Large deviations for empirical cycle counts of integer partitions and their relation to systems of Bosons

Stefan Adams

February 12, 2008

Abstract

VERSION 27.09.2007 Motivated by the Bose gas we introduce certain combinatorial structures. We analyse the asymptotic behaviour of empirical shape measures and of empirical path measures of N Brownian motions with large deviations techniques. The rate functions are given as variational problems which we analyse. A symmetrised system of Brownian motions, that is, for any i, the terminal location of the i-th motion is affixed to the initial point of the $\sigma(i)$-th motion, where σ is a uniformly distributed random permutation of $1, \dots, N$, is highly correlated and has to be formulated such that standard techniques can be applied. We review a novel spatial and a novel cycle structure approach for the symmetrised distributions of the empirical path measures. The cycle structure leads to a proof of a phase transition in the mean path measure.

1 Introduction

We study different aspects of combinatorial asymptotic large-N behaviour of distributions on the group $\mathfrak{S}_N$ of permutations of N elements and their cycles structures distributed on the set $\mathcal{P}_N$ of integer partitions of N. We combine this analysis with large deviations principles for certain empirical path measures of Brownian motions. We review two different approaches to analyse the large-N asymptotic of the mean path measure under symmetrised distributions. One is spatial structure of the symmetrisation and the other one is the cycle structure for concatenations of Brownian bridges to Brownian bridges whose time horizons equal the cycle lengths.

The main focus is to derive variational problems whose analyses will provide deeper insight into the probabilistic asymptotic behaviour for large systems of Brownian motions. This combination of combinatorial studies, large deviations techniques and variational analysis is novel and has its roots in the mathematical analysis of large systems of Bosons, and it is hence related to and carries forward the article Adams and König (2007a) in these proceedings. In Section 1.1 we review main features of studying systems of Bosons and in Section 1.2 we introduce probabilistic models and outline how these raise interesting combinatorial structures for permutations and integer partitions.

1.1 Motivation

The state of a large system of N identical quantum particles (subsystems) is described by the many-body wave function. Two many-body wave functions which result from each other by a permutation of the indices distinguishing the particles must describe the same state. Such a permutation can change the state vector (wave function) only by a numerical factor, and these factors must give a $1-$dimensional representation of the permutation group $\mathfrak{S}_N$ of N elements. Hence, there are only two possible choices, -1 and $+1$. That is, wave functions are antisymmetric or symmetric under permutations. Due to Pauli's exclusion principle, systems of Fermions are described by antisymmetric wave functions. If the wave functions are symmetric, i.e., they are elements in the image of the projection $P_N^+ \colon L^2(\mathbb{R}^{dN}) \to L^2_+(\mathbb{R}^{dN})$ of the N-particle Hilbert space $L^2(\mathbb{R}^{dN})$,

$$P_N^+(\Psi)(x) = \frac{1}{N!} \sum_{\sigma \in \mathfrak{S}_N} \Psi(x_{\sigma(1)}, \dots, x_{\sigma(N)}),$$

one calls the quantum particles Bosons. The Bosons are well-known because they show a phenomenon known as Bose-Einstein condensation. It was predicted by Einstein (1925) on the basis of ideas of the Indian physicist Bose (1924) concerning the statistical description of the quanta of light: In a system of particles described by symmetric many-body wave functions and whose total number is conserved, there should be a temperature below which a finite fraction of all the particles "condense" into the same one-particle state. Einstein's original prediction was for a non-interacting gas of particles. The predicted phase transition is associated with the condensation of atoms in the state of lowest energy and is the consequence of quantum statistical effects.

For a long time these predictions were considered as a curiosity of non-interacting gases and its statistics, called Bose statistics, and had no practical impact. But the ideal gas systems show that the above symmetrisation generates correlations among the non-interacting particles. We review the mathematics concerned with this symmetrisation and its relation to combinatorial studies of integer partitions and corresponding limit theorems. Our main objective is to derive variational formulae via large deviations principles for symmetrised systems of Brownian motions. We briefly motivate this ansatz in the following. N quantum particles are described by the N-particle Hamilton operator

$$H_N = \sum_{i=1}^N \big(- \Delta_i + W(x_i) \big) + \sum_{1 \le i < j \le N} v(|x_i - x_j|), \qquad x_1, \dots, x_N \in \mathbb{R}^d,$$

where the i-th Laplace operator, Δ_i, represents the kinetic energy of the i-th particle, and $W \colon \mathbb{R}^d \to [0, \infty]$ is the trap potential, and where the pair potential $v\colon \mathbb{R}_+ \to \mathbb{R}$ expresses the potential energy of two interacting particles. We do not specify here the assumptions on the trap potential W and on the pair potential v but refer to standard choices in Ruelle (1969) and Lieb et al. (2005). The ground state at zero temperature is the minimiser of the energy and is, due to the symmetry properties of the Hamilton operator, a symmetric N-particle wave function. The study of systems of Bosons at zero temperature is reviewed in the article Adams and König (2007a), more can be found in Lieb et al. (2005). Recent good references including experimental aspects are Pitaevskii and Stringari (2003) and Griffin et al. (1995).

To describe systems of Bosons at thermodynamic equilibrium with inverse temperature $\beta > 0$ one has to analyse the traces of the Boltzmann factor $\mathrm{e}^{-\beta H_N}$, like the free energy, or the pressure, where the trace is restricted to the subspace $L_+(\mathbb{R}^{dN})$ of symmetric N-particle wave functions. The trace class operator $\mathrm{e}^{-\beta H_N}$ is called the canonical ensemble for which the number of particles and the inverse temperature is fixed, see Khinchin (1960) and Thirring (1980) as standard references. The so-called quantum canonical partition function $Z_N^{(\mathrm{sym})}(\beta)$ is the trace of this operator, i.e.,

$$Z_N^{(\mathrm{sym})}(\beta) = \mathrm{Tr}_{\,L^2_+(\mathbb{R}^{dN})}\Big(\mathrm{e}^{-\beta H_N}\Big) = \mathrm{Tr}_{\,L^2(\mathbb{R}^{dN})}\Big(P_N^+ \mathrm{e}^{-\beta H_N}\Big).$$

However, these traces are very difficult to calculate because the spectral analysis of the Hamilton operator H_N with interaction is not known. We will discuss therefore only non-interacting Bosons in this article.

The genuine task of quantum statistical mechanics is to prove and analyse the thermodynamic limit $-\lim_{\Lambda\uparrow\mathbb{R}^d, N\to\infty} 1/\beta|\Lambda| \log Z_N^{(\mathrm{sym})}(\beta)$, which is the free energy, such that $N/|\Lambda| \to \rho \in (0,\infty)$, see Ruelle (1969) for a general introduction to the concept of thermodynmic limit. The quantum statistical mechanics of this ideal Bose gas is well understood (see for example Huang (1987)). One can calculate the specific free energy in the thermodynamic limit as a function of the inverse temperature and the density. The Bose-Einstein condensation transition can be identified here as a singularity in the specific free energy for certain parameter values. The pressure of the ideal Bose gas at finite temperature can be calculated in the so-called grandcanonical ensemble, where the particle number is a Poissonian random variable. However, since Einstein's work 1925 there has been no rigorous mathematical proof for interacting Bosons in the thermodynamic limit for finite density and positive temperature. The only exception is the proof of Bose-Einstein condensation on a lattice with hard-core exclusion and half filling, see Lieb et al. (2005). The main difficulties are the role of the symmetrisation, the role of the interaction and an appropriate definition/criterium of what Bose-Einstein is precisely. There have been three lines of rigorous mathematical attacks. One started with Landau (1941) and its description of superfluidity, which is considered as a Bose-Einstein condensation since London (1938), in terms of the spectrum of elementary excitations of the fluid. In 1947 Bogoliubov developed the first microscopic theory of interacting Bose gases, based on appromximations of the Hamilton operator and the concept of Bose-Einstein condensation. This initiated several theoretical studies; a recent account on the state of the art can be found in Adams and Bru (2004a,b) and on its contribution to superfluidity theory in Adams and Bru (2004c).

