ThermoDynamics for Economic Systems

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Introduction

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Introduction

- Thermodynamics is study of relations between heat and work.
- It started with the question of how much mechanical work can you get out of burning coal to boil water? Same question subsequently for electrical power stations.
- Even more interesting these days is the reverse: what heating rate can you get from given electrical power? If you plug a 30 ohm electrical resistance into 240V, you can heat your room at 1.92 kW. But you can heat it at 28kW with the same electrical power if you use an ideal heat pump from outside temperature of 0°C to inside of 20°C.
- Economics is about money, goods, preferences, labour and lots more.
- But raises questions like how much money can I make from some trade or manufacturing process? Or how much do I have to pay to get something done.
- Sounds rather similar.
- Is there a version of thermodynamics for economics?

Precursors of Thermoeconomics

- Irving Fisher's PhD in Economics (1891) "Mathematical Investigations in the Theory of Value and Prices", was supervised by Gibbs (together with sociologist Sumner)
- Utility theory (1890-1950): Fisher, Slutsky, Hotelling, Georgescu-Roegen, Houthakker, Samuelson
- Samuelson: "Pressure and volume, and for that matter absolute temperature and entropy, have to each other the same conjugate or dualistic relation that the wage rate has to labor or the land rent has to acres of land." (1970)
- Georgescu-Roegen, "The Entropy Law and the Economic Process" (1971); termed "thermoeconomics" by Corning & Kline 1998, but we don't use the word in this sense.
- Econophysics, e.g. Jaynes, 1982; Farjoun & Machover, 1983 (name coined by Stanley, 1996), Saslow, 1999; Raine, Foster, Potts, 2006; and Classical econophysics, Cockshott et al, 2009

Our approach

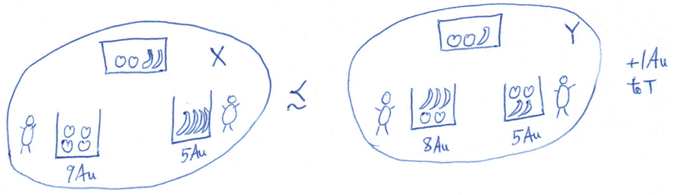
- Want a description of macroeconomics that predicts which trades will happen if barriers are removed or tariffs changed, what manufacturing processes are viable, to give meaning to the temperature of an economy...
- Take an axiomatic approach. Much loved by theoretical economists, maligned by practical ones.
- Krugman, 2009: "The economics profession went astray because economists, as a group, mistook beauty, clad in impressive-looking mathematics, for truth."
- Nonetheless, we attempt to justify the axioms from reality.
- Specifically, Lieb & Yngvason's axiomatic approach to thermodynamics (1998) avoids talking about heat and work: "Maybe an ingenious reader will find an application of this same logical structure to another field of science."
- We use their axiom numbering scheme A1-A16.
- Tell the beginnings of a coherent story and sketch areas still to be explored.

Setting

- Start with an exchange economy. People exchange different sorts of durable good with each other, including a "numéraire" we'll call *money*, measured in *Au* or $\odot$ ("aurum").
- Suppose any closed exchange economy settles to a statistical equilibrium state, determined by its amounts of goods and money. Not equilibrium in sense of Nash or Arrow & Debreu (a static state where it is to no-one's advantage to change anything, or prices such that supply $=$ demand).
- Irving Fisher, 1933: It is as absurd to assume that, for any long period of time, the variables in the economic organization, or any part of them, will "stay put," in perfect equilibrium, as to assume that the Atlantic Ocean can ever be without a wave.
- Even without production/consumption, envisage a dynamic state of continuing exchanges but a steady probability distribution (like Dynamic Stochastic General Equilibrium).
- *[Dubious, e.g. business cycles are plausibly endogenous, but perhaps depend on maintenance far from eqm.]

Accessibility

- Consider the effects on an economy that an external trader T with unlimited goods and money can achieve.
- Say state Y of the economy is *accessible* from state X, written $X \precsim Y$, if the trader can move the economy from X to Y. It is a pre-order (reflexive and A2: transitive).



