Mathematics for Fusion Power part 3

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General Hamiltonian symmetries of FGCM

Approximate symmetries

Relativistic GC motion

Velocity-dependent symmetries

- GC fibre bundle N over 3D M, $\pi : N \to M$, fibres $\mathbb{R}b^\flat$.
- What freedom do we gain in Hamiltonian symmetries if allow $U = (u, w)$ on N to depend on $p_\|$?
- FGCM $H = \frac{1}{2m}p_\|^2 + \mu|B|$, $\omega = -d(p_\| b^\flat) - e\pi^*\beta$.
- Suppose U does not depend on μ.
- Then $L_U H = 0$ implies $\frac{1}{m}p_\| w = 0$ and $L_u|B| = 0$. In particular, $w = 0$.
- $L_U\omega = 0$ with $w = 0$ implies $0 = L_u\omega = -d(L_u(p_\| b^\flat) + ei_u\beta)$ because $d\beta = 0$. So $L_u(p_\| b^\flat) + ei_u\beta = dL$ for some local function L (suppose global) and L is conserved.
- $dL = ei_u i_B\Omega + p_\| L_u b^\flat$.
- Nikos will develop further next week, in particular allowing approximate symmetries.

Approximate symmetries of GC approximations

- FGCM is only a first-order approximation, so allow approximate Hamiltonian symmetries of approximate systems.
- Can take small parameter $\varepsilon = m/e$ and scale variables.
- $H = \varepsilon(\frac{1}{2}p_{\|}^2 + \mu|B|)$, $\omega = -\beta - \varepsilon d(p_{\|} b^{\flat})$.
- Awkward feature that leading order of ω is degenerate.
- **Theorem**: $u_0 + \varepsilon u_1$ is an approximate Hamiltonian symmetry of FGCM on N iff $L_{u_0}\beta = 0$, $L_{u_0}\Omega = 0$, $L_{u_0}|B| = 0$ and $u_1 = \frac{b}{|B|} \times (p_{\|} X_0 - \nabla\psi_1)$ with $X_0 = \mathrm{curl}\,(b \times u_0) + \nabla(u_0 \cdot b)$ and a function ψ_1 such that $i_B d\psi_1 = 0$, $\partial_{p_{\|}}\psi_1 = p_{\|}\partial_{p_{\|}} b \cdot u_0$. It produces approximate conserved quantity $L = -\psi_0 - \varepsilon(\psi_1 - p_{\|} u_0 \cdot b)$.
- Case of u with u_0 independent of $p_{\|}$ is called a *weak quasi-symmetry* ($L_{u_0}|B| = 0$, $L_{u_0}\beta = 0$, $L_{u_0}\Omega = 0$): Rodriguez E, Helander P, Bhattacharjee A, Necessary and sufficient conditions for quasisymmetry, Phys Plasma 27 (2020) 062501
- Burby JW, Kallinikos N, MacKay RS, Approximate symmetries of guiding-centre motion, J Phys A 54 (2021) 125202

Weak QS & MHS implies a scaling is a QS

- **Theorem**: If $L_B\Omega = 0$, $i_B dB^\flat = dp$ (MHS), u is a vector field with $L_u\Omega = 0$, $L_u|B| = 0$, $i_u i_B\Omega = d\psi \neq 0$ a.e., $u \cdot B \neq 0$, & B has density of irrational surfaces (DIS), then $\exists\ \tau(\psi) \neq 0$ s.t. τu is a QS for B.
- **Proof**: $L_u\Omega = 0$, $i_u i_B\Omega = d\psi$, $L_B\Omega = 0$ imply $[u, B] = 0$. MHS $i_B i_J\Omega = dp$ implies $[J, B] = 0$. Also, $i_B dp = 0$ & DIS imply p a function of ψ, in particular J tangent to flux surfaces. $d\psi \neq 0$ implies u, B indpt tangents to flux surfaces, so $J = \kappa u + \lambda B$ for some functions κ, λ. Then $0 = [J, B] = (L_B\kappa)u + (L_B\lambda)B$, so $L_B\kappa = L_B\lambda = 0$. DIS implies κ and λ constant on flux surfaces. $i_B L_u B^\flat = L_u|B|^2 = 0$. Also, $L_B i_u B^\flat = i_u d(p + |B|^2) = 0$, so by DIS, $u \cdot B = C(\psi)$, some C. So $i_u L_u B^\flat = L_u C = 0$.
  Let $v = \tau u$, some $\tau(\psi)$. $L_v B^\flat = \tau L_u B^\flat + C d\tau$ so $i_B L_v B^\flat$ & $i_u L_v B^\flat$ are 0. For $n = \frac{\nabla\psi}{|\nabla\psi|^2}$, $i_n L_v B^\flat = \tau i_n i_u i_J\Omega + i_n d(\tau C) = \tau\lambda + \frac{d}{d\psi}(\tau C)$.
  We can make this 0 by choosing $\tau = \frac{1}{C}\exp(-\int^\psi \frac{\lambda}{C} d\psi)$.
  (B, u, n) span the tangent space a.e., hence $L_v B^\flat = 0$.
  $L_v\Omega = 0$, $L_v|B| = 0$ and $i_v i_B\Omega = \tau d\psi = d\Psi$ where $\Psi = \int^\psi \tau d\psi$ so has the same level sets, and v is a QS for B. □