A second line is devoted to the study of dilute interacting systems, compare the article Adams and König (2007a) for an overview on this. For these systems the first experimental realisation of Bose-Einstein condensation was derived in 1995. This was followed by rigorous studies in a series of papers by Lieb et al, see Lieb et al. (2005) for more information.

The third line of attack focuses on probabilistic representations of traces and interacting Brownian motions. This started with Adams et al. (2006a) and Adams et al. (2006b) for dilute systems and in Adams and König (2007b), Adams and Dorlas (2007) and Adams (2007) for symmetrised systems. In Section 1.2 below we outline this approach. This approach has two challenging task, one is to deal with the interaction of the Brownian motions and the other one is to resolve the correlations due to the symmetrisation. The symmetrisation correlations are the main subject of this article, and we will outline several aspects of combinatorial and stochastic analysis related with these.

In order to understand Bose-Einstein condensation as a quantum phase transition one needs to study correlation functions. In quantum statistical mechanics correlations can be expressed as reduced traces of the Boltzmann factor (see Thirring (1980) or Bratteli and Robinson (1997)). The reduced one-particle density matrix defines an integral kernel for the corresponding operator. Penrose (1951) and Onsager and Penrose (1956) introduced the concept of the non-diagonal long-range order of the one-particle reduced density matrix (the integral kernel) and defined this as a criterion for Bose-Einstein condensation.

Let us make some remarks on related literature. Scaling limits for shape measures of integer partitions in $\mathcal{P}_N$ under uniform distribution are obtained in Vershik (1996). Large deviations from this limit behaviour are in Dembo et al. (2000), where large deviations principles for scaled shape measures for partitions as well as for strict partitions under uniform distributions are derived. Motivated by the statistics of combinatorial partitions, illustrated by Vershik in Vershik (1996), Benfatto et al. (2005) derived limit theorems for statistics of combinatorial partitions for the case of a mean field Bose gas in the grand-canonical ensemble. Here, in contrast to the canonical ensemble, only the mean of the particle number is fixed. Benfatto et al. (2005) are using Fourier analysis of the corresponding traces to derive a complete description of the statistics of short and long cycles. For a perturbed mean-field model the density of long cycles for a perturbed mean-field model is analysed in Dorlas et al. (2005).

1.2 Systems of Bosons and Probabilistic models

Feynman 1953 introduced the functional integration methods for traces, see de Witt and Storaeds (1970) and Bratteli and Robinson (1997) for details. Since the 1960s, interacting Brownian motions are generally used for probabilistic representations for these traces. The parameter β, which is interpreted as the inverse temperature of the system, is then the length of the time interval of the Brownian motions. Difficulties arise for systems of Bosons due to the symmetrisation (see Bratteli and Robinson (1997)) and the interaction.

Let $\Omega := \{\omega\colon [0,\infty) \to \mathbb{R}^d\colon \omega \text{ continuous}\}$ be the set of continuous functions $[0,\infty) \to \mathbb{R}^d$. The elements in Ω are called trajectories or paths and we denote by $\Omega_k = \{\omega\colon [0,k\beta] \to \mathbb{R}^d\colon \omega \text{ continuous}\}, k \in \mathbb{N}$, the set of paths for time horizon $[0,k\beta]$. We write Ω_β for Ω_1. We equip Ω (respectively Ω_k) with the topology of uniform convergence and with the corresponding Borel σ-field $\mathcal{B}$ (respectively $\mathcal{B}_k$). We consider N Brownian motions, $B^{(1)},\dots,B^{(N)}$, with time horizon $[0,\beta]$ as N random variables taking values in Ω_β. For the reader's convenience, we repeat the definition of a Brownian bridge measure; see the Appendix in Sznitman (1998). We decided to work with Brownian motions having generator Δ instead of $\frac{1}{2}\Delta$. We write $\mathbb{P}_x$ for the probability measure under which $B = B^{(1)}$ starts from $x \in \mathbb{R}^d$. The canonical (non-normalised) Brownian bridge measure on the time interval $[0,\beta]$ with initial site $x \in \mathbb{R}^d$ and terminal site $y \in \mathbb{R}^d$ is defined as $\mu_{x,y}^{\beta}(A) = \mathbb{P}_x(B \in A; B_\beta \in \mathrm{d}y)/\mathrm{d}y$ for $A \subset \Omega_\beta$ measurable. Hence, the Brownian bridge measure for a Brownian bridge confined to a subset $\Lambda_N \subset \mathbb{R}^d$ is defined by

$$\mu_{x,y}^{\beta,N}(A) = \frac{\mathbb{P}_x(B \in A; B_\beta \in \mathrm{d}y, B_{[0,\beta]} \subset \Lambda_N)}{\mathrm{d}y} \qquad A \subset \Omega_\beta \text{ measurable.} \tag{1}$$

If the motions are not confined to stay in Λ_N we have

$$\mu_{x,y}^{\beta}(\Omega_\beta) = \frac{\mathbb{P}_x(B_\beta \in \mathrm{d}y)}{\mathrm{d}y} = (4\pi\beta)^{-d/2}\mathrm{e}^{-\frac{1}{4\beta}|x-y|^2}.$$

The Feynman-Kac formula gives an expression for the traces of Boltzmann factor. For that we define the following interaction Hamiltonian

$$G_{N,\beta} = \sum_{1 \le i < j \le N} \int_0^\beta v(|B_t^{(i)} - B_t^{(j)}|)\mathrm{d}t$$

for the N Brownian motions $B^{(1)},\dots,B^{(N)}$ with time horizon $[0,\beta]$.

For Dirichlet boundary conditions for the Hamilton operator (Laplace operator), i.e., the particles are enclosed in the box Λ_N, we have

$$\mathrm{Tr}\,(\mathrm{e}^{-\beta H_N}) = \int_{\Lambda_N} \mathrm{d}x_1 \cdots \int_{\Lambda_N} \mathrm{d}x_N \bigotimes_{i=1}^N \mu_{x_i,x_i}^{\beta,N}\Big(\mathrm{e}^{-G_{N,\beta}}\Big).$$

This trace describes so-called Boltzmann particles, which means classical particles for which no special statistics is required. The symmetrised trace is

$$\mathrm{Tr}_{\,L_+^2(\mathbb{R}^{dN})}(\mathrm{e}^{-\beta H_N}) = \frac{1}{N!}\sum_{\sigma \in \mathfrak{S}_N} \int_{\Lambda_N} \mathrm{d}x_1 \cdots \int_{\Lambda_N} \mathrm{d}x_N \bigotimes_{i=1}^N \mu_{x_i,x_{\sigma(i)}}^{\beta,N}\Big(\mathrm{e}^{-G_{N,\beta}}\Big). \tag{2}$$

The trace formula (2) is the starting point for the remaining sections, it defines transformed path measures for N Brownian motions. As mentioned above there are two aspects to deal with, the interaction and the symmetrisation. The interaction for dilute systems is handled in Adams and König (2007a) of these proceedings and the symmetrisation is studied here. In what follows we therefore do not handle interacting motions but focus on the symmetrisation. This symmetrisation is the origin for the Bose-Einstein condensation and has to be understood deeply. In Section 2 the cycle structure of the permutations raises interesting asymptotic combinatorial questions, and we derive our first large deviations principles for the discrete shape measures under various distributions. We give an overview of the combinatorial research on permutations and integer partitions. In Section 3 we combine the asymptotic combinatorics with certain path empirical measures of the Brownian motions. The objective there is to gain deeper insight in a probabilistic symmetrised model which can provide information on the corresponding systems of Bosons. For the first time we derive a phase transition for mean path measure for a model with no interaction. The proof of this transition requires complete information on the cycle structure, i.e., we will use our insights from Section 2. We contrast the cycle structure to a spatial structure, which we analyse in Section 3.1. This spatial approach is new, see Adams and König (2007b) and Adams and Dorlas (2007), and it gives an indirect proof for the Bose-Einstein condensation. In future, the spatial and the cycle method have to be combined to describe the transition behaviour also with interactions. This combination will enable one to prove Bose-Einstein condensation with the off-diagonal long range order behaviour criterion.

