i}\big) \, \mathsf{d}\mu \\ &= \prod_{i=1}^k \int_X T^{hi} f_i \cdot \overline{f_i} \, \mathsf{d}\mu. \end{aligned}$$

Finally, taking an average on h and using Theorem 5.14 and condition (4) from Theorem 5.11 we obtain

$$\left| C\text{-}\lim_h UC\text{-}\lim_n \langle u_{n+h}, u_n \rangle \right| = \left| C\text{-}\lim_h \prod_{i=1}^k \int_X T^{hi} f_i \cdot \overline{f_i} \, \mathsf{d}\mu \right| \le \prod_{i=1}^{k-1} \|f_i\|^2 \cdot C\text{-}\lim_h \left|\langle T^{hk} f_k, f_k \rangle\right| = 0$$

□

On the other end of the spectrum, we have Kronecker systems.

Theorem 6.6. *Let $\mathbf{X}$ be a Kronecker system, let $k \in \mathbb{N}$ and let $f \in L^\infty(X)$. Then for every $\epsilon > 0$, the set*

$$\left\{ n \in \mathbb{N} : \int_X \prod_{i=0}^k T^{ni} f \, \mathsf{d}\mu > \int_X f^{k+1} \, \mathsf{d}\mu - \epsilon \right\}$$

is syndetic.

To see why Theorem 6.6 implies that Theorem 6.1 holds for Kronecker systems, apply Theorem 6.6 to the indicator function 1_A of a set $A \in \mathcal{B}$ with $\mu(A) > 0$ and let S be the syndetic set of n for which $\mu(A \cap T^{-n} A \cap \dots \cap T^{-kn} A) > \mu(A)/2$. Let $L \in \mathbb{N}$ be bound on the gaps of S (so that every interval of length L contains an element of S). Then for $N-M$ large enough we have $|[M,N]\cap S|\geq (N-M-L)/L > (N-M)/(2L)$ and hence

$$UC\text{-}\lim_n \mu(A\cap\cdots\cap T^{-kn}A)\geq \lim_{N-M\to\infty}\frac{1}{N-M}\sum_{n\in S\cap[M,N]}\mu(A\cap\cdots\cap T^{-kn}A)\geq\frac{\mu(A)}{4L}>0.$$

Proof of Theorem 6.6. Let $\mathbf{X}$, k, f and $\epsilon>0$ be as in the statement of the theorem. By re-scaling we may assume that $\|f\|_\infty\leq 1$. Since f is a compact function, it follows from Exercise 5.23 that the set $S:=\{n\in\mathbb{N}:\|T^nf-f\|<\epsilon/k^2\}$ is syndetic. Observe that for each $n\in S$ and $i\in\{0,1,\dots,k\}$ we have

$$\|T^{in}f-f\|\leq\|T^{in}f-T^{(i-1)n}f\|+\|T^{(i-1)n}f-T^{(i-2)n}f\|+\cdots+\|T^nf-f\|=i\|T^nf-f\|\leq\frac{\epsilon}{k}.$$

Using the Cauchy-Schwarz inequality repeatedly we conclude that for every $n\in S$

$$\begin{aligned}\int_X\prod_{i=0}^kT^{ni}f\,\mathsf{d}\mu &= \int_X f\cdot\prod_{i=1}^kT^{ni}f\,\mathsf{d}\mu\geq\int_X f^2\cdot\prod_{i=2}^kT^{ni}f\,\mathsf{d}\mu-\frac{\epsilon}{k}\\ &\geq \int_X f^3\cdot\prod_{i=3}^kT^{ni}f\,\mathsf{d}\mu-\frac{2\epsilon}{k}\geq\cdots\geq\int_X f^{k+1}\,\mathsf{d}\mu-\epsilon.\end{aligned}$$

□

We have now shown that either a system $\mathbf{X}$ is weak mixing, and hence Theorem 6.1 holds, or it is not weak mixing, and hence it has a non-trivial Kronecker factor, where Theorem 6.1 holds. In either case, we have proved that any ergodic system has a non-trivial factor where Theorem 6.1 holds. The basic idea of the proof of Theorem 6.1 for general systems is to keep finding larger factors where the conclusion holds, ultimately covering all of $\mathbf{X}$. To make the necessary definitions we will take advantage of a useful theorem of Rokhlin.

6.3. **Rokhlin's skew-product lemma.** Two probability spaces $(X,\mathcal{B},\mu)$ and $(Y,\mathcal{D},\nu)$ are isomorphic if the (trivial) measure preserving systems $(X,\mathcal{B},\mu,Id)$ and $(Y,\mathcal{D},\nu,Id)$ are isomorphic. It is a known result that if μ is a Borel measure on a compact metric space with no point masses, then $(X,\mathcal{B},\mu)$ is isomorphic to $[0,1]$ with the Borel σ-algebra and the Lebesgue measure. In particular any two such probability spaces are isomorphic! It follows in particular that, when understood as probability spaces with the (appropriate) Lebesgue measure, $[0,1]$ is isomorphic to $[0,1]^2$. The following lemma improves upon these ideas and provides a useful way to understand factors.

Lemma 6.7. *Let $\mathbf{X}=(X,\mathcal{B},\mu,T)$ be an ergodic measure preserving system and let $\mathbf{Y}=(Y,\mathcal{C},\nu,S)$ be a factor, with factor map $\pi:X\to Y$. Then there exists a probability space $(Z,\mathcal{D},\lambda)$ and a measurable map $\rho:Y\to\mathrm{Aut}(Z)$ (called a co-cycle) taking values in the set of measure preserving transformations of $(Z,\mathcal{D},\lambda)$ such that $\mathbf{X}$ is isomorphic to $(Y\times Z,\mathcal{C}\otimes\mathcal{D},\nu\otimes\lambda,R)$, where $R(y,z)=\big(Sy,\rho(y)(z)\big)$ (in other words, $\mathbf{X}$ is a skew-product over $\mathbf{Y}$).*

Example 6.8. *Take $X=[0,1]^2$ with the (Borel) Lebesgue measure and let $T:(y,x)\mapsto(y+\alpha,x+y)$ for some fixed irrational α. Let $Y=[0,1]$, also endowed with the (Borel) Lebesgue measure and let $S:y\mapsto y+\alpha$. Then the projection $\pi:X\to Y$ onto the first coordinate is a factor map of measure preserving systems.*

In this case we can take $Z=[0,1]$ and $\rho(y)$ to be the rotation on Z by y (in other words $\rho(y):z\mapsto z+y\bmod 1$).

6.4. **Relative weak mixing and compactness.** Recall that a m.p.s. $\mathbf{X}$ is weak mixing if and only if the product $\mathbf{X}\times\mathbf{X}$ is ergodic. The following definition extends this concept to a relative notion.

Definition 6.9. *Let $\mathbf{X}=(X,\mathcal{B},\mu,T)$ and $\mathbf{Y}=(Y,\mathcal{C},\nu,S)$ be ergodic systems and let $\pi:\mathbf{X}\to\mathbf{Y}$ be a factor map. Let $(Z,\mathcal{D},\lambda)$ and ρ be given by Lemma 6.7. The* **relative product** *of $\mathbf{X}$ with itself over $\mathbf{Y}$ is the system*

$$\mathbf{X}\times_{\mathbf{Y}}\mathbf{X}:=(Y\times Z\times Z,\mathcal{C}\otimes\mathcal{D}\otimes\mathcal{D},\nu\otimes\lambda\otimes\lambda,T\times_{\mathbf{Y}}T),$$

where $T\times_{\mathbf{Y}}T:(y,z_1,z_2)\mapsto(Sy,\rho(y)z_1,\rho(y)z_2)$.