Remark

- $u \cdot B \neq 0$ is unnecessary. Instead of solving $i_n L_{\tau u} B^\flat = 0$ for τ, define τ to be the period function of the u-lines. It is a flux function by existence of LA coordinates for $[u, B] = 0$. Then let $v = \tau u$ and circle average $L_v B^\flat = i_v i_J \Omega + d i_v B^\flat = \tau \lambda d\psi + d(\tau C)$ over the flow of v: $0 = (\tau\lambda + (\tau C)')\langle d\psi \rangle = (\tau\lambda + (\tau C)')d\psi$. So $i_n L_v B^\flat = 0$ for this choice of τ.

Triple product criterion for weak QS

- Recall that $L_u\beta = 0$ plus bounded u-lines and $\operatorname{div} B = 0$ implies $i_u i_B \Omega = d\psi$ for some function ψ.
- **Theorem** [Rodriguez et al]: If $B \cdot \nabla|B| \neq 0$ a.e. then $u = \frac{\nabla\psi \times \nabla|B|}{B\cdot\nabla|B|}$ is a weak QS with flux function ψ iff $\Omega(\nabla\psi, \nabla|B|, \nabla(B \cdot \nabla|B|)) = 0$ and $B \cdot \nabla\psi = 0$.
- **Proof**: "if": $L_u|B| = 0$, $B \times u = \nabla\psi$ (using $B \cdot \nabla\psi = 0$), $\mathrm{div} u = -\Omega(\nabla\psi, \nabla|B|, \nabla(B \cdot \nabla|B|))/(B \cdot \nabla|B|)^2 = 0$.
  "only if": $B \times u = \nabla\psi$ implies $B \cdot \nabla\psi = 0$.
  $\Omega(\nabla\psi, \nabla|B|, \nabla(B \cdot \nabla|B|)) = -(B \cdot \nabla|B|)^2 \mathrm{div} u = 0$. □
- Question about continuity of u where $B \cdot \nabla|B| = 0$ (which must occur).

Relativistic version

- DT fusion alphas have $|v|/c \approx 4.3\%$.
- Relativistic charged particle motion in a steady EM field in 3D space M has Hamiltonian formulation on T^*M: $H = \gamma(p)mc^2 + e\Phi(q)$, $\omega = -d(\pi^* p) - e\pi^*\beta$, with $\gamma = \sqrt{1 + (|p|/mc)^2}$. Note $p = \gamma m v^\flat$ and $\gamma = (1 - |v|^2/c^2)^{-1/2}$.
- Reduction by gyro-rotation produces adiabatic invariant $\mu = \frac{|p_\perp|^2}{2m|B|}$, $H = c\sqrt{m^2c^2 + p_\parallel^2 + 2m\mu|B(Q)} + e\Phi(Q)$, $\omega = -d(p_\parallel b^\flat) - e\beta$.
- Quasi-symmetry u: $L_u|B| = 0$, $L_u\Phi = 0$, $L_u b^\flat = 0$, $L_u\beta = 0$.
- Alternatively, motion in general EM field in space-time $\tilde{M}$ wrt particle's proper time τ: $\frac{dP}{d\tau} = -e i_V F$, where in Minkowski coordinates $g = -c^2dt^2 + \sum_i (dx^i)^2$, $V = \gamma(1, v)$, $P = mV^\flat = (-\mathcal{E}, p)$ and F is the Faraday (closed) 2-form $F = \sum_\sigma B^i dx^j \wedge dx^k + \sum_i \frac{E_i}{c} dx^i \wedge dt$ for cyclic perms σ of 123.
- Hamiltonian form on $T^*\tilde{M}$: $H = |P|^2/2m$, $\omega = -d(\pi^* P) - e\pi^* F$, restricted to $H = -\frac{1}{2}mc^2$.
- Can reduce by gyro-rotation to 3DoF. Time-translation symmetry or generalisations reduce to 2DoF. Quasi-symmetries reduce to 1DoF.

Plan for remaining weeks

- week 4: Nikos to present generalised and approximate symmetries of GCM
- week 5: relax to omnigenity
- week 6: relax to isodrastic plus KAM tori
- week 7&8: interaction of two charges
- plus perhaps pressure-jump Hamiltonian, divertors