2 Large deviations for Cycle counts

The cycle structure of permutations allows to replace in (2) the sum over permutations by a sum over integer partitions. This in turn defines probability distributions on permutations and on integer partitions. We introduce some basic facts on integer partitions.

For any integer N, a partition λ of N is the collection of integers $n_1 \geq n_2 \geq \cdots \geq n_k \geq 1, k \in \{1,\dots,N\}$, such that $\sum_{i=1}^k n_i = N$. We denote the set of all partitions of N by $\mathcal{P}_N$. Any partition $\lambda \in \mathcal{P}_N$ is determined by the sequence $\{r_k\}_{k=1}^N$ of positive integers r_k such that $\sum_{k=1}^N kr_k = N$, where we write $r_k(\lambda) = r_k$. We call the number r_k an *occupation number* or *cycle count* of the partition, and we denote the whole tuple of the cycle counts by $R_N = (r_1,\dots,r_N)$ The multiplicity $\sharp\lambda$ of a partition is the number of cycles, i.e., $\sharp\lambda = \sum_{k=1}^N r_k$. A cycle of length k is a chain of permutations, such as 1 goes to 2, 2 goes to 3, 3 goes to 4, etc. until $k-1$ goes to k and finally k goes to 1. A permutation with exactly r_k cycles of length k is said to be of type $\{r_k\}_{k=1}^N$. Hence, each partition $\lambda \in \mathcal{P}_N$ corresponds to a conjugacy class $A(\lambda)$ of permutations, i.e., those of the same type, with exactly

$$\sharp A(\lambda) = \frac{N!}{\prod_{k=1}^N r_k! k^{r_k}}$$

elements.

If a permutation is chosen uniformly and at random from the $N!$ possible permutations in $\mathfrak{S}_N$, then the counts r_k of cycles of length k are dependent random variables. The joint distribution of the cycle counts is given by

$$\mathbb{P}(R_N = r) = 1\!\!1\{\sum_{k=1}^N kr_k = N\} \prod_{k=1}^N \Big(\frac{1}{k}\Big)^{r_k} \frac{1}{r_k!} \tag{3}$$

with $r = (r_1, \ldots, r_N) \in \mathbb{Z}_+^N$. The uniform distribution in (3) is called the Ewens Sampling formula with parameter $\Theta = 1$. The Ewens Sampling formula (Ewens (1972)) reads

$$\mathbb{P}(R_N = r) = \frac{N!}{\Theta(\Theta+1)\cdots(\Theta+N-1)} 1\!\!1\{\sum_{k=1}^N kr_k = N\} \prod_{k=1}^N \Big(\frac{\Theta}{k}\Big)^{r_k} \frac{1}{r_k!}$$

with $r = (r_1, \ldots, r_N) \in \mathbb{Z}_+^N$. This sampling formula was analysed intensively by Kingman (1975), Kingman (1978b) and Kingman (1978a), see also Watterson (1976) for a diffusion model of the allele frequencies. There exist an extensive literature on questions related with this sampling formula and random discrete partitions. See the recent monographs Pitman (2002) and Arratia et al. (2003) for an overview and further references. These studies go back to Goncharov (1944), who studied the asymptotic behaviour of the distribution of cycle counts for the uniform (Ewens sampling with $\Theta = 1$) distribution. For permutations of single points of point process clouds in $\mathbb{R}^d$ or graphs we refer to Kolchin (1986), see further the Section 3.1 below.

We focus on large deviations of different distributions of the the following functional of integer partitions, the so-called *discrete empirical shape measure*, or *empirical cycle count* distribution, defined as

$$Q_N \colon \mathcal{P}_N \to \mathcal{M}_1(\mathbb{N}), \lambda \mapsto Q_N^\lambda(\cdot) = \frac{1}{N} \sum_{k=\cdot}^N r_k(\lambda), \tag{4}$$

where $\mathcal{M}_1(\mathbb{N})$ is the set of probability measures on $\mathbb{N}$. We will write $Q_N^\lambda = Q_N$ in the following. The name shape measure has its roots in the two conjugate representations of integer partitions, the so-called Ferrer diagram and the Young Tableau. Define $\widehat{Q}_N(k) = Q_N(k) - Q_N(k+1)$ for any $k \in \mathbb{N}$. Then the occupation numbers are given by $r_k = N\widehat{Q}_N(k), k = 1, \ldots, N$, which define uniquely the integer partition λ. In a Ferrer diagram the partition $\{r_k\}_{k=1}^N$ is represented by r_k rows of k horizontal blocks. They are placed in a diagram in descending order with the longest or largest k at the top. It can also be viewed as a block diagram in the $(NQ_N(1), \ldots, NQ_N(N))$ space. Here $NQ_N(k)$ blocks are put vertically in the k-th column. The total number of rows is the multiplicity $\sharp\lambda$ of the partition and the area (the total number of blocks) of this diagram is N. A Young tableau is similar but, here, $NQ_N(k)$ blocks are put horizontally in the k-th row. One can obtain a Young tableau from a Ferrer diagram by first turning the diagram upside down and then by rotating it through $90°$

clockwise. The notion "shape" comes now from the study of the asymptotic shapes of this diagrams/tableaux as $N \to \infty$ under suitable continuous scaling, see Vershik (1996) and for a recent overview Pitman (2002). Large deviations from the expected shape of the diagrams are studied in Dembo et al. (2000) for the uniform distribution.

We do not scale the discrete shape measure because we need in the limit the whole discrete cycle count distribution for our large deviations results in Section 3. That is we are interested in the large N-behaviour of the discrete shape measures Q_N under different distributions of the integer partitions. Beside the uniform distribution in (3) and general Ewens sampling distribution our main interest is in the following distribution.