We say that $\mathbf{X}$ is **weak mixing relative to** *(or a* **weak mixing extension** *of) $\mathbf{Y}$ is the relative product $\mathbf{X}\times_{\mathbf{Y}}\mathbf{X}$ is ergodic.*

Exercise 6.10. *Show that a system is weak mixing if and only if it is relative weak mixing with respect to the trivial factor (i.e. the factor to the one-point system).*

Next recall that a system is a Kronecker system if and only if every L^2 function is compact, i.e., for any $f \in L^2$ the orbit $\{T^n f : n \in \mathbb{N}\}$ is pre-compact. For a subset of a Hilbert space, pre-compact is equivalent to totally bounded, so f is compact if and only if for any $\epsilon > 0$ there are finitely many functions $g_1, \ldots, g_r \in L^2$ such that $\{T^n f : n \in \mathbb{N}\} \subset \bigcup_{i=1}^r B(g_i, \epsilon)$. This inclusion can be written as

$$\forall n \in \mathbb{N} \quad \min_{1 \le i \le r} \|T^n f - g_i\|_{L^2(\mu)} < \epsilon.$$

We can now relativize the notion of a compact (or Kronecker) factor.

Definition 6.11. *Let $\mathbf{X} = (X, \mathcal{B}, \mu, T)$ and $\mathbf{Y} = (Y, \mathcal{C}, \nu, S)$ be ergodic systems and let $\pi : \mathbf{X} \to \mathbf{Y}$ be a factor map. Let $(Z, \mathcal{D}, \lambda)$ be given by Lemma 6.7 and consider the measures $\mu_y = \delta_y \otimes \lambda$ for each $y \in Y$.*

A function $f \in L^2(X)$ is **compact relative to Y** *if for every $\epsilon > 0$ there are finitely many functions $g_1, \ldots, g_r \in L^2(X)$ such that*

$$\forall n \in \mathbb{N} \quad \min_{1 \le i \le r} \|T^n f - g_i\|_{L^2(\mu_y)} < \epsilon \quad \text{for } \nu\text{-a.e. } y.$$

The system $\mathbf{X}$ is **compact relative to** *(or a* **compact extension of***) $\mathbf{Y}$ if there is a dense set of relatively compact functions in $L^2(X)$.*

Exercise 6.12. *Show that an ergodic system is a Kronecker system if and only if it is compact relative to the trivial (one point) factor.*

Exercise 6.13. *Show that the system $\mathbf{X} = (X, \mathcal{B}, \mu, T)$ given by $X = [0,1]^2$, $\mathcal{B} = \text{Borel}$, $\mu = \text{Lebesgue}$ and $T : (x, y) \mapsto (x + \alpha, y + x)$, where α is irrational, is a compact extension of the rotation by α (i.e. the system $\mathbf{Y} = (Y, \mathcal{D}, \nu, S)$ where $Y = [0,1]$, $\mathcal{D} = \text{Borel}$, $\nu = \text{Lebesgue}$ and $S : x \mapsto x + \alpha$).*

Note that in Definition 6.11 we do not require that every function $f \in L^2(X)$ be compact relative to $\mathbf{Y}$ but only that a dense subset of L^2 has this property. The next exercise helps explain why.

The notation $\lfloor x \rfloor$ for a real number x, denotes the largest integer n such that $n \le x$.

Exercise 6.14. *Let $\mathbf{X}$ be as in the previous exercise. Show that the function $f(x, y) = e(y \lfloor 1/x \rfloor)$ is not conditionally compact with respect to $\mathbf{Y}$.*

It is often helpful to think of "compact" as a generalization of "finite". The next exercise explains in which sense the notion of "relatively compact" generalizes the notion of "relatively finite".

Exercise 6.15. *Let $\mathbf{X}$ and $\mathbf{Y}$ be ergodic systems and let $\pi : \mathbf{X} \to \mathbf{Y}$ be a factor map. Let $(Z, \mathcal{D}, \lambda)$ be given by Lemma 6.7 and suppose that Z is finite. Show that $\mathbf{X}$ is a compact extension of $\mathbf{Y}$.*

6.5. Sketch of the proof.

Definition 6.16 (Sz systems)**.** *An ergodic system $(X, \mathcal{B}, \mu, T)$ is called* **Sz** *(for Szemerédi) if it satisfies the conclusion of the Theorem 6.1*

We already saw that every ergodic system has a non-trivial factor that is ***Sz*** (either the whole system if it is weak mixing, or its non-trivial Kronecker factor otherwise). The idea of the proof of Theorem 6.1 is that any proper factor which is Sz is contained in a strictly larger factor which is also Sz. There are 3 main components. The first states that the Sz property can be lifted by weak mixing extensions.

Theorem 6.17. *If an ergodic system $\mathbf{X}$ is a weak mixing extension of a system $\mathbf{Y}$ and $\mathbf{Y}$ is Sz, then so is $\mathbf{X}$.*

The proof of Theorem 6.17 is very similar to the proof of Theorem 6.5 which dealt with the "absolute" case. In particular it combines a similar induction with the van der Corput trick.

Similarly, the next results states that the Sz property can be lifted by compact extensions.

Theorem 6.18. *If an ergodic system $\mathbf{X}$ is a compact extension of a system $\mathbf{Y}$ and $\mathbf{Y}$ is Sz, then so is $\mathbf{X}$.*

The proof of Theorem 6.18 draws on the ideas from the proof of Theorem 6.6 which dealt with the "absolute" case, but is more complicated and requires additional insights. One useful (but not entirely necessary) tool is the van der Waerden theorem, in the finitistic form described in Exercise 1.5.

The third major step is a relative version of the Jacobs-de Leeuw-Glicksberg decomposition. Recall that a consequence of this decomposition is that whenever a system $\mathbf{X}$ is not weak mixing, it has a non-trivial factor $\mathbf{Y}$ which is a Kronecker system.

Theorem 6.19. *If a non-trivial extension $\pi : \mathbf{X} \to \mathbf{Y}$ of ergodic systems is not relatively weak mixing, then there exists an intermediate relatively compact extension, i.e., there exists a system $\mathbf{Z}$ and factor maps $\pi_1 : \mathbf{X} \to \mathbf{Z}$ and $\pi_2 : \mathbf{Z} \to \mathbf{Y}$ such that $\pi = \pi_2 \circ \pi_1$, π_2 is non-trivial and $\mathbf{Z}$ is a compact extension of $\mathbf{Y}$.*

We are now ready to finish the proof of Theorem 6.1. The idea is to consider a maximal factor of $\mathbf{X}$ which is a Sz system. Here maximal means with respect to the natural partial order on all factors of $\mathbf{X}$ given by $\mathbf{Y} \prec \mathbf{Z}$ if $\mathbf{Y}$ is a factor of $\mathbf{Z}$.[5] An equivalent description of this partial order is obtained by corresponding each factor to a subset of $L^2(X)$ (as explained after Theorem 6.2); then $\mathbf{Y} \prec \mathbf{Z}$ if and only if $L^2(Y) \subset L^2(Z)$.

To consider a maximal Sz factor of $\mathbf{X}$ one can use Zorn's lemma (or, alternatively, a transfinite induction). To be able to apply this lemma, one needs to show that the Sz property is "closed", in the following sense.

Lemma 6.20. *Let $\mathbf{X}$ be an ergodic system and suppose that there is a totally ordered family of factors $\mathbf{Y}_\alpha$ such that $\bigcup_\alpha L^2(\mathbf{Y}_\alpha)$ is dense in $L^2(\mathbf{X})$. If every $\mathbf{Y}_\alpha$ is Sz, then so is $\mathbf{X}$.*

Lemma 6.20 implies that Zorn's lemma can be applied and hence that $\mathbf{X}$ has a maximal Sz factor, say $\mathbf{Y}$. If $\mathbf{X}$ is a non-trivial extension of $\mathbf{Y}$ there are two cases. In the first case, $\mathbf{X}$ is a weak mixing extension of $\mathbf{Y}$ and hence by Theorem 6.17 $\mathbf{X}$ is Sz. In the second case, $\mathbf{X}$ is not a weak mixing extension of $\mathbf{Y}$, and hence by Theorem 6.19 there is a non-trivial extension $\mathbf{Z}$ of $\mathbf{Y}$ which is a compact extension of $\mathbf{Y}$ and a factor of $\mathbf{X}$. By Theorem 6.18, $\mathbf{Z}$ is Sz, but this contradicts the fact that $\mathbf{Y}$ was the maximal factor of $\mathbf{X}$ that was Sz.

Therefore, $\mathbf{X}$ must be Sz itself and this finishes the proof.

7. Extensions of Szemerédi's theorem

Shortly after Furstenberg published his ergodic theoretic proof of Szemerédi's theorem, in joint work with Katznelson they established a multidimensional version. For many years, the only known proofs of this multidimensional Szemerédi theorem (Theorem 7.1 below) involved ergodic theory.