- Say Y is *equivalent* to X and write $X \sim Y$, if $X \precsim Y$ & $Y \precsim X$. It is an equivalence relation (symmetric, A1: reflexive, and transitive), representing reversible accessibility.

continued

- Write $X \prec Y$ if $X \precsim Y$ but $X \not\sim Y$.
- Denote the states of a system consisting of two unconnected economies A, B by (X_A, Y_B).
  A3: If $X_A \precsim X'_A$ and $Y_B \precsim Y'_B$ then $(X_A, Y_B) \precsim (X'_A, Y'_B)$.
- Similarly, one can consider a scaled version λA of an economy A, for any $\lambda > 0$.
  A4: If $X \precsim Y$ for A then $\lambda X \precsim \lambda Y$ for λA.
  *[but scaling symmetry is probably not valid]

Financial equilibrium

- Given two economies A and B, define their *financial join* to be the joint economy where money is allowed to flow between them but nothing else (*financial contact*), e.g. send to relatives or invest in an enterprise.
- Denote by $\theta(X_A, Y_B)$ the state of the financial join of A, B reached from initial states X_A, Y_B after they've come to equilibrium. We deduce A11: $(X_A, Y_B) \precsim \theta(X_A, Y_B)$, because the trader doesn't have to do anything.
- Also A12: there exist X'_A, Y'_B s.t. $\theta(X_A, Y_B) \sim (X'_A, Y'_B)$, because trader could provide flow of money between A and B.
- If there is no nett flow of money between them on making financial contact, say their states are in *financial equilibrium*: $X_A \equiv Y_B$. Equivalently, after money flow, $X'_A \equiv Y'_B$.
- Suppose each state of an economy to be in financial equilibrium with itself (i.e. there are no internal barriers to money flow in an economy), so $\equiv$ is reflexive.
- Also A13: $\equiv$ is transitive, so an equivalence relation.

Relation between $\precsim$ and $\equiv$

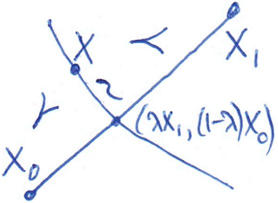
- A key axiom of [LY] is A14: for each state X of a system A there are states X_0, X_1 with $X_0 \equiv X_1$ such that $X_0 \prec X \prec X_1$. This requires that one can make a copy of a system to talk about the financial join of a system with itself. To justify A14
  - First let X_0', X_1' be $X \mp 1\odot$. Then $X_0' \prec X \prec X_1'$ because, assuming that an economy accepts money and does not give it away for nothing, the trader can just give $1\odot$ to move X_0' to X and $1\odot$ to move X to X_1', and each change is irreversible.
  - Then clone the system into A_0, A_1 with initial states X_0', X_1'. The trader can buy goods from A_0 at its market price and sell them to A_1 at their market price, which are in general different. Trading at market prices is reversible and the trader ends up in the same state except for amount of money.
  - Eventually, the trader moves the states of the two copies so that the market price for some (and any) good is the same in both. It implies financial equilibrium between A_0 and A_1. Let $X_0 \equiv X_1$ be states achieved, then $X_0 \sim X_0' \prec X \prec X_1' \sim X_1$.

Subdivision

- Another axiom is A5: any system can be subdivided into two parts in an arbitrary ratio $\lambda : 1-\lambda$ by cutting connections, and in particular the state X into a state $(\lambda X, (1-\lambda)X)$ of the system, with $X \sim (\lambda X, (1-\lambda)X)$.
  *[It is not clear that this holds in the economic context: cutting internal connections might make a substantial change; but we proceed nonetheless.]
- Then following [LY], for all states $Y = (\lambda_1 Y_1, \ldots \lambda_n Y_n)$ and Y' (with not necessarily the same subdivisions) of a system, $Y \precsim Y'$ or $Y' \precsim Y$ (including $Y \sim Y'$).
- Also assume A6: if $(X, \varepsilon Z_0) \precsim (Y, \varepsilon Z_1)$ for some Z_0, Z_1 and a sequence of $\varepsilon \to 0$ then $X \precsim Y$.

Consequences: 1. Entropy

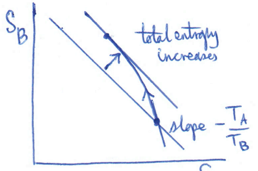
- By Theorem 1 of [LY], deduce existence of a function S of state, unique up to affine transformation, such that for $\sum_i \lambda_i = \sum_j \lambda'_j$, $Y \precsim Y'$ iff $\sum_i \lambda_i S(Y_i) \leq \sum_j \lambda'_j S(Y'_j)$.

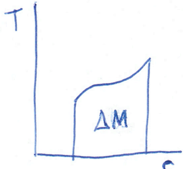


- We call S *entropy*.
- **Second law of Economics**: On putting two or more economies in contact the total entropy can not decrease.
- So we interpret S as aggregate utility

2. Temperature

- With a few more axioms, we also deduce existence of a function T of state such that when two systems are in financial contact there is nett money flow from A to B iff $T(X_A) > T(Y_B)$, and A, B are in financial equilibrium iff $T(X_A) = T(Y_B)$. Call T *temperature*.
- T satisfies $\frac{1}{T} = \frac{\partial S}{\partial M}$ holding all fixed except money M.
- So it has units $\odot/S$, which depends on entropy scale, but ratios of temperatures are uniquely defined.
- Following Skilling, call $\beta = \frac{1}{T}$ *coolness*. Its economic interpretation is marginal utility of money (Hotelling, 1932).
- Money flows, with $T_B dS_B = dM_{AB} = -T_A dS_A$ for reversible changes, until temperatures equalise. A loses entropy, B gains. Note $\frac{dT}{dS} = \frac{T}{C}$ where *money capacity* $C = \frac{dM}{dT}$.



