Mathematics for Fusion Power part 2

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Quasisymmetry

QS in Magnetohydrostatic plasma

FGCM

- Recall FGCM: $H = \frac{1}{2m} p_{\|}^2 + \mu |B(Q)|$, $\omega = -d(p_{\|} b^\flat) - e\pi^* \beta$.
- It is on a fibre bundle, subbundle of T^*M with fibres $\mathbb{R} b^\flat$ (assume $|B| \neq 0$).
- For axisymmetric B we reduced to 1DoF and hence found simple principle for confinement: make some bounded level sets of μ, L, H.
- But requires strong toroidal current.
- Can we find other B fields for which FGCM has a continuous symmetry?
- If so, we get reduction to 1DoF, simple principle for confinement, and perhaps cases with small toroidal current?

Quasisymmetry (QS)

- Say 3D vector field u is a *quasisymmetry* for B if $L_u\beta = 0$, $L_u|B| = 0$, $L_u b^\flat = 0$. Lift u to $U = (u, 0)$ on the GC phase space. Then for all μ, $L_U H = 0$ and $L_U\omega = 0$.
- A formal way to lift a vector field u to the GC phase space is $U = (u, -p_\parallel i_b L_u b^\flat)$, chosen to preserve $p_\parallel b^\flat$, but gives same result.
- So FGCM conserves L defined by $i_U\omega = dL$: $i_u\beta$ is closed so assuming no global obstacle, $i_u\beta = d\psi$ for some function ψ, and then $L = p_\parallel u \cdot b - e\psi$.
- Particles on suitable bounded level sets of μ, L, H are confined.
- Examples: For an axisymmetric B in Euclidean space, rotation about the axis is a quasisymmetry. Helical symmetry $u = k\partial_z + h\partial_\phi$ gives others, but has unbounded u-orbits; and quotient in vertical can't be realised in Euclidean space.
- QS was proposed in 1983 but still no non-axisymmetric examples known in Euclidean space!
- We'll study their properties and deduce many restrictions.
- JW Burby, N Kallinikos, RS MacKay, Some mathematics for quasi-symmetry, J Math Phys 61 (2020) 093503

Open questions

- Maybe $L_u g = 0$? (in which case, for Euclidean g and bounded u-orbits, u has to be rotation about an axis), or
- Kovalevskaya found a class of integrable cases for spinning tops (rigid body with one fixed point in a gravitational field) distinct from the Poisson-Euler and Lagrange cases (and

proved that there are no others).
- So maybe there are non-axisymmetric magnetic fields for which GC motion is integrable?

Some consequences of QS

- Flux function ψ: $L_u\beta = 0$ implies $di_u i_B\Omega = 0$, so $i_u i_B\Omega = d\psi$ for some local function ψ. Assume there are orbits of u, B spanning H_1, then ψ is global.
- If u, B are independent (equivalently, $d\psi \neq 0$) on a component of a level set of ψ, then it is a submanifold (called a *flux surface*) and u, B are tangent to it. The bounded components are 2-tori because orientable (use u, B as frame) and support a nowhere-zero vector field (u or B).
- $L_u\Omega = 0$: $b^\flat \wedge \beta = |B|\Omega$, thus $L_u(b^\flat \wedge \beta) = L_u b^\flat \wedge \beta + b^\flat \wedge L_u\beta = 0$. So $0 = L_u(|B|\Omega) = (L_u|B|)\Omega + |B|L_u\Omega$. So $L_u\Omega = 0$.
- $L_u B^\flat = 0$: $L_u B^\flat = L_u(|B|b^\flat) = (L_u|B|)b^\flat + |B|L_u b^\flat = 0$.
- $L_u C = 0$ where $C = u \cdot B$: $L_u(u \cdot B) = L_u i_u B^\flat = i_u L_u B^\flat = 0$.
- $L_u B = [u, B] = 0$: $i_{[u,B]}\Omega = L_u i_B\Omega - i_B L_u\Omega$ and Ω is non-degenerate. This leads to...