Let $d \in \mathbb{N}$. Given a set $A \subset \mathbb{N}^d$, its upper density is defined by

$$\bar{d}(A) = \limsup_{N\to\infty} \frac{1}{N^d} \big|A \cap \{1, \ldots, N\}^d\big|.$$

Theorem 7.1 (Furstenberg-Katznelson [11])**.** *If $A \subset \mathbb{N}$ has $\bar{d}(A) > 0$, then for every finite set $F \subset \mathbb{N}^d$ there exists $n \in \mathbb{N}$ and $x \in \mathbb{N}^d$ such that*

$$A \supset x + nF := \{x + nv : v \in F\}.$$

For instance, if $d = 2$ and $F = \{0, 1, \ldots, k\}^2$, it follows from Theorem 7.1 that any subset of $\mathbb{N}^2$ with positive upper density contains a square $k \times k$ grid.

Exercise 7.2. *Show that, using only Szemerédi's theorem, one can deduce that any subset of $\mathbb{N}^2$ with positive upper density contains a rectangular $k \times k$ grid, i.e. a set of the form*

$$\big\{(x_1, x_2) + (in, jm) : 1 \le i, j \le k\big\}$$

for some $x_1, x_2, n, m \in \mathbb{N}$.

Here's the multiple recurrence theorem they established.

Theorem 7.3. *Let $(X, \mathcal{B}, \mu)$ be a probability space and let $T_1, \ldots, T_d : X \to X$ be commuting measure preserving transformations. Then for any $A \subset \mathcal{B}$ with $\mu(A) > 0$ there exists $n \in \mathbb{N}$ such that*

$$\mu\big(A \cap T_1^{-n} A \cap \cdots \cap T_d^{-n} A\big) > 0.$$

[5]To more precise we also need the factor maps between $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$ to be compatible.

Exercise 7.4. *Show that Theorem 7.3 implies that, under the same conditions, for every $k \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that*

$$\mu\left(A \cap \bigcap_{i=1}^{d} \bigcap_{j=1}^{k} T_i^{-jn} A\right) > 0.$$

To show that Theorem 7.3 implies Theorem 7.1, one needs a suitable extension of the Correspondence Principle.

Proposition 7.5. *Let $d \in \mathbb{N}$ and $E \subset \mathbb{N}^d$. Then there exists a probability space $(X, \mathcal{B}, \mu)$, commuting measure preserving transformations $T_1, \ldots, T_d$ on X and a set $A \in \mathcal{B}$ such that $\mu(A) = \bar{d}(E)$ and for any $n_1, \ldots, n_k \in \mathbb{N}^d$, say $n_i = (n_{i,1}, \ldots, n_{i,d})$, we have*

$$\bar{d}\big(E \cap (E - n_1) \cap \cdots \cap (E - n_k)\big) \geq \mu\left(A \cap \bigcap_{i=1}^{k} T_1^{-n_{i,1}} T_2^{-n_{i,2}} \cdots T_d^{-n_{i,d}} A\right)$$

Exercise 7.6. *Adapt the proof of Theorem 3.4 to give a proof of Proposition 7.5. [Hint: Take $X = \{0,1\}^{N_0^d}$, let T_i be the shift in the i-th direction and let $A = \{x \in X : x_{(0,\ldots,0)} = 1\}$.]*

Exercise 7.7. *Show that Theorem 7.1 follows from combining Theorem 7.3 with Proposition 7.5.*

The proof of Theorem 7.3 follows the same basic structure as the proof of Theorem 3.3. In particular, it uses the idea of exhausting the system $(X, \mathcal{B}, \mu, T_1, \ldots, T_d)$ by weak mixing and compact extensions; although in this situation one also needs to consider more general behaviour.

Later Bergelson and Leibman proved the polynomial version of Szemerédi's theorem, Theorem 1.18. In fact they proved a multidimensional version as well. The polynomial Szemerédi theorem is deduced (using the Correspondence Principle) from the following polynomial multiple recurrence result:

Theorem 7.8. *Let $(X, \mathcal{B}, \mu, T)$ be an invertible measure preserving system and let $p_1, \ldots, p_k \in \mathbb{Z}[x]$ satisfy $p_i(0) = 0$. Then for any $A \in \mathcal{B}$ with $\mu(A) > 0$ there exists $n \in \mathbb{N}$ such that*

$$\mu\big(A \cap T^{-p_1(n)} A \cap \cdots \cap T^{-p_k(n)} A\big) > 0.$$

The proof of Theorem 7.8 follows the strategy implemented by Furstenberg, and in particular uses directly Theorem 6.19 and analogues of Theorems 6.17 and 6.18. To lift the polynomial recurrence property over weak-mixing extensions one can use the van der Corput trick and a similar argument to the linear case. However, in order to lift the polynomial recurrence property over compact extensions, one requires a suitable version of the van der Warden theorem (Theorem 1.4).

Theorem 7.9. *Let $p_1, \ldots, p_k \in \mathbb{Z}[x]$ satisfy $p_i(0) = 0$. For any finite partition $\mathbb{N} = C_1 \cup \cdots \cup C_r$ of $\mathbb{N}$ there exists $x, n \in \mathbb{N}$ and $C \in \{C_1, \cdots, C_r\}$ such that*

$$\{x, x + p_1(n), \cdots, x + p_k(n)\} \subset C.$$

It is clear the Theorem 7.9 is a corollary of Theorem 1.18; however it is required to prove Theorem 1.18, so one needs to be able to prove Theorem 7.9 directly.

8. Coloring theorems and Topological Dynamics

It turns out that to prove coloring results such as van der Waerden's theorem, ergodic theory isn't as suitable as another branch of dynamics, called topological dynamics.

Definition 8.1. *A* **topological dynamical system** *(or simply* **system***) is a pair (X, T) where X is a compact metric space and $T : X \to X$ is continuous.*

Given a system (X, T), any closed set $Y \subset X$ satisfying $TY \subset Y$ gives rise to a subsystem (Y, T).

Definition 8.2. *A system (X, T) is* **minimal** *if there is no proper subsystem.*

An application of Zorn's lemma shows that any topological dynamical system has a minimal subsystem.

Exercise 8.3. *Show that a system (X, T) is minimal if and only if every point $x \in X$ has a dense orbit (the orbit of a point $x \in X$ is the set $\{T^n x : n \in \mathbb{N}\}$).*

Exercise 8.4. *Show that if a system (X, T) is minimal then $T : X \to X$ is surjective.*

Proposition 8.5. *If (X, T) is minimal and $A \subset X$ is open and non-empty, then there exists $n \in \mathbb{N}$ such that $A \cap T^{-n}A \neq \emptyset$.*

Proof. The set $B := X \setminus \bigcup_{n\in\mathbb{N}} T^{-n}A$ is closed and $TB \subset B$. Since $A \neq \emptyset$, $B \neq X$, and hence by minimality $B = \emptyset$. It follows that $\bigcup_{n\in\mathbb{N}} T^{-n}A = X$ and hence some $T^{-n}A$ must have non-empty intersection with A. □

The connection between coloring theorems and topological dynamics is given by the following instance of the correspondence principle.

Proposition 8.6. *Let $\mathbb{N} = C_1 \cup \cdots \cup C_r$ be an arbitrary finite coloring of $\mathbb{N}$. There exists a minimal topological dynamical system (X, T) and a cover $X = A_1 \cup \cdots \cup A_r$ by open sets such that for any $n_1, \ldots, n_k \in \mathbb{N}$ and any $i \in \{1, \ldots, r\}$,*

$$A_i \cap T^{-n_1}A_i \cap \cdots \cap T^{-n_k}A_i \neq \emptyset \quad \Rightarrow \quad C_i \cap (C_i - n_1) \cap \cdots \cap (C_i - n_k) \neq \emptyset$$

Proof. Let $X_0 = \{1, \ldots, r\}^{\mathbb{N}_0}$, let $T : X_0 \to X_0$ be the left shift and let $\chi \in X_0$ be the function $\chi = \sum i 1_{C_i}$. Let $X_1 = \overline{T^n\chi : n \in \mathbb{N}}$ be the orbit closure of χ, notice that (X_1, T) is a subsystem of (X_0, T), and let $X \subset X_1$ be a minimal subsystem.