$$\nu_N^{\rm (Bose)}(\lambda) = \frac{1}{Z_N^{\rm (Bose)}(\beta)} \prod_{k=1}^N \Big(\frac{\big(\varrho^{-1}N\big)^{r_k}}{r_k! k^{r_k}} \Big) \Big(\frac{1}{4\pi\beta k} \Big)^{{\rm d}/2 r_k} \qquad \lambda \in \mathcal{P}_N \tag{5}$$

with normalisation

$$Z_N^{\rm (Bose)}(\beta) = \sum_{\lambda \in \mathcal{P}_N} \prod_{k=1}^N \Big(\frac{\big(\varrho^{-1}N\big)^{r_k}}{r_k! k^{r_k}} \Big) \big(\frac{1}{4\pi\beta k} \big)^{d/2 r_k}$$

for given $d \in \mathbb{N}$ and $\beta, \varrho > 0$. This distribution is motivated from the non-interacting Bose gas enclosed in $\Lambda_N \subset \mathbb{R}^d$ with particle density $\varrho = |\Lambda_N|/N$. We outline this in the following. Going back to trace formula (2) note that the conjugacy classes $A(\lambda)$ of permutations are the ones where the trace operation is constant because it is a cyclic operation. For each partition $\lambda \in \mathcal{P}_N$ we can regroup the product of the Brownian bridge measures $\mu_{x,y}^{\beta,N}$ in such a way that we concatenate r_k times k Brownian bridges to obtain r_k Brownian bridges of time horizon $[0, k\beta]$. This is possible because we integrate out the intermediate spatial points. Hence we get for the canonical partition function of the non-interacting Bose gas

$$Z_N^{\rm (sym)}(\beta) = \sum_{\lambda \in \mathcal{P}_N} \prod_{k=1}^N \Big(\frac{1}{r_k! k^{r_k}} \Big) \bigotimes_{k=1}^N \big(\int_{\Lambda_N} {\rm d}x \mu_{x,x}^{k\beta,N} \big)^{\otimes r_k} (\Omega_\beta^{\otimes N}). \tag{6}$$

The difference with (5) is that there we are using the free Brownian bridge measure, i.e., the motions are not confined to stay in Λ_N. However, both expression are close and coincide in the limit $\Lambda_N \uparrow \mathbb{R}^d$ as $N \to \infty$, because of the estimation

$$(4\pi\beta k)^{-d/2}(1 - {\rm e}^{-dN/4\beta}) \le \mu_{x,x}^{k\beta,N}(\Omega_k) \le (4\pi\beta k)^{-d/2},$$

which compares the measure with Dirichlet boundary condition for the box Λ_N with the free Brownian bridge measure. It is technically easier here to work on a torus and with periodic boundary conditions for the Laplacian.

To formulate the rate functions we need some notations. Let

$$\mathcal{M} = \{Q \in [0,1]^{\mathbb{N}} \colon \sum_{l\in\mathbb{N}} Q(l) \le 1, Q(l) \ge Q(l+1)\,\forall l \in \mathbb{N}\}$$

be the set of monotonously non-increasing sub-probability functions on $\mathbb{N}$. For $Q \in \mathcal{M}$ define $\widehat{Q}(k) = Q(k) - Q(k+1)$ for any $k \in \mathbb{N}$. For $d \ge 1$ let

$$\widehat{Q}^*(k) = \frac{1}{\rho(4\pi\beta)^{d/2} k^{1+\mathrm{d}/2}}, \quad k \in \mathbb{N}, \tag{7}$$

be given, and define the functional

$$\mathcal{S}^{\text{(Bose)}}(Q) = \sum_{k=1}^{\infty} \widehat{Q}(k)\Big(\log\frac{\widehat{Q}(k)}{\widehat{Q}^*(k)} - 1\Big) \qquad Q \in \mathcal{M}. \tag{8}$$

The corresponding functional for the uniform distribution $\nu_N^{\text{(u)}}$ is given as

$$\mathcal{S}^{\text{(u)}}(Q) = \sum_{k=1}^{\infty} \widehat{Q}(k)\big(\log \widehat{Q}(k)k - 1\big) \qquad Q \in \mathcal{M}.$$

The uniform distribution is defined through (3), i.e.,

$$\nu_N^{\text{(u)}}(\lambda) = \frac{1}{Z_N}\prod_{k=1}^{N}\Big(\frac{1}{k}\Big)^{r_k}\frac{1}{r_k!} \qquad \lambda \in \mathcal{P}_N,$$

with normalisation $Z_N = \sum_{\lambda\in\mathcal{P}_N}\prod_{k=1}^{N}\Big(\frac{1}{k}\Big)^{r_k}\frac{1}{r_k!}$.

The main results follow in the next theorem.

Theorem 2.1 (Adams (2008b).) *(a) Under the uniform measure $\nu_N^{\text{(u)}}$ the empirical discrete shape measures Q_N satisfy a large deviations principle on $\mathcal{M}$ with speed N and rate function*

$$I^{\text{(u)}}(Q) = \mathcal{S}^{\text{(u)}}(Q) - \chi \qquad \textit{with } \chi = \inf_{Q\in\mathcal{M}} \mathcal{S}^{\text{(u)}}(Q). \tag{9}$$

(b) Let $\varrho \in (0,\infty)$ and $\Lambda_N \subset \mathbb{R}^d$ with $\Lambda_N \uparrow \mathbb{R}^d$ and $N/|\Lambda_N| \to \varrho$ as $N \to \infty$. Under the measure $\nu_N^{\text{(Bose)}}$ the empirical discrete shape measures Q_N satisfy a large deviations principle on $\mathcal{M}$ with speed N and rate function

$$I^{\text{(Bose)}}(Q) = \mathcal{S}^{\text{(Bose)}}(Q) - \chi(\beta,\varrho) \qquad \textit{with } \chi(\beta,\varrho) = \inf_{Q\in\mathcal{M}} \mathcal{S}^{\text{(Bose)}}(Q). \tag{10}$$

Remark 2.2 (Free energy, Adams (2007).) *The variational formula* (10) *gives the specific free energy $f(\beta,\rho) := \lim_{N\to\infty} -1/\beta|\Lambda_N|\log Z_N^{\text{(sym)}}(\beta)$ for inverse temperature β and density ρ of the non-interacting Bose gas, i.e.,*

$$f(\beta,\rho) = \frac{\rho}{\beta}\inf_{Q\in\mathcal{M}}\Big\{\sum_{k=1}^{\infty}\widehat{Q}(k)\log\Big(\frac{\widehat{Q}(k)}{\widehat{Q}^*(k)} - 1\Big)\Big\}.$$

We analyse the variational formulae for χ and $\chi(\beta,\rho)$, and we derive an expression for the specific free energy f as a function of β and ρ. Define a dimension dependent critical density

$$\rho_c = \begin{cases} \frac{1}{(4\pi\beta)^{d/2}}\zeta\Big(\frac{d}{2}\Big) , & \text{for } d \geq 3 \\ +\infty , & \text{for } d = 1,2 \end{cases} , \tag{11}$$

where ζ is the Riemann zeta function,

$$\zeta\Big(\frac{d}{2}\Big) = \sum_{k=1}^{\infty} k^{-\frac{d}{2}}.$$

Furthermore, denote by $g_s(\alpha)$ the so-called Bose functions (see (20) in Appendix 4)

$$g_s(\alpha) = \sum_{k=1}^{\infty} k^{-s}\mathrm{e}^{-\alpha k} \qquad \text{for all } \alpha > 0 \text{ and all } s > 0.$$

For any $\rho < \rho_c$ we denote by $\alpha = \alpha(\beta,\rho)$ the unique root of

$$\rho = \frac{1}{(4\pi\beta)^{d/2}} \sum_{k=1}^{\infty} k^{-d/2}\mathrm{e}^{-\alpha k}. \tag{12}$$

The essential difference in $d \geq 3$ and $d = 1,2$ lies in the fact that in the latter two cases the corresponding Bose functions, $g_1(\alpha)$ respectively $g_{\frac{1}{2}}(\alpha)$, diverge as $\alpha \to 0$ (see Appendix 4 and Gram (1925)). For $d = 1,2$ there is a unique α for any density $\rho < \infty$. For $d \geq 3$ there is such an unique α given only for densities $\rho < \rho_c$. Hence, this is the mathematical origin of the so-called Bose-condensation, where for $d \geq 3$ and $\rho > \rho_c$ particles condense in the zero mode state.