Liouville-Arnol'd coordinates

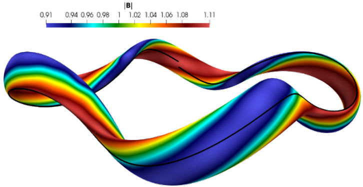
- **Theorem**: u, B linearly indpt commuting vector fields on a compact surface S imply $\exists$ coordinates $(\theta^1, \theta^2) : S \to \mathbb{T}^2$ such that u, B are indpt constant combinations of $\partial_{\theta^1}, \partial_{\theta^2}$.
- **Proof**: Let ϕ^u, ϕ^B be the flows of u and B. They commute, so we can combine them into an action ϕ of $\mathbb{R}^2$ on S. Flowing for a time t_1 along u and t_2 along B from an initial point x_0 produces a local diffeomorphism ϕ from $t = (t_1, t_2)$ near 0 to a neighbourhood of x_0. S is compact so there are $t = (t_1, t_2) \neq (0, 0)$ such that $\phi_t(x_0) = x_0$. The set of such pairs forms a 2D lattice. Choose a pair of generators T^1, T^2 and let A be the matrix with these as columns. We obtain an action of $\theta = (\theta^1, \theta^2) \in T^2$ on S by $\phi_{A\theta}$. Applying to a fixed x_0, this gives a diffeomorphism of $\mathbb{T}^2$ to S. In these coordinates, u, B are the first and second columns of A^{-1}. $\square$
- Idea was rediscovered by Hamada to make such coordinates on constant pressure surfaces for magnetohydrostatic (MHS) fields ($J \times B = \nabla p$), from $[J, B] = 0$.

continued

- Can extend smoothly by ψ as third coordinate. So $u = u^1(\psi)\partial_{\theta_1} + u^2(\psi)\partial_{\theta_2}$ and similarly for B.
- If on each flux surface there is a level set of $|B|$ that is a closed curve, then by $L_u|B| = 0$, it is a u-line. Then all the u-lines on it are closed. So $u^1 : u^2$ is rational, and by continuity the ratio is independent of ψ.
- We'll see that u is constant in such coordinates.

more

- Choose toroidal & poloidal cycles on flux surfaces; distinguish
  1. QA (quasiaxisymmetric): u-lines are homologous to toroidal, as for a tokamak; NCSX was to be substantially non-AS QA but not completed; CFQS likely to be first.
  2. QP (quasipoloidal): u-lines homologous to the poloidal cycle.
  3. QH(N,M) (quasihelical): u-lines are homologous to N poloidal loops plus M toroidal loops, for some non-zero integers N, M (wlog in lowest terms and with $M \geq 0$), e.g. HSX is QH(4,1)



- In the case of MHS in Euclidean space with a magnetic axis, we'll see that $M = 1$, in particular QP is impossible.
- Define *winding ratio* $\iota(\psi)$ for B to be limit of ratio of number of poloidal turns to toroidal turns on level set of ψ.

$L_u g$

- In set where u, B are independent, $d\psi \neq 0$. Let $n = \frac{\nabla\psi}{|\nabla\psi|^2}$ (so $i_n d\psi = 1$, $n \cdot B = 0$, $n \cdot u = 0$). Then (B, u, n) form a basis.
- **Theorem**: In this basis, $L_u g$ has matrix $\begin{bmatrix} 0 & 0 & 0 \\ 0 & L_u|u|^2 & i_n L_u u^\flat \\ 0 & u \cdot [n, u] & L_u|n|^2 \end{bmatrix}$ and $L_u|n|^2 = -|B|^2|n|^4 L_u|u|^2$.
- Note: symmetric but alternative expressions for off-diagonal.
- **Lemma**: For any vector fields u, X and covariant 2-tensor g, $i_X L_u g = L_u i_X g - i_{[u,X]} g$.
- **Proof**: For any vector field Y, $(L_u g)(X, Y) = L_u(g(X, Y)) - g(L_u X, Y) - g(X, L_u Y)$. So $i_Y i_X L_u g = L_u i_Y i_X g - i_Y i_{[u,X]} g - i_{[u,Y]} i_X g$. Apply $L_u i_Y X^\flat = i_Y L_u X^\flat + i_{[u,Y]} X^\flat$ to 1-form $X^\flat = i_X g$, and obtain $i_Y i_X L_u g = i_Y L_u i_X g - i_Y i_{[u,X]} g$. Y arbitrary, hence result. □