Let $A_i := \{x \in X : x_0 = i\}$. If $y \in A_i \cap T^{-n_1}A_i \cap \cdots \cap T^{-n_k}A_i$ for some i and $n_1, \ldots, n_k$, then $y_0 = y_{n_1} = \cdots = y_{n_k} = i$. Since $y \in X \subset X_1$, there exists a point $T^n\chi$ in the orbit of χ such that $(T^n\chi)_m = y_m$ for every $m \leq n_k$. In particular $(T^n\chi)_{n_j} = i$ for every $j = 0, \ldots, k$ (where for convenience we define $n_0 = 0$) which means that $\chi_{n+n_j} = i$ for every j and hence that $n \in C_i \cap (C_i - n_1) \cap \cdots \cap (C_i - n_k)$. □

In view of Proposition 8.6, van der Warden's theorem follows from the following multiple recurrence theorem.

Theorem 8.7. *Let (X, T) be a minimal system and $X = C_1 \cup \cdots \cup C_r$ a finite open cover of X. Then for every $k \in \mathbb{N}$ there exists $n \in \mathbb{N}$ and $i \in \{1, \ldots, r\}$ such that*

$$C_i \cap T^{-n}C_i \cap \cdots \cap T^{-kn}C_i \neq \emptyset.$$

Exercise 8.8. *Using Proposition 8.6, show that Theorems 1.4 and 8.7 are equivalent.*

[Hint: To show that Theorem 1.4 implies Theorem 8.7, take any point x in X and construct a coloring of $\mathbb{N}$ by looking at the orbit of x.]

We assume minimality in the statement of Theorem 8.7 because it makes the proof easier (similar to how we assume ergodicity in the proof of Theorem 6.1). However it can be shown directly that this assumption can be discarded.

Exercise 8.9. *Show that in Theorem 8.7, the assumption that (X, T) is minimal is not needed.*

It turns out that Theorem 8.7 is equivalent to a version closer to Theorem 3.3.

Lemma 8.10. *Suppose Theorem 8.7 holds for some $k \in \mathbb{N}$. Then for any minimal system (X, T) and any open $A \subset X$, if $A \neq \emptyset$ then there exists $n \in \mathbb{N}$ such that $A \cap T^{-n}A \cap \cdots \cap T^{-kn}A \neq \emptyset$.*

Proof. As we've seen above, $X = \bigcup_{i\in\mathbb{N}} T^{-i}A$. By compactness it follows that $X = \bigcup_{i=1}^{N} T^{-i}A$ for some $N \in \mathbb{N}$. Using Theorem 8.7 we find $i \leq N$ and $n \in \mathbb{N}$ such that $\emptyset \neq T^{-i}A \cap T^{-n}T^{-i}A \cap \cdots \cap T^{-kn}T^{-i}A = T^{-i}\big(A \cap T^{-n}A \cap \cdots \cap T^{-kn}A\big)$, which implies that $A \cap T^{-n}A \cap \cdots \cap T^{-kn}A \neq \emptyset$. □

Proof of Theorem 8.7. The proof goes by induction over k. The case $k = 1$ follows immediately from Proposition 8.5.

Next, suppose $k > 1$ and the result has been established for any smaller value of k. Some C_i must be non-empty; suppose WLOG $C_1 \neq \emptyset$. Then apply the induction hypothesis and Lemma 8.10 to find $n_1 \in \mathbb{N}$ such that $B_1 := C_1 \cap T^{-n_1}C_1 \cap \cdots \cap T^{-(k-1)n_1}C_1 \neq \emptyset$.

We now consider two cases. In the first case $T^{-n_1}B_1 \cap C_1 \neq \emptyset$. But then $C_1 \cap T^{-n_1}C_1 \cap \cdots \cap T^{-kn_1}C_1 \neq \emptyset$ and we are done.

The second case is when $T^{-n_1}B_1 \cap C_1 = \emptyset$. In this case, $T^{-n_1}B_1$ must have a non-empty intersection with some other C_i; WLOG suppose $D_2 := T^{-n_1}B_1 \cap C_2 \neq \emptyset$. We can now invoke again the induction hypothesis and Lemma 8.10 to find $n_2 \in \mathbb{N}$ such that $B_2 := D_2 \cap T^{-n_2} D_2 \cap \cdots \cap T^{-(k-1)n_2} D_2 \neq \emptyset$. We consider three new subcases.

In the first case $T^{-n_2} B_2 \cap C_2 \neq \emptyset$. But then (since $D_2 \subset C_2$), $C_2 \cap T^{-n_2} C_2 \cap \cdots \cap T^{-kn_2} C_2 \neq \emptyset$ and we are done.

In the second case, $T^{-n_2} B_2 \cap C_1 \neq \emptyset$. But then (since $D_2 \subset T^{-n_1} B_1 \subset T^{-in_1} C_1$ for each $i \in \{1, \dots, k\}$), $C_1 \cap T^{-(n_1+n_2)} C_1 \cap \cdots \cap T^{-k(n_1+n_2)} C_1 \neq \emptyset$ and we are done.

In the third case $T^{-n_2} B_2$ must have a non-empty intersection with some other C_i; WLOG suppose $D_3 := T^{-n_2} B_2 \cap C_3 \neq \emptyset$.

We can continue in this manner, but since we start with a finite open cover, after r steps we do not have a final case and the proof will finish. $\square$

Using a similar strategy, we can establish directly the following coloristic corollary of Sárközy's theorem.

Theorem 8.11. *If $\mathbb{N} = C_1 \cup \cdots \cup C_r$ there exists $C \in \{C_1, \dots, C_r\}$ and $n, x \in \mathbb{N}$ such that $\{x, x+n^2\} \subset C$.*

The dynamical version of Theorem 8.11 is the following.

Theorem 8.12. *Let (X, T) be a minimal system and suppose $X = C_1 \cup \cdots \cup C_r$ is an open cover. Then there exists $C \in \{C_1, \dots, C_r\}$ and $n \in \mathbb{N}$ such that $T^{-n^2} C \cap C \neq \emptyset$.*

Similarly to Lemma 8.10, one can write an equivalent formulation of Theorem 8.11 using a single open set.

Theorem 8.13. *Let (X, T) be a minimal system and let $A \subset X$ be open and non-empty. Then for some $n \in \mathbb{N}$, $A \cap T^{-n^2} A \neq \emptyset$.*

Exercise 8.14. *Prove that the following are all equivalent statements:*

(1) *Theorem 8.11.*
(2) *Theorem 8.12.*
(3) *Theorem 8.13.*
(4) *Theorem 8.12 without the minimality assumption.*

8.1. **Piecewise syndetic sets.** We've encountered above the notion of syndetic sets: subsets of $\mathbb{N}$ with bounded gaps. The dual notion to syndetic sets is that of thick sets.

Definition 8.15. *A set $T \subset \mathbb{N}$ is* **thick** *if it contains arbitrarily long intervals, i.e.,*

$$\forall N \in \mathbb{N}\ \exists m_N \in \mathbb{N} \text{ s.t. } \{m_N, m_N + 1, \dots, m_N + N\} \subset T.$$

Exercise 8.16. (1) *Show that a set $T \subset \mathbb{N}$ is thick if and only if its complement $\mathbb{N} \setminus T$ is not syndetic.*
(2) *Show that a set $S \subset \mathbb{N}$ is syndetic if and only if its complement $\mathbb{N} \setminus S$ is not thick.*
(3) *Show that a set $T \subset \mathbb{N}$ is thick if and only if for any syndetic set $S \subset \mathbb{N}$, the intersection $S \cap T \neq \emptyset$.*
(4) *Show that a set $S \subset \mathbb{N}$ is syndetic if and only if for any thick set $T \subset \mathbb{N}$, the intersection $S \cap T \neq \emptyset$.*

Definition 8.17. *A set $A \subset \mathbb{N}$ is* **piecewise syndetic** *if $A = S \cap T$ for a syndetic set $S \subset \mathbb{N}$ and a thick set $T \subset \mathbb{N}$.*

Note that all three notions of syndetic, thick and piecewise syndetic are upwards closed, i.e. if A possesses one of those properties and $B \supset A$, then B also possesses the same property.