Theorem 2.3 (Analysis for $\boldsymbol{\chi}$, Adams (2008b)). *The functional $\mathcal{S}^{(\mathrm{u})}$ is convex and there is a unique minimiser Q^* for $\chi = \inf_{Q\in\mathcal{M}} \mathcal{S}^{(\mathrm{u})}(Q)$, and it is defined through*

$$\widehat{Q}^*(k) = \frac{\mathrm{e}^{-\alpha k}}{k} \qquad k \in \mathbb{N} \textit{ and } \alpha = \log 2.$$

The analysis for the Bose distribution gives the proof of the Bose-Einstein condensation for non-interacting Bose gas depending on the parameters d, ρ and β.

Theorem 2.4 (Analysis for $\boldsymbol{\chi(\beta,\varrho)}$, Adams (2007)). *For any $\rho < \infty$ in dimensions $d = 1,2$, and $\varrho < \varrho_c$ in dimensions $d \geq 3$, there is a unique minimiser $Q \in \mathcal{M}$ of the variational formula for $\chi(\beta,\varrho)$ in* (10) *having probability mass one with*

$$\widehat{Q}(k) = \frac{\mathrm{e}^{-\alpha k}}{\rho(4\pi\beta)^{d/2}k^{1+\frac{d}{2}}} \qquad \textit{for } k \in \mathbb{N},$$

whereas for dimensions $d \geq 3$ and densities $\varrho > \varrho_c$, there is no minimiser for the variational problem (10) *with probability mass one, but the infimum is attained for any minimising sequence $(Q_n)_{n\in\mathbb{N}}$ of $Q_n \in \mathcal{M}$ such that $Q_n \to Q^*$ as $n \to \infty$.*

The specific free energy for $d \geq 3$ is given by

$$f(\beta,\varrho) = \begin{cases} -\frac{1}{(4\pi\beta)^{d/2}\beta} g_{\frac{d+2}{2}}(\alpha) - \frac{1}{\beta}\varrho\alpha\ , & \text{for } \varrho < \varrho_c \\ -\frac{1}{(4\pi\beta)^{d/2}\beta}\zeta\Big(\frac{d+2}{2}\Big)\ , & \text{for } \varrho > \varrho_c, \end{cases}$$

and for $d = 1, 2$ by

$$f(\beta,\varrho) = -\frac{1}{(4\pi\beta)^{d/2}\beta} g_{\frac{d+2}{2}}(\alpha) - \frac{\varrho\alpha}{\beta},$$

where α is the unique root of (12).

3 Large deviations for empirical path measures

In this section we present our large deviations results for the empirical path measures for N Brownian motions $B^{(1)}, \dots, B^{(N)}$ in $\mathbb{R}^d$ with time horizon $[0,\beta]$. The empirical path measures

$$L_N = \frac{1}{N}\sum_{i=1}^{N} \delta_{B^{(i)}}$$

are random elements in the set $\mathcal{M}_1(\Omega_\beta)$ of probability measures on the set Ω_β of continuous paths $[0,\beta] \to \mathbb{R}^d$. We analyse the large-$N$ behaviour of the distributions of L_N under different symmetrised measures in Section 3.1 and Section 3.2 respectively. In both cases we derive large deviations principles whose rate functions are given as variational problems. In Section 3.2 the analysis for the variational problem for the cycle structure gives the proof of a phase transition in the empirical path measure.

3.1 Spatial structure

We analyse the large-N asymptotic of the empirical path measure L_N under the following symmetrised probability measure

$$\mathbb{P}^{(\mathrm{sym})}_{m,N} = \frac{1}{N!}\sum_{\sigma\in\mathfrak{S}_N} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} m(\mathrm{d}x_1)\cdots m(\mathrm{d}x_N) \bigotimes_{i=1}^{N} \mathbb{P}^{\beta}_{x_i,x_{\sigma(i)}}, \qquad (13)$$

where $m \in \mathcal{M}_1(\mathbb{R}^d)$ is a probability measure and where $\mathbb{P}^\beta_{x,y}$ is the Brownian bridge probability measure

$$P^\beta_{x,y} = \mu^\beta_{x,y}/\mu^\beta_{x,y}(\Omega_\beta) = \frac{\mu^\beta_{x,y}}{(4\pi\beta)^{d/2}},$$

i.e., a probability measure on Ω_β. The expectation with respect to the measure $P^\beta_{x,y}$ is denoted by $\mathbb{E}^\beta_{x,y}$. We can conceive $\mathbb{P}^{(\mathrm{sym})}_{m,N}$ as a two-step random mechanism: First we pick uniformly a random permutation σ, then we pick N Brownian motions with initial distribution m, and the i-th motion is conditioned to terminate at the initial point of the $\sigma(i)$-th motion, for any i.

The main idea in resolving the combinatorics of the measure (13) is to rewrite it as a sum over pair frequencies $NQ(x,y), x,y \in \mathbb{R}^d$. Here Q is a pair probability measure with equal marginals, and $NQ(x,y)$ is the number of Brownian motions which are sent from location x to location y due to the symmetrisation. We shall count the number of permutations which are admissible for a given pair probability measure Q. Furthermore for Brownian motions we need to work with open sets of positive Lebesgue measure instead of single points. But this is a technical point and it is analysed in detail in Adams and König (2007b), where one needs an additional assumption on the probability measure $m \in \mathcal{M}_1(\mathbb{R}^d)$. We will neglect these details and refer to Adams and Dorlas (2007), where symmetrised systems of random walks on graphs are analysed and applied to certain mean-field type interacting systems.

The core idea, performed in Adams and König (2007b), Adams and Dorlas (2007) and Adams (2008a), is that the rewriting gives a sum over pair probability measures with two terms, one part is counting permutations for a given pair probability measure, and for any given pair probability measure the other part is a probability measure for N Brownian motions. This probability measure is now a product of not necessarily identically distributed Brownian bridge probability measures. Hence, we resolved the correlations due to the symmetrisation in a two level large deviations setting (see for example Dawson and Gärtner (1994)). Our rate functions consist of two parts, one deals with the combinatorics and is therefore a function of a pair probability measure and the initial measure m, the other part governs the large deviations for the empirical path measures under the corresponding probability measure.

The motivation for this novel approach is threefold. First, it is an appealing method from the mathematical point of view and originated from combinatorial methods for microcanonical ensembles in Adams (2001). Second, Bose-Einstein condensation in Onsager and Penrose (1956) is defined as an off-diagonal long range behaviour of the one-particle reduced density matrix, which measures the correlation between two spatial points. Third, we are informed by Schrödinger (1931) who considered the question of how any two spatial points are connected by random paths. The crucial observation is that this aspect of the problem can be described by pair measures. Schrödinger (1931) raised the question of the most probable behaviour of a large system of diffusing particles in thermal equilibrium. Föllmer (1988) gave a mathematical formulation of these ideas in terms of large deviations. He applied Sanov's theorem to obtain a large deviations principle for L_N when $B^{(1)}, B^{(2)}, \dots$ are i.i.d. Brownian motions with initial distribution m and no condition at time β. The rate function is the relative entropy with respect to $\int_{\mathbb{R}^d} m(\mathrm{d}x)\,\mathbb{P}_x \circ B^{-1}$, where the motions start in x under $\mathbb{P}_x$. Then Schrödinger's question amounts to identifying the minimiser of that rate function under given fixed independent initial and final distributions.

It turns out that the unique minimiser is of the form $\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathrm{d}x\mathrm{d}y\, f(x)g(y)\, \mathbb{P}^\beta_{x,y}\circ B^{-1}$, i.e. a Brownian bridge with independent initial and final distributions. The probability densities f and g are characterised by a pair of dual variational equations, which originally appeared in Schrödinger (1931) for the special case that both the initial and the final measures are Lebesgue measure.