continued

- **Proof of Theorem**: Apply the Lemma to $X = B, u, n$:
  1. $X = B$ gives $i_B L_u g = 0$, hence first row and column are 0.
  2. $X = u$ gives $i_u L_u g = L_u i_u g = L_u u^\flat$. Apply i_u or i_n to get $i_u i_u L_u g = L_u i_u u^\flat = L_u |u|^2$ and $i_n i_u L_u g = i_n L_u u^\flat$.
  3. $X = n$ gives $i_n L_u g = L_u i_n g - i_{[u,n]} g$. Then $i_u i_n L_u g = L_u i_u i_n g - i_u i_{[u,n]} g = u \cdot [n, u]$.

  $d\psi = i_u i_B \Omega$ so $\nabla\psi = B \times u$, so $|\nabla\psi|^2 = i_{B\times u} i_u i_B \Omega$, but $i_{B\times u}\Omega = B^\flat \wedge u^\flat$, so $|\nabla\psi|^2 = i_u i_B (B^\flat \wedge u^\flat) = |B|^2|u|^2 - (B \cdot u)^2$.
  So $L_u|\nabla\psi|^2 = |B|^2 L_u |u|^2$, hence the last result.
  Alternatively, define rate of strain tensor $E = \frac{1}{2} g^{-1} L_u g$ and use $0 = L_u \Omega = \mathrm{tr}\, E$. □
- For a QS u, $L_u g = 0$ iff $L_u u^\flat = 0$. True for axisymmetry: $u = \partial_\phi$, $u^\flat = r^2 d\phi$, $L_u u^\flat = r^2 d i_u d\phi = r^2 d(1) = 0$.
- Notes: Can show $n \cdot [n, u] = B \cdot [n, u] = 0$, so $[n, u]$ parallel to $u_\perp = u - \frac{u\cdot B}{|B|^2} B$. Also, $i_v d\psi = i_B d|u|^2$ for $v = \mathrm{curl}\, u$. And $i_B L_u g = 0$ implies $\det E = 0$.

Case of Euclidean metric

- **Theorem**: If vector field u preserves Euclidean metric g ($L_u g = 0$) then $u(x) = U + Ax$ for some vector U and antisymmetric matrix A.
- **Proof**: $|x - y|^2$ constant under the flow of u implies $(u(x) - u(y)) \cdot (x - y) = 0$ (u "equiprojective"). Let $U = u(0)$ and $v(x) = u(x) - U$. Then v is equiprojective and taking $x = 0$, $\forall\, y\ v(y) \cdot y = 0$. So $\forall\, x, y$,

$$\begin{aligned} v(x) \cdot y + v(y) \cdot x &= v(x) \cdot (y - x) + v(y) \cdot (x - y) \\ &= (v(x) - v(y)) \cdot (y - x) = 0 \end{aligned} \tag{1}$$

Thus $\forall\, x, y, z$ and $\lambda, \mu \in \mathbb{R}$,

$$\begin{aligned} v(\lambda x + \mu y) \cdot z = -(\lambda x + \mu y) \cdot v(z) &= -\lambda x \cdot v(z) - \mu y \cdot v(z) \\ &= \lambda v(x) \cdot z + \mu v(y) \cdot z, \end{aligned} \tag{2}$$

so $v(x) = Ax$ for some matrix A. By (1), A is antisymmetric. □

- So u is a translation plus a rotation.