The relation between piecewise syndetic sets and partition Ramsey theory is made apparent by the following lemma.

Lemma 8.18 (Brown's lemma)**.** *Let A be piecewise syndetic, and suppose that $A = A_1 \cup \cdots \cup A_r$. Then at least one of the A_i is piecewise syndetic.*

Proof. By an inductive argument it suffices to prove the lemma when $r = 2$. Suppose $A = S \cap T = A_1 \cup A_2$ where S is syndetic and T is thick. Let $\tilde{S} = S \setminus A_2$. If $\tilde{S}$ is syndetic, then $A_1 = \tilde{S} \cap T$ is piecewise syndetic. If $\tilde{S}$ is not syndetic, then its complement $\tilde{T} := \mathbb{N} \setminus \tilde{S}$ is thick, and hence $A_2 = \tilde{T} \cap S$ is piecewise syndetic. $\square$

Since $\mathbb{N}$ is piecewise syndetic, for any coloring of $\mathbb{N}$ one of the colors is piecewise syndetic. Therefore, if one seeks to show that any finite coloring of $\mathbb{N}$ contains a certain monochromatic pattern, it suffices to show that every piecewise syndetic set contains it.

8.2. **Minimal systems and (piecewise) syndetic sets.** Let (X,T) be a topological dynamical system, let $U \subset X$ be open and let $x \in X$. We denote by $V(x,U) := \{n \in \mathbb{N} : T^n x \in U\}$ the set of visit times of x to U. The connection between minimal systems and syndetic sets is given in the following lemma.

Lemma 8.19. *A system (X,T) is minimal if and only if for every non-empty open set $U \subset X$ and every $x \in X$, the set $V(x,U)$ is syndetic.*

Proof. If for every non-empty open set $U \subset X$ and every $x \in X$, the set $V(x,U)$ is syndetic, then in particular $V(x,U) \neq \emptyset$ and it follows that every point has a dense orbit. In view of Exercise 8.3, (X,T) is minimal.

Conversely suppose that (X,T) is minimal and let $U \subset X$ be open and non-empty, and let $x \in X$. Then $Y := X \setminus \bigcup_{i=0}^{\infty} T^{-i}U$ is a closed and T-invariant subset of X which is not all of X since $U \cap Y = \emptyset$. By minimality it follows that $Y = \emptyset$ and hence $X = \bigcup_{i=0}^{\infty} T^{-i}U$. By compactness there exists $r \in \mathbb{N}$ such that $X = \bigcup_{i=0}^{r} T^{-i}U$. Given any $n \in \mathbb{N}$, the point $T^n x \in X$ must belong to one of the $T^{-i}U$ and hence $T^{n+i}x \in U$. In other words, for every $n \in \mathbb{N}$ there exists $i \in \{0,\dots,r\}$ such that $n+i \in V(x,U)$, and this implies that $V(x,U)$ is syndetic. □

A topological dynamical system (X,T) is called ***transitive*** if there exists at least one point with a dense orbit. In a transitive system, we can replace sets of visits with the closely related sets of return times (sets of visits $V(x,U)$ where $x \in U$).

Exercise 8.20. *Show that a transitive system (X,T) is minimal if and only if for every for every non-empty open set $U \subset X$ and every $x \in U$, the set $V(x,U)$ is syndetic.*

There is a version of Lemma 8.19 that applies to non-minimal systems. Recall that every system (X,T) has a minimal subsystem.

Lemma 8.21. *Let (X,T) be a transitive system, suppose $x \in X$ has a dense orbit, let $Y \subset X$ be a minimal subsystem and let $U \subset X$ be an open set such that $U \cap Y \neq \emptyset$. Then $V(x,U)$ is piecewise syndetic.*

Proof. Let $y \in Y$ and let $S = V(y,U)$. By Lemma 8.19, S is syndetic, so there exists $r \in \mathbb{N}$ such that $S - \{1,\dots,r\} = \mathbb{N}$.

For each $N \in \mathbb{N}$ let $m_N \in \mathbb{N}$ be such that $T^{m_N}x$ is so close to y that for each $n \in \{0,1\dots,N\}$, whenever $T^n y \in T^{-n}U$, also $T^n(T^{m_N}x) \in T^{-n}U$. Therefore $V(x,U) \supset m_N + \big(S \cap \{0,\dots,N\}\big)$ for every $N \in \mathbb{N}$. We claim that the union $A := \bigcup_{N\in\mathbb{N}} m_N + \big(S \cap \{0,\dots,N\}\big)$ is piecewise syndetic, and this will finish the proof.

Indeed, the union $T = \bigcup_{N\in\mathbb{N}} m_N + \{0,\dots,N\}$ is thick, and letting $\tilde{S} := (\mathbb{N} \setminus T) \cup A$ we clearly have $A = \tilde{S} \cap T$, so it suffices to prove that $\tilde{S}$ is syndetic. Take any $x \in \mathbb{N}$. If $x \notin T$, then $x \in \tilde{S}$. Otherwise, $x \in m_N + \{0,\dots,N\}$ for some $N \in \mathbb{N}$, so that $x = m_N + n$ for some $n \in \{0,\dots,N\}$. We can then find $i \in \{1,\dots,r\}$ such that $n+i \in S$, and hence $x+i = m_N+n+i \in \tilde{S}$. We conclude that $\tilde{S}$ is syndetic and hence A is piecewise syndetic. □

8.3. **Partition regular patterns.** We will use Lemma 8.21 to derive a strengthening of the van der Waerden theorem (Theorem 1.4). In fact, we will develop a more general framework: call a ***pattern on*** $\mathbb{N}$ a collection $\mathcal{P}$ of finite subsets of $\mathbb{N}$. Elements of a pattern $\mathcal{P}$ may be called ***configurations***. The pattern is ***shift invariant*** if for every $C \in \mathcal{P}$ and $n \in \mathbb{N}$ also $C+n \in \mathcal{P}$. We say that the pattern $\mathcal{P}$ is ***monochromatic*** if for every finite coloring of $\mathbb{N}$ there exists $C \in \mathcal{P}$ which is monochromatic.

Example 8.22. *Let $R \subset \mathbb{N}$ and let $\mathcal{P} := \{\{x, x+r\} : x \in \mathbb{N}, r \in R\}$. Then $\mathcal{P}$ is a shift invariant pattern. If R is the set of perfect squares, then $\mathcal{P}$ is monochromatic, in view of Theorem 8.11.*

Theorem 8.23. *Let $\mathcal{P}$ be a shift invariant pattern. Then the following are equivalent:*

1. *$\mathcal{P}$ is monochromatic.*
2. *For every minimal system (X,T) and every non-empty open $U \subset X$, there is a configuration $C \in \mathcal{P}$ such that $\bigcap_{n\in C} T^{-n}U \neq \emptyset$.*
3. *For every topological system (X,T) and every finite open cover $X = C_1 \cup \dots \cup C_r$, there is a configuration $C \in \mathcal{P}$ and $i \in \{1,\dots,r\}$ such that $\bigcap_{n\in C} T^{-n}C_i \neq \emptyset$.*
4. *For every $r \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that for any coloring of the interval $[1,N]$ with r colors there exists $C \in \mathcal{P}$ contained in $[1,N]$ which is monochromatic.*

(5) *For every syndetic set $S \subset \mathbb{N}$ there exists $C \in \mathcal{P}$ such that $C \subset S$.*
(6) *For every piecewise syndetic set $A \subset \mathbb{N}$ there exists $C \in \mathcal{P}$ such that $C \subset A$.*
(7) *For every piecewise syndetic set $A \subset \mathbb{N}$ there exists $C \in \mathcal{P}$ such that the set*

$$\{n \in \mathbb{N} : C + n \subset A\}$$

is piecewise syndetic.

Proof. The implication (3)⇒(1) follows at once from Proposition 8.6. To see why (1)⇒(2), take a minimal system (X, T), an open set $\emptyset \neq U \subset X$ and let $x \in X$ be arbitrary. As we saw before, finitely many pre-images of U cover X (by minimality and compactness) so we can color $n \in \mathbb{N}$ according to which pre-image of U contains the point $T^n x$. Applying (1) to this coloring it follows that (2) holds. The implication (2)⇒(3) follows immediately from the fact that every system has a minimal subsystem.