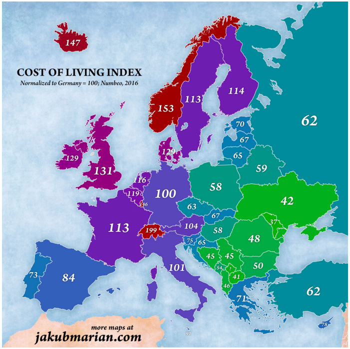


3. Market prices

- For each type of good, with amount G, let $\nu = \frac{\partial S}{\partial G}$ ('ceteris paribus'), called *potential* for the good.
- Then $dS = \beta dM + \nu dG$.
- So we deduce a *market price* $\mu = \nu/\beta$ for the good, i.e. for reversible exchange between M and G.
- By entropy law (taking trader to have constant entropy), economy will buy from external trader at prices below μ, sell at prices above.
- Can write in conventional form $dM = TdS - \mu dG$.
- Can extend to a vector G of amounts of goods and (co)vector ν of potentials.

Temperature estimates

- Can one estimate temperature ratio between two economies not in financial contact?
- Many countries impose money or foreign exchange controls or capital controls (and in the past), so there is scope for this.
- Even for countries in financial contact there are large variations in purchasing power, e.g. Penn World Tables, World Bank's International Comparison Program. Presumably other forces hold them out of financial equilibrium



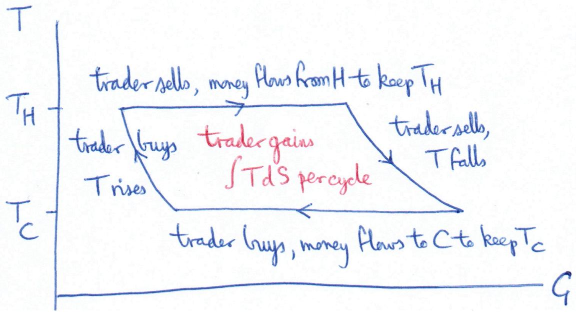
(thermoelectric effect).

Carnot cycle

- We illustrate by an economic Carnot cycle.
- Suppose an island economy A can be put in financial contact with one of two mainland economies H (hot) and C (cold) at a time, $T_H > T_C$.
- The trader can make money by moving the island's economy near-reversibly to the temperature of H (buying goods slightly above the market price so that free island goods decrease, money increases, hence temperature rises), putting it in financial contact with H, moving it isothermally with nett transfer of money from H to A (by selling goods to the island slightly under the market price), disconnecting the bridge to H, moving A reversibly to the temperature of C (selling goods so temperature falls), putting A in contact with C, moving it isothermally by buying from the island with nett flow of money to C,...

continued

- The nett effect of each cycle is to move some money from H and split it between C and the trader: a money pump.



Financial thermometer

- We can make a financial thermometer by putting a (small) ship economy s into financial contact with the economy A of interest but with no transfer of goods.
- If $T_s < T_A$, people in A deposit money in banks that invest in s because goods are cheap there; it is lent out in s to buy goods and drives up their price until T_s reaches T_A.
- If $T_s > T_A$, people on s deposit money in banks that invest in A because money is more useful in A; with the outflow of money from s, the price of goods fall on s until T_s reaches T_A.
- The market price of each good on the ship will settle down to a function of the temperature of the mainland, so one can read off the temperature from the market price after calibration (or could use total money on s).
- A toy example is a ship economy with a single good and N independent identical Cobb-Douglas agents, for which $S = N \log(\frac{G}{N})^{\alpha}(\frac{M}{N})^{\gamma}$. Then $\mu = \alpha \frac{N}{G} T$ (compare Kelvin).

Trade between economies

- We suppose an intermediary trader and that for some good, $\mu_A < \mu_B$. Trader can buy goods from A at any price above μ_A and sell them to B at any price below μ_B. So trader can end up with no change in goods, but an increase in money.
- If there are costs like transport, tariffs, then $\mu_B - \mu_A$ has to be big enough to cover them.