ϕ^u is a circle action

- Assume closed regular level set S of ψ (so a torus), and $d|B|$, $d\psi$ independent on a component C of a level set of $|B|$ on S.
- Then C is a circle and a closed u-line. From LA, all u-lines on the same flux surface are closed, have the same period $\tau(\psi)$ and are non-contractible.
- The same holds for all nearby flux surfaces.
- **Theorem**: If $u \cdot B \neq 0$ a.e. on this union of flux surfaces then τ is constant.
- **Proof**: Let $v = \tau(\psi)u$, ϕ be the flow of v (period 1) and $f = 1/\tau$. For forms α, define *circle-average* $\langle \alpha \rangle = \int_0^1 \phi_t^* \alpha \, dt$. $0 = L_u B^\flat = L_{fv} B^\flat = v \cdot B \, df + f \, L_v B^\flat$. Take $\langle \ \rangle$: $v \cdot B$ and f are constant along each u-line, and $\langle L_v \alpha \rangle = 0$ for any α, so $0 = v \cdot B \, \langle df \rangle$. And $\langle df \rangle = d\langle f \rangle = df$, so if $v \cdot B \neq 0$ a.e. we get f is constant. $\square$

Comments

- In case of axisymmetry, $\tau = 2\pi$.
- Relate to proof that if every orbit on an energy level of a Hamiltonian system is periodic then they have a common period? J Moser, CPAM 23 (1970) 609
- Magnetic flux through annulus S bounded by u-circles $\gamma_2 - \gamma_1$ is $\tau[\psi]$, where $[\psi] = \psi(\gamma_2) - \psi(\gamma_1)$:
  $\int_S i_B\Omega = \int_0^\tau dt \int_{\phi_t^u L} i_u i_B \Omega$ for an arc L from γ_1 to γ_2 and time t along u. $i_u i_B \Omega = d\psi$ and $\int dt = \tau$.
- Current through S is $\int_S i_J\Omega = [\int_\gamma B^\flat] = [\int_0^\tau u \cdot B\, dt] = \tau[C]$.

Alternative fibration by tori

- Instead of using u, B commuting vector fields on level sets of ψ, can use u, J commuting vector fields on level sets of C.
- $i_{[u,J]}\Omega = L_u i_J \Omega - i_J L_u \Omega = L_u dB^\flat = 0$ so $[u, J] = 0$.
- Already have $i_u dC = 0$.
  $i_u i_J \Omega = i_u dB^\flat = L_u B^\flat - di_u B^\flat = -dC$, so $i_J dC = 0$.
- Thus have LA coordinates on regular level sets of C.
- If there is a regular joint level set of (C, ψ) then get common period for u by propagating constant period on level set of C and that for ψ.
- Not useful in MHS (where we'll show C constant on flux surfaces), but might be useful more generally.

Conditions for a QS

- Can reduce to conditions on just u and the metric g.
- Let the rate of strain tensor $E = g^{-1} L_u g$.
- **Theorem**: u a QS implies $\mathrm{div}\, u = 0$ (equivalently $\mathrm{tr}\, E = 0$), and E has a unit null field e (in particular $\det E = 0$) with $[u, e] = 0$ independent of u a.e.
- **Proof**: $L_u \Omega = 0$, $[u, b] = 0$, & $i_b L_u g = 0$. □
- $\mathrm{tr}\, E = \det E = 0$ implies $\mathrm{rank}\, E = 0$ or 2. The conditions can be written as 2 or 3 homogeneous PDEs for u: $\mathrm{div}\, u = 0$ is first order, $\det E = 0$ is third order. In the rank-2 case, for suitable ordering of components, a null vector is $x = \begin{bmatrix} E_{12}E_{23} - E_{22}E_{13} \\ E_{21}E_{13} - E_{11}E_{23} \\ E_{11}E_{22} - E_{12}E_{21} \end{bmatrix}$. Let $e = x/|x|$, then require $[u, x] = |x|^{-2} g(x, [u, x])x$, which can be written as $x \times [u, x] = 0$. It is of fifth order.
- If $L_u g = 0$ then can choose any u-invariant functions ψ, C and get a QS field $B = (u \times \nabla\psi + Cu)/|u|^2$ (& $[u, b] = 0$).
- For rank 2 under conditions of Thm, $\exists$ compatible B & general formula for it by $\mathbb{T}^2$-averaging over flow of (u, e) [Burby].