It is clear that (4) implies (1); the converse implication follows from the "compactness principle" discussed in Section 1.

To prove that (1)⇒(5), notice that any syndetic set S induces a coloring of $\mathbb{N}$ by covering it with finitely many shifts; since $\mathcal{P}$ is shift invariant, if $S - i$ contains a configuration in $\mathcal{P}$, then so does S. Conversely, if (5) holds, then in view of Lemma 8.19 so does (2).

Using Lemma 8.18 we deduce that (6)⇒(1). It is trivial that (7)⇒(6) so to finish the proof it will suffice to show that (2)⇒(7). Let $A = S \cap T$ for a syndetic S and a thick T. For each $N \in \mathbb{N}$ let $m_N \in \mathbb{N}$ such that $\{m_N, \dots, m_N + N\} \subset T$.

Consider the left shift $T : \{0,1\}^{\mathbb{N}_0} \to \{0,1\}^{\mathbb{N}_0}$ and let $X \subset \{0,1\}^{\mathbb{N}_0}$ be the orbit closure of the point 1_A. Passing to a subsequence of (m_N) if needed, we can assume that the limit $y = \lim_{N\to\infty} T^{m_N} 1_A$ exists. Then $y \in X$ and hence the orbit closure X_1 of y is a subsystem of X. It can be proved that the point $(0, 0, \dots)$ does not belong to X_1 (cf. Exercises 8.24 and 8.25 below). Therefore the clopen set $U := \{x \in X : x_0 = 1\}$ has non-empty intersection with any subsystem of X_1, and in particular U has non-empty intersection with a minimal subsystem Y of (X, T).

Using part (2) on the open subset $U \cap Y$ of Y we find a configuration $C \in \mathcal{P}$ such that $W := \bigcap_{i\in C} T^{-i} U$ satisfies $W \cap Y \neq \emptyset$. We can now apply Lemma 8.21 to deduce that $B := \{n \in \mathbb{N} : T^n 1_A \in W\}$ is piecewise syndetic. For every $n \in B$ and $i \in C$ we have $T^n 1_A \in T^{-i} U$, so $T^{n+i} 1_A \in U$ so $n + i \in A$. We conclude that $n + C \subset A$ and this finishes the proof. □

Exercise 8.24. *Show that the point y constructed at the end of the proof of Theorem 8.23 is the indicator function of a syndetic set.*

Exercise 8.25. *Show that if $y \in \{0,1\}^{\mathbb{N}_0}$ is the indicator function of a syndetic set then $(0, 0, \dots)$ does not belong to the orbit closure of y under the shift.*

Combining Theorem 8.23 with van der Waerden's theorem we obtain the following strengthening.

Corollary 8.26. *Let $A \subset \mathbb{N}$ be piecewise syndetic and let $k \in \mathbb{N}$. Then there exists $n \in \mathbb{N}$ such that the intersection $A \cap (A - n) \cap \dots \cap (A - kn)$ is piecewise syndetic.*

Proof. Theorem 1.4 can be reformulated as stating that $\mathcal{P} := \{\{x, x+n, \dots, x+kn\} : x, n \in \mathbb{N}\}$ is monochromatic. In view of condition (7) in Theorem 8.23, there exists $x, n \in \mathbb{N}$ such that the set $B := \{m \in \mathbb{N} : m+\{x, x+n, \dots, x+kn\} \subset A\}$ is piecewise syndetic. The desired conclusion now follows from the observation that $A \cap (A - n) \cap \dots \cap (A - kn) \supset B + x$. □

8.4. **Monochromatic sums and products.** In this subsection we use the facts established above to prove the following theorem.

Theorem 8.27. *If $\mathbb{N} = C_1 \cup \dots \cup C_r$, there exists $x, y \in \mathbb{N}$ and $t \in \{1, \dots, r\}$ such that*

$$\{x, x + y, xy\} \subset C_t. \tag{8.1}$$

Proof. We will construct inductively four sequences:

- an increasing sequence $(y_i)_{i\geq 1}$ of natural numbers,
- two sequences $(B_i)_{i\geq 0}$ and $(D_i)_{i\geq 1}$ of piecewise syndetic subsets of $\mathbb{N}$,
- a sequence $(t_i)_{i\geq 0}$ of colors in $\{1, \dots, r\}$,

such that $B_i \subset C_{t_i}$ for every $i \geq 0$.

Initiate by choosing $t_0 \in \{1, \ldots, r\}$ such that C_{t_0} is piecewise syndetic, and let $B_0 := C_{t_0}$. Assume now that $i \geq 1$ and that we have already defined $(t_j)_{j=0}^{i-1}$, $(y_j)_{j=1}^{i-1}$, $(B_j)_{j=0}^{i-1}$ and $(D_j)_{j=1}^{i-1}$. We apply Corollary 8.26 to find $y_i \in \mathbb{N}$ such that

$$D_i := B_{i-1} \cap \bigcap_{j=1}^{i} \Big(B_{i-1} - y_j^2 \cdots y_{i-1}^2 y_i\Big) \tag{8.2}$$

is piecewise syndetic (with the convention that for $i = j$, the (empty) product $y_j^2 \cdots y_{i-1}^2$ equals 1). Observe that $y_i D_i$ is also piecewise syndetic, and therefore Lemma 8.18 provides some $t_i \in \{1, \ldots, r\}$ such that $B_i := y_i D_i \cap C_{t_i}$ is piecewise syndetic. This finishes the construction of the sequences.

Note that $B_i \subset y_i D_i \subset y_i B_{i-1}$; iterating this fact we obtain

$$\forall\ 0 \leq j < i, \qquad B_i \subset y_{j+1} y_{j+2} \cdots y_i B_j. \tag{8.3}$$

Since the sequence (t_i) takes only finitely many values, there exist (infinitely many) $j < i$ such that $t_i = t_j$. Let $\tilde{x} \in B_i$, let $y := y_{j+1} \cdots y_i$, and let $x := \tilde{x}/y$. We claim that $\{x, x+y, xy\} \subset C_{t_i}$, which will complete the proof. Indeed $xy = \tilde{x} \in B_i \subset C_{t_i}$ and from (8.3) we have $xy \in B_i \subset yB_j$ so $x \in B_j \subset C_{t_j} = C_{t_i}$. Finally we have

$$\begin{array}{rcl} y(x+y) & = & \tilde{x} + y^2 \in B_i + y^2 \subset y_i D_i + y^2 \\ \text{using (8.2)} & \subset & y_i\big(B_{i-1} - y_{j+1}^2 \cdots y_{i-1}^2 y_i\big) + y^2 \\ \text{using (8.3)} & \subset & y_i\big(y_{j+1} \cdots y_{i-1} B_j - y_{j+1}^2 \cdots y_{i-1}^2 y_i\big) + y^2 \\ & = & yB_j - y^2 + y^2 = yB_j, \end{array}$$

which implies that $x + y \in B_j \subset C_{t_j} = C_{t_i}$. □

9. Infinite Ramsey Theory

So far we have discussed only finite configurations that appear monochromatically whenever one partitions $\mathbb{N}$, or in every set with positive upper density. In this section, we mention some infinite configurations that have similar properties.

We start by recalling Ramsey's theorem. Given a set X and $m \in \mathbb{N}$, denote by $\binom{X}{m} := \{A \subset X : |A| = m\}$ the collection of all subsets of X with exactly m elements. Recall that a complete graph is a pair $(V, \binom{V}{2})$ for a set V.

Theorem 9.1 (Ramsey's theorem for graphs)**.** *For any $r, k \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that whenever V is a set with $|V| = n$ and we finitely color the edges of the complete graph with vertices V, i.e. $\binom{V}{2} = C_1 \cup \cdots \cup C_r$, there exists a color C_i and a set $S \subset V$ with $|S| = k$ such that $\binom{S}{2} \subset C_i$.*

There is an infinite version of Ramsey's theorem, which is the version we will be interested in. For convenience, we use $\mathbb{N}$ to stand for the countably infinite set of vertices, but naturally any other set of the same cardinality could play the same role.