We introduce now the rate functions for our method. With

$$H(Q|P) = \int_{\mathbb{R}^d\times\mathbb{R}^d} Q(\mathrm{d}x)\log\frac{Q(\mathrm{d}x)}{P(\mathrm{d}x)}$$

we denote the relative entropy of the pair probability measure $Q\in\mathcal{M}_1(\mathbb{R}^d\times\mathbb{R}^d)$ with respect to $P\in\mathcal{M}_1(\mathbb{R}^d\times\mathbb{R}^d)$. Let $\mathcal{M}_1^{(\mathrm{s})}(\mathbb{R}^d\times\mathbb{R}^d)$ be the set of shift-invariant probability measures Q on $\mathbb{R}^d\times\mathbb{R}^d$, i.e., measures whose first and second marginals coincide and are both denoted by $\overline{Q}$. Note that $Q\mapsto H(Q|\overline{Q}\otimes m)$ is strictly convex.

Define the functional $I_m^{(\mathrm{sym})}$ on $\mathcal{M}_1(\Omega_\beta)$ by the following variational problem

$$I_m^{(\mathrm{sym})}(\mu) = \inf_{Q\in\mathcal{M}_1^{(\mathrm{s})}(\mathbb{R}^d\times\mathbb{R}^d)}\Big\{H(Q|\overline{Q}\otimes m) + I^{(Q)}(\mu)\Big\},$$

where

$$I^{(Q)}(\mu) = \sup_{\Phi\in\mathcal{C}_\mathrm{b}(\Omega_\beta)}\Big\{\langle\Phi,\mu\rangle - \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} Q(\mathrm{d}x,\mathrm{d}y)\log\mathbb{E}^\beta_{x,y}\big(\mathrm{e}^{\Phi(B)}\big)\Big\} \tag{14}$$

for $\mu\in\mathcal{M}_1(\Omega_\beta)$ and $\langle\Phi,\mu\rangle = \int_{\Omega_\beta}\Phi(\omega)\,\mu(\mathrm{d}\omega)$. Here $\mathcal{C}_\mathrm{b}(\Omega_\beta)$ is the space of bounded continuous functions on Ω_β. Hence, $I^{(Q)}$ is a Legendre-Fenchel transform, but *not* the one of a logarithmic moment generating function of any random variable. In particular, $I^{(Q)}$, and therefore also $I_m^{(\mathrm{sym})}$, are nonnegative, and $I^{(Q)}$ is convex as a supremum of linear functions. There seems to be no way to represent $I^{(Q)}(\mu)$ as the relative entropy of μ with respect to any measure.

Let us explore briefly the variational problem connected with the rate function $I_m^{(\mathrm{sym})}$. By $\pi_s\colon\Omega_\beta\to\mathbb{R}^d$ we denote the projection $\pi_s(\omega)=\omega_s$. The marginal measure of $\mu\in\mathcal{M}_1(\Omega_\beta)$ is denoted by $\mu_s=\mu\circ\pi_s^{-1}\in\mathcal{M}_1(\mathbb{R}^d)$; analogously we write $\mu_{0,\beta}=\mu\circ(\pi_0,\pi_\beta)^{-1}\in\mathcal{M}_1(\mathbb{R}^d\times\mathbb{R}^d)$ for the joint distribution of the initial and the terminal point of a random process with distribution μ. It is easy to see that $Q=\mu_{0,\beta}$ if $I^{(Q)}(\mu)<\infty$. Indeed, in (14) relax the supremum over all $\Phi\in\mathcal{C}_\mathrm{b}(\Omega_\beta)$ to all functions of the form $\omega\mapsto f(\omega_0,\omega_\beta)$ with $f\in\mathcal{C}_\mathrm{b}(\mathbb{R}^d)$. This gives that

$$\begin{aligned}\infty > I^{(Q)}(\mu) &\geq \sup_{f\in\mathcal{C}_\mathrm{b}(\mathbb{R}^d)}\Big(\big\langle\mu_{0,\beta},f\big\rangle - \big\langle Q,\log\mathbb{E}^\beta_{\pi_0,\pi_\beta}\big(\mathrm{e}^{f(B_0,B_\beta)}\big)\Big)\\ &= \sup_{f\in\mathcal{C}_\mathrm{b}(\mathbb{R}^d)}\big\langle\mu_{0,\beta}-q,f\big\rangle,\end{aligned}$$

and this implies that $\mu_{0,\beta}=Q$. In particular, the infimum in the variational problem for $I_m^{(\mathrm{sym})}$ is uniquely attained at this Q, i.e.,

$$I_m^{(\mathrm{sym})}(\mu) = \begin{cases} H\big(\mu_{0,\beta}|\mu_0\otimes m\big) + \sup_{\Phi\in\mathcal{C}_\mathrm{b}(\mathcal{C})}\Big\langle\mu,\Phi-\log\mathbb{E}^\beta_{\pi_0,\pi_\beta}\big(\mathrm{e}^{\Phi(B)}\big)\Big\rangle & \text{if } \mu_0=\mu_\beta,\\ +\infty & \text{otherwise.}\end{cases}$$

In particular, $I_m^{(\mathrm{sym})}$ is convex.

Our main large deviations result reads as follows.

Theorem 3.1 (Large deviations for L_N) *Fix $\beta \in (0,\infty)$ and $m \in \mathcal{M}_1(\mathbb{R}^d)$. Then, as $N \to \infty$, under the symmetrised measure $\mathbb{P}_{m,N}^{(\mathrm{sym})}$, the empirical path measures L_N satisfy a large deviations principle on $\mathcal{M}_1(\Omega_\beta)$ with speed N and rate function $I_m^{(\mathrm{sym})}$.*

Simplifying the large deviations principle says that, as $N \to \infty$,

$$\mathbb{P}_{m,N}^{(\mathrm{sym})}\big(L_N = \mu\big) \approx \mathrm{e}^{-N I_m^{(\mathrm{sym})}(\mu)}, \qquad \mu \in \mathcal{M}_1(\Omega_\beta).$$

Proof. If m has compact support the proof is in Adams and König (2007b). Arbitrary initial distributions are handled in Adams (2008a). A corresponding result for symmetrised systems of random walks on graphs with applications to mean-field models is given in Adams and Dorlas (2007). $\square$

There are also analogous results for the mean

$$Y_N = \frac{1}{N}\sum_{i=1}^N \mu_\beta^{(i)},$$

of the N occupation measures,

$$\mu_\beta^{(i)}(\mathrm{d}x) = \frac{1}{\beta}\int_0^\beta \delta_{B_s}(\mathrm{d}x)\,\mathrm{d}s, \qquad i = 1,\dots,N.$$

We will present below these results for the very special case that m is the Lebesgue measure of finite set in $\mathbb{R}^d$. The general version can be found in Adams and König (2007b) and Adams and Dorlas (2007).

Let us comment briefly on the shape of the rate functions above. The symmetrised measure $\mathbb{P}_{m,N}^{(\mathrm{sym})}$ arises from a two-step probability mechanism. This is reflected in the representation of the rate function $I_m^{(\mathrm{sym})}$: in a peculiar way the entropy term $H(Q|\overline{Q}\otimes m)$ describes the large deviations of the uniformly distributed random permutation σ, together with the integration over $m^{\otimes N}$. The measure Q governs a particular distribution of N independent, but not identically distributed, Brownian bridges. Under this distribution, L_N satisfies a large deviations principle with rate function $I^{(Q)}$, which also can be guessed from the Gärtner-Ellis theorem (Dembo and Zeitouni, 1998, Th. 4.5.20).