QS in MHS: C constant on flux surfaces

- A basic desire for plasma confinement is an equilibrium between the charged particles and the magnetic field.
- Simplest is *magnetohydrostatic*: $J \times B = \nabla p$ for some function p (pressure), equivalently $i_B i_J \Omega = dp$.
- Use $i_J \Omega = dB^\flat$; write as $i_B dB^\flat = dp$ or $L_B B^\flat = d(p + |B|^2)$.
- Note that $L_J p = L_B p = 0$ and $[J, B] = 0$: $i_{[J,B]}\Omega = i_J L_B \Omega - L_B i_J \Omega = 0 - L_B dB^\flat = -dL_B B^\flat = 0$.
- Also $L_u p = 0$: Apply L_u to $i_B dB^\flat = dp$ to get $dL_u p = 0$. So $L_u p$ is constant k on connected components. But the orbits of u are closed so $k = 0$. Thus, p is constant on flux surfaces.
- **Theorem**: If u is a QS for an MHS field B then $u \cdot B$ is constant $C(\psi)$ on flux surfaces.
- **Proof**: $0 = L_u B^\flat = i_u dB^\flat + di_u B^\flat$, so $d(u \cdot B) = -i_u dB^\flat$. Applying i_u gives $L_u(u \cdot B) = 0$. Applying i_B gives $L_B(u \cdot B) = i_u dp = 0$. As u, B span the tangent plane to a flux surface then $u \cdot B$ is constant on it. □
- For QS vacuum ($dB^\flat = 0$), C is constant because $i_u dB^\flat = 0$.

Current

- How much toroidal current is there in a QS MHS plasma?
- **Theorem**: $J = -p'(\psi)u - C'(\psi)B$
- **Proof**: $i_B i_J \Omega = dp$, $i_u i_B \Omega = d\psi$, and $L_u B^\flat = 0$ can be written as $i_u i_J \Omega + dC = 0$. $i_J d\psi = i_J i_u i_B \Omega = i_u i_B dB^\flat = i_u dp = 0$, so $J = \kappa u + \lambda B$ for some functions κ, λ. Putting this into the first gives $-\kappa d\psi = dp$, so $\kappa = -p'$. And into the third gives $\lambda d\psi + dC = 0$, so $\lambda = -C'$. □
- Choosing poloidal & toroidal LA coordinates θ, ϕ for $[u, B] = 0$, then $J^\phi = -p' u^\phi - C' B^\phi$. This is a function of ψ. u^ϕ is a constant. Maybe could choose the rest to cancel?
- But maybe the real point is to reduce $\int_S i_J \Omega$ over a poloidal disk S. That equals $\int_{\partial S} B^\flat$. How to get hold of that?

QS Grad-Shafranov equation

- A PDE for ψ for a QS MHS plasma.
- Contracting J with $u^\flat$: $J \cdot u + CC' + |u|^2 p' = 0$.
- $i_u i_B \Omega = d\psi$ and $u \cdot B = C$ imply $B = (Cu + u \times \nabla\psi)/|u|^2$.
- $i_{\nabla\psi}\Omega = i_{B\times u}\Omega = B^\flat \wedge u^\flat$. Let $v = \operatorname{curl} u$. Then $di_{\nabla\psi}\Omega = dB^\flat \wedge u^\flat - B^\flat \wedge du^\flat = i_J\Omega \wedge u^\flat - B^\flat \wedge i_v\Omega$.
- For a vector field X and volume-form Ω, $\mathrm{div} X$ is defined by $L_X\Omega = (\mathrm{div} X)\Omega$.
- So Laplacian $\Delta\psi = \mathrm{div}\nabla\psi = u \cdot J - B \cdot v$.
- $B \cdot v = (Cu \cdot v - u \times v \cdot \nabla\psi)/|u|^2$.
- QSGSE: $\Delta\psi - \frac{u\times v}{|u|^2} \cdot \nabla\psi + C\frac{u\cdot v}{|u|^2} + CC'(\psi) + |u|^2 p'(\psi) = 0$, with $L_u\psi = 0$.