Theorem 9.2 (Infinite Ramsey's theorem for graphs)**.** *Whenever $\binom{\mathbb{N}}{2} = C_1 \cup \cdots \cup C_r$, there exists a color C_i and an infinite subset $S \subset \mathbb{N}$ such that $\binom{S}{2} \subset C_i$.*

As is often the case, the infinite version has a shorter formulation, and it implies the finitistic formulation.

Exercise 9.3. *Show that Theorem 9.2 implies Theorem 9.1.*

However, unlike in many other Ramsey theoretic examples, the finite and infinite versions are not equivalent. This is made precise by the Paris-Harrington Theorem.

As a corollary of Theorem 9.2 we obtain the following arithmetic corollary. For a set $I \subset \mathbb{N}$ we denote by $I \oplus I$ the ***restricted sumset*** $\{x + y : x, y \in I; x \neq y\}$.

Corollary 9.4. *For any finite coloring of $\mathbb{N}$ there exists an infinite set $I \subset \mathbb{N}$ such that $I \oplus I$ is monochromatic.*

Proof. Suppose $\mathbb{N} = C_1 \cup \cdots \cup C_r$. Let $\tilde{C}_i := \left\{\{a,b\} \in \binom{\mathbb{N}}{2} : a+b \in C_i\right\}$. Observe that $\binom{\mathbb{N}}{2} = \tilde{C}_1 \cup \cdots \cup \tilde{C}_r$, so we can apply Theorem 9.2 to find an infinite set $I \subset \mathbb{N}$ such that $\binom{I}{2}$ is contained in a single $\tilde{C}_i$. But this means that $I \oplus I \subset C_i$. ☐

Exercise 9.5. *Find a 3-coloring of $\mathbb{N}$ without a monochromatic set of the form $I + I = \{x+y : x, y \in I\}$. [Hint: Color $\mathbb{N}$ with longer and longer intervals of alternating colors to avoid any pair $\{x, 2x\}$. By using the third color one can avoid pairs $\{x+y, 2x\}$ when $y \ll x$.]*

Ramsey's theorem has a version for hypergraphs:

Theorem 9.6 (Infinite Ramsey's theorem for hypergraphs)**.** *Let $m \in \mathbb{N}$. Whenever $\binom{\mathbb{N}}{m} = C_1 \cup \cdots \cup C_r$, there exists a color C_i and an infinite subset $S \subset \mathbb{N}$ such that $\binom{S}{m} \subset C_i$.*

Similarly, we can extend Corollary 9.7.

Corollary 9.7. *Let $m \in \mathbb{N}$. For any finite coloring of $\mathbb{N}$ there exists an infinite set $I \subset \mathbb{N}$ such that $I^{\oplus m} := \{x_1 + \cdots + x_m : x_1, \ldots, x_m \in I, x_1 < \cdots < x_m\}$ is monochromatic.*

I turns out that a much more spectacular result than Corollary 9.7 holds. To state we need the concept of an IP-set

Definition 9.8. *A set $A \subset \mathbb{N}$ is an* **IP-set** *if there exists an infinite set $I \subset \mathbb{N}$ such that*

$$A = \left\{ \sum_{n \in F} n : F \subset I; 0 < |F| < \infty \right\}.$$

Note that an equivalent characterization of IP-sets are sets of the form $\bigcup_{m=1}^{\infty} I^{\otimes m}$.

Theorem 9.9 (Hindman)**.** *For any finite coloring of $\mathbb{N}$ there is a monochromatic IP-set.*

Exercise 9.10. *Show that Theorem 9.9 is equivalent to the statement that any finite coloring of an IP-set yields a monochromatic IP-set. [Hint: $\mathbb{N}$ is the IP-set generated by the set $I = \{2^n : n \in \mathbb{N}\}$.]*

We will present a very simple proof of Theorem 9.9 based on the existence of idempotent ultrafilters on $\mathbb{N}$.

Definition 9.11. *An* **idempotent ultrafilter** *is a collection p of subsets of $\mathbb{N}$ satisfying*

1. *$\emptyset \notin p$ and $\mathbb{N} \in p$,*
2. *If $A \in p$ and $B \supset A$, then $B \in p$,*
3. *If $A, B \in p$ then $A \cap B \in p$.*
4. *If $A \in p$ and $A = A_1 \cup \cdots \cup A_r$ then one of the A_i is in p.*
5. *$A \in p$ if and only if $\{n \in \mathbb{N} : A - n \in p\} \in p$.*

A collection satisfying conditions (1)-(3) is called a ***filter***. Given a set $A \subset \mathbb{N}$, the collection $p := \{B \subset \mathbb{N} : A \subset B\}$ is a filter. A collection satisfying conditions (1)-(4) is called an ***ultrafilter***. Ultrafilters are maximal filters (for the inclusion relation); therefore Zorn's lemma implies that any filter is contained in an ultrafilter. Given $n \in \mathbb{N}$, the collection $p_n := \{A \subset \mathbb{N} : n \in A\}$ is an ultrafilter. Ultrafilters of that form are called ***principal***. The existence of non-principal ultrafilters requires at least some weak form of the axiom of choice.

Proposition 9.12. *There exist idempotent ultrafilters. In fact, for any IP-set $A \subset \mathbb{N}$ there exists an idempotent ultrafilter p such that $A \in p$.*

The proof of Proposition 9.12 requires some background on ultrafilters and is beyond the scope of this lecture. We will take it for granted and use it to give a quick proof of Hindman's theorem.

Proof of Theorem 9.9. Using Proposition 9.12, let p be an idempotent ultrafilter. Given a finite coloring of $\mathbb{N}$ there is a color, call it C which belongs to p. Now let $A_1 = C$ and choose $x_1 \in A_1$ such that $A_1 - x_1 \in p$; such x_1 exists because the set $A_1 \cap \{x \in \mathbb{N} : A_1 - x \in p\}$ is in p and hence non-empty. Let $A_2 := A_1 \cap (A_1 - x_1)$ and note that $A_2 \in p$. We now proceed recursively for each $n = 2, 3, \ldots$, finding $x_n \in A_n$ such that $A_n - x_n \in p$ (which exists because $A_n \cap \{x \in \mathbb{N} : A_n - x \in p\}$) and letting $A_{n+1} = A_n \cap (A_n - x_n) \in p$.

We claim that for any nonempty $F \subset \mathbb{N}$, the sum $\sum_{i\in F} x_i \in A$ which finishes the proof. To prove the claim, let $n = \max F$ and $\tilde{F} = F \setminus \{n\}$ and note that $x_n \in A_n \subset A - \sum_{i\in\tilde{F}} x_i$, so that $\sum_{i\in F} x_i = x_n + \sum_{i\in\tilde{F}} x_i \in A$ □

Exercise 9.13. *Let $k \in \mathbb{N}$ and let A be an IP-set. Show that A contains a multiple of k.*

Observe that, in view of Exercise 9.13, the most obvious density analogue of Hindman's theorem is false, i.e. there are sets with positive upper density which do not contain an IP-set. However, these so-called local obstructions can be easily avoided if one is allowed to shift to the following question of Erdős:

Question 9.14. *Is is true that whenever $A \subset \mathbb{N}$ has positive upper density, there exists $t \in \mathbb{N}$ such that $A - t$ contains an IP-set?*

It turns out that even this density version of Hindman's theorem is false, and the answer to Question 9.14 is negative.

Exercise 9.15. *Show that for every $\epsilon > 0$ there exists a set $A \subset \mathbb{N}$ with upper density $\bar{d}(A) > 1 - \epsilon$ and such that for any $t \in \mathbb{N}$ there exists $k = k(t)$ such that $A - t$ has no multiples of k.*

In view of the negative answer to Question 9.14 (obtained by combining Exercises 9.13 and 9.15) Erdős made the following conjecture, which is still open.

Conjecture 9.16. *If $A \subset \mathbb{N}$ has $\bar{d}(A) > 0$, there exists an infinite set $B \subset \mathbb{N}$ and a shift $t \in \mathbb{N}$ such that $A - t \supset B \oplus B$.*

Exercise 9.17. *Find a set $A \subset \mathbb{N}$ with $\bar{d}(A) > 0$ and such that for any t and any infinite set $B \subset \mathbb{N}$, $A - t$ does not contain $B + B$, and hence a restricted sum $B \oplus B$ is required in Conjecture 9.16.*

Exercise 9.18. *Show that if Conjecture 9.16 holds, then one can always take $t \in \{0, 1\}$.*

Erdős made another related conjecture.