Let us contrast this to the case of i.i.d. Brownian bridges $B^{(1)},\dots,B^{(N)}$ with starting distribution m, i.e., we replace $\mathbb{P}_{m,N}^{(\mathrm{sym})}$ by $(\int m(\mathrm{d}x)\,\mathbb{P}_{x,x}^\beta)^{\otimes N}$. Here the empirical path measure L_N satisfies a large deviations principle with rate function

$$I_m(\mu) = \sup_{\Phi \in \mathcal{C}_{\mathrm{b}}(\Omega_\beta)} \Big\{\langle \Phi,\mu\rangle - \log \int_{\mathbb{R}^d} m(\mathrm{d}x)\,\mathbb{E}_{x,x}^\beta\big(\mathrm{e}^{\Phi(B)}\big)\Big\},$$

as follows from an application of Cramér's theorem (Dembo and Zeitouni, 1998, Theorem 6.1.3). Note that $I_m(\mu)$ is the relative entropy of μ with respect to $\int m(\mathrm{d}x)\,\mathbb{P}^{\beta}_{x,x}\circ B^{-1}$. Although there is apparently no reason to expect a direct comparison between the distributions of L_N under $\mathbb{P}^{(\mathrm{sym})}_{m,N}$ and under $(\int m(\mathrm{d}x)\,\mathbb{P}^{\beta}_{x,x})^{\otimes N}$, the rate functions admit a simple relation: it is easy to see that $I^{(Q)}\geq I_m$ for the measure $Q(\mathrm{d}x,\mathrm{d}y)=m(\mathrm{d}x)\delta_x(\mathrm{d}y)\in\mathcal{M}_1^{(\mathrm{s})}(\mathbb{R}^d\times\mathbb{R}^d)$, since

$$-\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}Q(\mathrm{d}x,\mathrm{d}y)\log\mathbb{E}^{\beta}_{x,y}\big(\mathrm{e}^{\Phi(B)}\big)\geq-\log\int_{\mathbb{R}^d}m(\mathrm{d}x)\,\mathbb{E}^{\beta}_{x,x}\big(\mathrm{e}^{\Phi(B)}\big).$$

In particular, $I_m^{(\mathrm{sym})}\geq I_m$.

An interesting question is what happens if we replace in the definition of the symmetrised probability measure $\mathbb{P}^{(\mathrm{sym})}_{m,N}$ the Brownian bridge probability measure $\mathbb{P}^{\beta}_{x,y}$ by $g(x,y)\mathbb{P}^{\beta}_{x,y}$, when $g\colon\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$ is a continuous function? The motivation to multiply the Brownian bridge probability measure by the spatial function g is to model (see Adams (2008a)) the spatial correlations for permutations of finitely many points of graphs or finitely many points of point process clouds in $\mathbb{R}^d$. Compare Fichtner (1991), who studied permutations of random point configurations in $\mathbb{R}^d$ and introduced the spatial weight $\mathrm{e}^{-c|x-y|^2}$ for permutations that sent the spatial point x to the spatial point y. We shall discuss no further details at this stage but formulate our general result.

Proposition 3.2 (Adams and König (2007b)) *Let $g\colon\mathbb{R}^d\to\mathbb{R}^d\to\mathbb{R}$ be continuous and define $\mathbb{P}^{(\mathrm{sym})}_{m,N}$ with $\mathbb{P}^{\beta}_{x,y}$ replaced by $g(x,y)\mathbb{P}^{\beta}_{x,y}$. Then the following holds.*

(a) *Theorem 3.1 remains true under the replacement . The corresponding rate function is $\mu\mapsto I_m^{(\mathrm{sym})}(\mu)-\langle\mu_{0,\beta},\log g\rangle$.*

(b)

$$\begin{aligned}\lim_{N\to\infty}\frac{1}{N}\log\Big(\frac{1}{N!}\sum_{\sigma\in\mathfrak{S}_N}\int_{(\mathbb{R}^d)^N}\prod_{i=1}^N m(\mathrm{d}x_i)\prod_{i=1}^N g(x_i,x_{\sigma(i)})\Big)\\ =-\inf_{Q\in\mathcal{M}_1^{(\mathrm{s})}(\mathbb{R}^d\times\mathbb{R}^d)}\Big\{H(Q|\overline{Q}\otimes m)-\langle Q,\log g\rangle\Big\}.\end{aligned}\tag{15}$$

(c) *The unique minimiser of the rate function $\mu\mapsto I_m^{(\mathrm{sym})}(\mu)-\langle\mu_{0,\beta},\log g\rangle$ is given by*

$$\mu^*=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}Q^*(\mathrm{d}x,\mathrm{d}y)\,\mathbb{P}^{\beta}_{x,y}\circ B^{-1},\tag{16}$$

where $Q^*\in\mathcal{M}_1^{(\mathrm{s})}$ is the unique minimiser of the formula on the right hand side of (15)*.*

(d) Law of large numbers: *Under the measure $g\mathbb{P}^{(\mathrm{sym})}_{m,N}$, normalised to a probability measure, the sequence $(L_N)_{N\in\mathbb{N}}$ converges in distribution to the measure μ^* defined in* (16)*.*

Setting $g\equiv 1$ we derive the easily the following law of large numbers for our previous case.

Corollary 3.3 *Under the measure* $\mathbb{P}^{(\mathrm{sym})}_{m,N}$*, normalised to a probability measure, the sequence* $(L_N)_{N\in\mathbb{N}}$ *converges in distribution to the measure* μ^* *given by*

$$\mu^* = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} m\otimes m(\mathrm{d}x,\mathrm{d}y)\,\mathbb{P}^\beta_{x,y}\circ B^{-1}.$$

That is, in spite of strong correlations for fixed N under $\mathbb{P}^{(\mathrm{sym})}_{m,N}$, the initial and terminal locations $B^{(1)}_0$ and $B^{(1)}_\beta$ of the first motion become independent in the limit $N\to\infty$. One can prove this also in an elementary way, and also the fact that, for any $k\in\mathbb{N}$ and for all $i_1<i_2<\cdots<i_k$, the Brownian motions $B^{(i_1)},\dots,B^{(i_k)}$ under $\mathbb{P}^{(\mathrm{sym})}_{m,N}$ become independent in the limit $N\to\infty$.

We finish the section with the following special case as promised above. We replace the initial distribution $m\in\mathcal{M}_1(\mathbb{R}^d)$ by the Lesbesgue measure of a set $\Lambda\subset\mathbb{R}^d$ having finite Lesbesgue measure. That is we study the non-normalised measure

$$\mu^{(\mathrm{sym})}_{\Lambda,N} = \frac{1}{N!}\sum_{\sigma\in\mathfrak{S}_N}\int_\Lambda\cdots\int_\Lambda \mathrm{d}x_1\cdots\mathrm{d}x_N\bigotimes_{i=1}^N \mu^\beta_{x_i,x_{\sigma(i)}}. \qquad (17)$$