Axisymmetric case

- $u = r\hat{\phi}, v = 2\hat{z}$, so $u \times v = 2r\hat{r}, u \cdot v = 0, |u|^2 = r^2$. Gives GSE $\Delta^*\psi + CC' + r^2 p' = 0$ with $\Delta^*\psi = \partial_z^2\psi + \partial_r^2\psi - \frac{1}{r}\partial_r\psi$.
- Known also as Hicks 1899 equation for ideal fluid flows.
- Under nice conditions, specify functions p and C of ψ and get existence & uniqueness of a solution ψ as a function of (r, z).
- Variational formulation $\delta \int \left(\frac{|\nabla\psi|^2 - C^2}{2r^2} - p\right) r\, dr\, dz = 0$.
- Solov'ev equilibria:
  $\psi(r, z) = (bR^2 + c_0 r^2)\frac{z^2}{2} + \frac{1}{8}(a - c_0)(r^2 - R^2)^2$ is a solution for $p = p_0 - a\psi,\ C^2 = C_0^2 - 2bR^2\psi$.

QSGSE continued

- When $L_u g \neq 0$, the QSGSE needs supplementing by 3 other PDEs to enforce $L_B \Omega = 0, L_u B^\flat = 0$.
- We didn't find a variational principle for it.
- There is an alternative QSGSE using circle-averaged metric, which does have a variational principle.
- JW Burby, N Kallinikos, RS MacKay, Generalised Grad-Shafranov equation for non-axisymmetric MHD equilibria, Phys Plasmas 27 (2020) 102504

QS in MHS with magnetic axis

- If a foliation by toroidal flux surfaces degenerates to a circle, call it a *magnetic axis*.
- **Theorem**: For an MHS field in Euclidean space with a magnetic axis γ, QS must have $M = 1$.
- **Proof**: $dp = i_B dB^\flat = |B| i_b d(|B| b^\flat) = |B| i_b (|B| db^\flat + d|B| \wedge b^\flat) = |B|^2 i_b db^\flat - |B| d_\perp |B|$, where $d_\perp |B| = d|B| - (i_b d|B|) b^\flat$. Now $i_b db^\flat = L_b b^\flat = \kappa^\flat$, where κ is the curvature vector for b. On γ, $dp = 0$ and $\kappa \neq 0$ somewhere. So $d_\perp |B| \neq 0$ there.
  • If $M > 1$ then the nearby u-lines cross plane perpendicular to γ in at least M points (actually, precisely M, using $u \cdot B = C(\psi)$) and have same $|B|$ at each, so unless they cluster then $d_\perp |B| = 0$. They can't all cluster because they are equally spaced in LA coordinates.
  • If $M = 0$ then $u \cdot B$ constant on flux surfaces implies u close to perpendicular to γ, so again $d_\perp |B|$ goes to zero on γ. □
- But perhaps we don't care about integrability near the magnetic axis, as long as we have it further out.

Beyond MHS

- Even staying at the level of one-fluid models for a plasma, there is scope for more equilibria than MHS.
- Can allow a mean flow velocity v (density ρ) and an electrostatic potential Φ.
- Ideal equilibrium: $L_{\rho v}\Omega = 0$, $\rho i_v dv^\flat = i_B dB^\flat - dp$, $L_v(p\rho^{-5/3}) = 0$, $d\Phi = i_B i_v \Omega$. Leaves out resistivity, Hall effect, Nernst effect, heat flow...
- Single particle Hamiltonian $H = \frac{p_\parallel^2}{2m} + \mu|B| + e\Phi$. Natural to require $L_u\Phi = 0$ for a QS: says u, B, v are linearly dependent, so write $v = xB + yu$.
- Probably want to require $L_u\rho = 0$. Then $L_{\rho v}\Omega = 0$ implies $L_B(\rho x) + L_u(\rho y) = 0$.
- Should we require $[u, v] = 0$? Then $L_u x = L_u y = 0$ by independence of u, B.
- What more can one deduce?
- Extend to anisotropic pressure, cf Rodriguez & Bhattacharjee PRE 104 (2021) 015213