Conjecture 9.19. *If $A \subset \mathbb{N}$ has $\bar{d}(A) > 0$, then there are infinite sets $B, C \subset \mathbb{N}$ such that $A \supset B + C$.*

Exercise 9.20. *Show that Conjecture 9.16 implies Conjecture 9.19.*

Conjecture 9.19 has been recently established in [21] using ergodic theory. The full proof is beyond the scope of this notes, but we will look at one special case that was established earlier in [6].

One could try to use Furstenberg's correspondence principle in the form of Theorem 3.4. Note that

$$\exists B, C \subset \mathbb{N}: \ |B|, |C| = \infty, \ B + C \subset A \qquad \iff \qquad \exists B \subset \mathbb{N}: \ |B| = \infty, \ \left|\bigcap_{b\in B} A - b\right| = \infty.$$

Unfortunately Theorem 3.4 as stated does not allow one to take infinite intersections of the form $\bigcap_{b\in B} A - b$. This issue could actually be addressed; however the problem is that in a probability space (and hence in a measure preserving system) infinite sets of measure zero are negligible, so in order to obtain from the Correspondence Principle a conclusion of the form $\left|\bigcap_{b\in B} A - b\right| = \infty$ one would need to show that the corresponding set $\bigcap_{b\in B} T^{-b}A$ has positive measure. However, this is not always the case.

Exercise 9.21. *Consider the doubling map (i.e. the transformation $x \mapsto 2x \bmod 1$ on $[0, 1)$ with the Lebesgue measure) and let $A = [0, 1/2)$. Show that for any infinite set $B \subset \mathbb{N}$, the intersection $\bigcap T^{-b}A$ has zero measure.*

Exercise 9.22. $(*)$ *Consider the doubling map (i.e. the transformation $x \mapsto 2x \bmod 1$ on $[0, 1)$ with the Lebesgue measure) and let $A \subset [0, 1)$ be any Borel set. Show that for any infinite set $B \subset \mathbb{N}$, the intersection $\bigcap T^{-b}A$ has zero measure.*

As these exercises illustrate, we can not, in general, use the Correspondence Principle in the form presented in Theorem 3.4 (although it turns out that one can still use a different version of the more general Correspondence Principle encapsulated in the beginning of Section 3). However, as mentioned above, Theorem 3.4 suffices to prove a special case of Conjecture 9.19.

Definition 9.23. *A set $E \subset \mathbb{N}$ is* **weak-mixing** *if there exists a measure preserving system $(X, \mathcal{B}, \mu, T)$ and a set $A \in \mathcal{B}$ satisfying the conclusion of Theorem 3.4 (i.e. $\mu(A) = \bar{d}(E)$ and (3.1) holds), such that the function $1_A - \mu(A)$ is a weak-mixing function (cf. Definition 5.15).*

Note that a weak-mixing function in a m.p.s. must have 0 integral, which is why we consider the function $1_A - \mu(A)$ instead of 1_A in this definition. In this sense the Jacobs-de Leeuw-Glicksberg decomposition (cf. Theorem 5.24) of 1_A is given by the sum of a weak mixing function and a constant function.

The following special case of Theorem 3.4 was first obtained in [6].

Theorem 9.24. *Let $A \subset \mathbb{N}$ with $\bar{d}(A) > 0$. If A is weak mixing, then there exist infinite sets $B, C \subset \mathbb{N}$ such that $A \supset B + C$.*

The relevance of the weak mixing property is captured by the following property.

Exercise 9.25. *Let $(X, \mathcal{B}, \mu, T)$ be a m.p.s. and let $A \in \mathcal{B}$ be such that $\mu(A) > 0$ and the function $1_A - \mu(A)$ is weak mixing. Show that for any $B \in \mathcal{B}$ with $\mu(B) > 0$, the set*

$$R := \{n \in \mathbb{N} : \mu(A \cap T^{-n}B) > 0\}$$

has full natural density, i.e. $d(R) = 1$ (which is stronger than just $\bar{d}(R) = 1$).

Proof of Theorem 9.24. Let $A \subset \mathbb{N}$ have $\bar{d}(A) > 0$ and be weak mixing. Let $(X, \mathcal{B}, \mu, T)$ be a m.p.s. and let $D \in \mathcal{B}$ be such that $\mu(D) = \bar{d}(A)$, (3.1) holds and the function $1_D - \mu(D)$ is weak mixing. Using Exercise 9.25 it follows that for any set $E \in \mathcal{B}$ with $\mu(E) > 0$ we have

$$d\big(\{n \in \mathbb{N} : \mu(T^{-n}D \cap E) > 0\}\big) = 1. \tag{9.1}$$

We will construct recursively two increasing sequences $(b_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ such that for any $n \in \mathbb{N}$ we have

$$b_n \in \bigcap_{i=1}^{n-1}(A - c_i), \qquad c_n \in \bigcap_{i=1}^{n}(A - b_i), \qquad \mu\left(\bigcap_{i=1}^{n} T^{-c_i}D\right) > 0, \qquad \mu\left(\bigcap_{i=1}^{n} T^{-b_i}D\right) > 0. \tag{9.2}$$

The first two properties in (9.2) imply that the sets $B := \{b_n : n \in \mathbb{N}\}$ and $C := \{c_n : n \in \mathbb{N}\}$ satisfy $B + C \subset A$ which is the desired conclusion.

Let $b_1 \in \mathbb{N}$ be arbitrary and let $c_1 \in A - b_1$ be arbitrary. Note that (9.2) holds for $n = 1$. Next suppose that $m > 1$ and $b_1, \dots, b_{m-1}$ and $c_1, \dots, c_{m-1}$ have been chosen satisfying (9.2) for every $n < m$. Using Eq. (9.1) with $E = \bigcap_{i=1}^{m-1} T^{-b_i}D$ (which from (9.2) has positive measure) it follows that $R := \{b \in \mathbb{N} : \mu(T^{-b}D \cap E) > 0\}$ has full density. In view of the correspondence property (3.1), and then again the induction hypothesis (9.2),

$$\bar{d}\left(\bigcap_{i=1}^{m-1}(A - c_i)\right) \geq \mu\left(\bigcap_{i=1}^{m-1} T^{-c_i}D\right) > 0$$

Therefore the intersection $R \cap \bigcap_{i=1}^{m-1}(A - c_i)$ has positive upper density, and in particular it is infinite. Choose $b_m > b_{m-1}$ in that intersection. With this choice of b_m, both the first and last property in (9.2) hold for $n = m$. Next use Eq. (9.1) with $E = \bigcap_{i=1}^{m-1} T^{-c_i}D$ and again the correspondence principle and the induction hypothesis to conclude that the intersection

$$\bigcap_{i=1}^{m}(A - b_i) \ \cap \ \left\{c \in \mathbb{N} : \mu\left(T^{-c}D \cap \bigcap_{i=1}^{m-1} T^{-c_i}D\right) > 0\right\}$$

has positive upper density. Choosing $c_m > c_{m-1}$ in this intersection, both the second and third properties in (9.2) are satisfied for $n = m$. This finishes the construction of the sequences (b_n) and (c_n) and hence the proof. □

It turns out that the proof of Theorem 9.24 can be simplified to yield a stronger result.

Exercise 9.26. *Show that any weak-mixing set $A \subset \mathbb{N}$ with positive upper density contains $B \otimes B$ for some infinite set $B \subset \mathbb{N}$.*

At this point, it would seem natural to prove Conjecture 9.19 by using the Jacobs-de Leeuw-Glicksberg decomposition together with Theorem 9.24 and an analysis of Kronecker systems. However, a delicate subtlety related to the fact that the Jacobs-de Leeuw-Glicksberg decomposition produces measurable but not necessarily continuous components prevents this approach from working directly.

The interested reader may find the full proof in the original manuscript [21], or in a more streamlined ergodic rendition discovered by Host in [15]. Both proofs make use (in a way or another) of methods from Ergodic Ramsey Theory, including a Correspondence Principle, and the Jacobs-de Leeuw-Glicksberg decomposition.

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