Investment between economies

- What happens if two economies are put in contact allowing transfer of some types of good, including money?
- Then $dS_i = \beta_i dM_i + \nu_i dG_i$ for each economy $i = A, B$.
- The possible transfers dM_{AB}, dG_{AB} from A to B are those that do not decrease total entropy, i.e. so that $(\beta_B - \beta_A)dM_{AB} + (\nu_B - \nu_A)dG_{AB} \geq 0$.
- Flows can continue until all potentials agree. This corresponds to maximising total entropy subject to constraints. "Gains of trade" = increase in aggregate utility.
- External trader can extract money (compare smuggler). Maximise profit by making every step quasi-reversible.

continued

- If A is large compared to B, the maximum profit is given by the "exergy"

$$W = M_B - T_A S_B + \mu_A G_B - M_0 \geq 0,$$

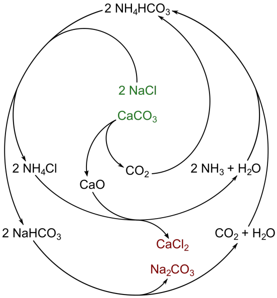
  with M_0 to make $W = 0$ if $T_B = T_A, \mu_B = \mu_A$.
- For two finite economies A, B, there are values β, ν at which they will equilibrate. Then the maximum profit that a trader could make is the sum of the exergies for the two economies relative to an environment at these values.
- If B imposes a tariff τ per unit, for entry of good, then decrease ν_B by $\beta_B \tau$?
- Exploitation? one economy loses entropy at the benefit of the other? Already seen if only money is free to flow.

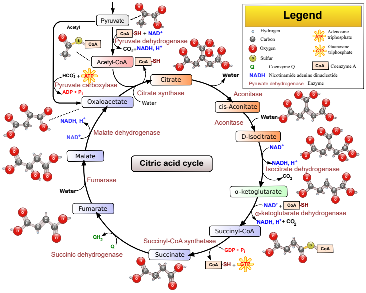
Foreign exchange

- Each economy might have its own currency: €, £, $, ¥.
- Can infer exchange rate via price of each currency relative to "money".
- Insights into Soros pushing the UK out of the ERM.??

Production, Consumption, Manufacturing

- A real economy has production and consumption, not just exchange. This makes it out of equilibrium.
- Furthermore it has manufacturing, analogous to chemical reactions (input-output matrices = stoichiometric matrices) except requires intertemporal comparison and rarely reversible.





- Could perhaps render manufacturing reversible by Arrow-Debreu securities (contracts to pay a specified amount if a specified condition is satisfied at a specified future time)?

Thermodynamic relations

- Maxwell: Symmetry of second derivatives leads to relations between first partial derivatives.
- e.g. $\frac{\partial \beta}{\partial G}|_M = \frac{\partial \nu}{\partial M}|_G$ ($\frac{\partial T}{\partial G}|_M = \mu \frac{\partial T}{\partial M}|_G - T \frac{\partial \mu}{\partial M}|_G$) or $\frac{\partial \mu}{\partial T}|_G = \frac{\partial S}{\partial G}|_T$.
- Compare Slutsky 1915 cross-elasticities, Hotelling integrability conditions, 1932, for individual utility
- Phenomenological relations between fluxes and potential differences?
- Onsager symmetry relations in linear regime?

Further interpretations

- Is entropy = log(available volume of possible configurations)? Thus corresponds to "liberty" (J.S.Mill).
- If so, do we get interpretation of T as a standard deviation of fluctuations in equilibrium, e.g. $\mathrm{Var}(M) = CT^2$ (for money capacity $C = \frac{\partial M}{\partial T}$ of a subsystem)?

Discussion

- Implications for trade negotiations, management of economy, foreign exchange, investment...
- Practical thermometers.
- Continuum (or at least, distributed) economies with local thermoeconomic equilibrium.
- Enterprise as enzymes; free energy for changes at constant T
- Issues of extensivity (maybe need Tsallis entropies), printing of money by central banks, interest rates, growth, technological improvement, frictions, labour, births & deaths, migration (yet more non-equilibrium),...
- Connections to microeconomics, e.g. via Markov processes: Schnakenberg and analogies to chemical reaction theory (de Donder, Prigogine,...), Maes&Netocny..., which way reactions go, cycles turn (Krebs, Solvay)...

Computational Challenge

- It would be great to test the ideas on some in-silico economies. One could make various toy agent-based models and try to measure their temperatures and market prices and see which trades occur.

PhD opportunities at Warwick

- PhD in Mathematics or Interdisciplinary Mathematics:
  next online open day 26 January
  https://warwick.ac.uk/fac/sci/maths/postgrad/prospective/openday
- MSc and PhD in Mathematics of Systems
  https://warwick.ac.uk/fac/sci/mathsys/apply/openday/

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Comments welcome!