Apart from questions motivated from physics, this measure is also mathematically interesting. According to an analogous result of Theorem 3.1 for the mean of occupation measures, the distribution of the mean of the normalised occupation measures Y_N, under $(Z^{(\mathrm{sym})}_{\Lambda,N})^{-1}\mu^{(\mathrm{sym})}_{\Lambda,N}$, satisfies a large deviations principle. Here $Z^{(\mathrm{sym})}_{\Lambda,N}$ is the normalisation for the the measure (17). That is, we have

$$\lim_{N\to\infty}\frac{1}{N}\log\big(\mu^{(\mathrm{sym})}_{\Lambda,N}\circ Y_N^{-1}(\cdot)\big) = -\inf_{p\in\,\cdot} J^{(\mathrm{sym})}_\Lambda(p),$$

in the weak sense on subsets of $\mathcal{M}_1(\mathbb{R}^d)$, where we introduced

$$J^{(\mathrm{sym})}_\Lambda(p) = \inf_{Q\in\mathcal{M}_1^{(\mathrm{s})}(\mathbb{R}^d\times\mathbb{R}^d)}\Big\{H(Q|\overline{Q}\otimes\mathrm{Leb}_\Lambda)+J^{(Q)}(p)\Big\} - \inf_{p\in\mathcal{M}_1(\mathbb{R}^d)}\big\{\widetilde{J}^{(\mathrm{sym})}_\Lambda(p)\big\}$$

with $\widetilde{J}^{(\mathrm{sym})}_\Lambda(p) = \inf_{Q\in\mathcal{M}_1^{(\mathrm{s})}(\mathbb{R}^d\times\mathbb{R}^d)}\big\{H(Q|\overline{Q}\otimes\mathrm{Leb}_\Lambda)+J^{(Q)}(p)\big\}$ and

$$J^{(Q)}(p) = \sup_{f\in\mathcal{C}_{\mathrm{b}}(\mathbb{R}^d)}\Big\{\beta\langle f,p\rangle - \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} Q(\mathrm{d}x,\mathrm{d}y)\,\log\frac{\mathbb{E}_x\big(\mathrm{e}^{\int_0^\beta f(B_s)\,\mathrm{d}s};B_\beta\in\mathrm{d}y\big)}{\mathrm{d}y}\Big\}.$$

The main goal is to express $J^{(\mathrm{sym})}_\Lambda$ in much easier and more familiar terms. It turns out that $J^{(\mathrm{sym})}_\Lambda(p)$ is identical to the energy of the square root of the density of p, in the jargon of large deviations theory also sometimes called the *Donsker-Varadhan rate function*, $I_\Lambda\colon\mathcal{M}_1(\mathbb{R}^d)\to[0,\infty]$ defined by

$$I_\Lambda(p) = \begin{cases}\big\|\nabla\sqrt{\frac{\mathrm{d}p}{\mathrm{d}x}}\big\|_2^2, & \text{if } p \text{ has a density with square root in } H^1_0(\Lambda^\circ),\\ \infty & \text{otherwise.}\end{cases}$$

Theorem 3.4 (Adams and König (2007b)) *Let $\Lambda \subset \mathbb{R}^d$ be a bounded closed box. Then $\beta^{-1} J_\Lambda^{(\mathrm{sym})}(p) = I_\Lambda(p) - \inf_{p\in\mathcal{M}_1(\mathbb{R}^d)} I_\Lambda(p)$ for any $p \in \mathcal{M}_1(\mathbb{R}^d)$.*

In the theory and applications of large deviations, I_Λ plays an important role as the rate function for the normalised occupation measure of one Brownian motion (or, one Brownian bridge) in Λ, in the limit as time to tends infinity (see Gärtner (1977) and Donsker and Varadhan (1983)). It is remarkable that this function turns out also to govern the large deviations for the mean of the normalised occupation measures under the symmetrised measure $\mu_{\Lambda,N}^{(\mathrm{sym})}$, in the limit of a large number of motions. Let us give an informal discussion and interpretation of this fact.

The measure $\mu_{\Lambda,N}^{(\mathrm{sym})}$ in (17) admits a representation which goes back to Feynman (1953) and which we want to briefly discuss. Every permutation $\sigma \in \mathfrak{S}_N$ can be written as a concatenation of cycles. Given a cycle $(i, \sigma(i), \sigma^2(i), \dots, \sigma^{k-1}(i))$ with $\sigma^k(i) = i$ and precisely k distinct indices, the contribution coming from this cycle is independent of all the other indices. Furthermore, by the fact that $\mu^\beta_{x_i,x_{\sigma(i)}}$ is the conditional distribution given that the motion ends in $x_{\sigma(i)}$, this contribution (also executing the k integrals over $x_{\sigma^l(i)} \in \Lambda$ for $l = k-1, k-2, \dots, 0$) turns the corresponding k Brownian bridges of length β into one Brownian bridge of length $k\beta$, starting and ending in the same point $x_i \in \Lambda$ and visiting Λ at the times $\beta, 2\beta, \dots, (k-1)\beta$. Hence,

$$\mu_{\Lambda,N}^{(\mathrm{sym})} = \frac{1}{N!} \sum_{\sigma\in\mathfrak{S}_N} \bigotimes_{k\in\mathbb{N}} \Big(\int_\Lambda \mathrm{d}y_k \, \mu^{k,\beta,\Lambda}_{y_k,y_k} \Big)^{\otimes f_k(\sigma)},$$

where $f_k(\sigma)$ denotes the number of cycles in σ of length precisely equal to k, and $\mu^{k,\beta,\Lambda}_{x,y}$ is the Brownian bridge measure $\mu^{k\beta}_{x,y}$ as in (1), restricted to the event $\bigcap_{l=1}^k \{B_{l\beta} \in \Lambda\}$. (See (de Witt and Storaeds, 1970, Lemma 2.1) for related combinatorial considerations.) If $f_N(\sigma) = 1$ (i.e., if σ is a cycle), then we are considering just one Brownian bridge B of length $N\beta$, with uniform initial measure on Λ, on the event $\bigcap_{l=1}^N \{B_{l\beta} \in \Lambda\}$. Furthermore, Y_N is equal to the normalised occupation measure of this motion. For such a σ, the limit $N \to \infty$ turns into a limit for diverging time, and the corresponding large-deviation principle of Donsker and Varadhan formally applies. This reasoning applies for permutations σ having only cycles whose lengths are growing with N unboundedly. Presumably, the contribution from those permutations whose bounded cycles sum up to something of order N is strictly smaller. A thorough investigation of the large deviation properties of the cycle structure and the distribution of the cycle lengths is contained in Adams (2007) and in Section 3.2 below for the case of boxes $\Lambda = \Lambda_N$ having volume of order N. There, a phase transition in β for the mean path is obtained. This phase transition is absent in the present case; the fixed box Λ forces all cycle lengths to grow unbounded with N.

3.2 Cycle structure

We analyse the large-N behaviour of a system of N Brownian motions with time horizon $[0,\beta]$ in $\mathbb{R}^d$ confined in subsets $\Lambda_N \subset \mathbb{R}^d$, i.e., the behaviour of the system under the symmetrised measure

$$\mathbb{P}_N^{(\mathrm{sym})} = Z_N^{(\mathrm{sym})}(\beta)^{-1} \frac{1}{N!} \sum_{\sigma \in \mathfrak{S}_N} \int_{\Lambda_N} \mathrm{d}x_1 \cdots \int_{\Lambda_N} \mathrm{d}x_N \bigotimes_{i=1}^N \mu^{\beta,N}_{x_i, x_{\sigma(i)}}, \tag{18}$$

and $Z_N^{(\mathrm{sym})}(\beta)$ is the normalisation

$$Z_N^{(\mathrm{sym})}(\beta) = \frac{1}{N!} \sum_{\sigma \in \mathfrak{S}_N} \int_{\Lambda_N} \mathrm{d}x_1 \cdots \int_{\Lambda_N} \mathrm{d}x_N \bigotimes_{i=1}^N \mu^{\beta,N}_{x_i, x_{\sigma(i)}}(\Omega_\beta^N).$$

The measure in (18) is different from the measure (13) in the previous section. Here, we want to exploit our results for the discrete empirical shape measure and the formula (6). The core idea in formula (6) is to concatenate Brownian bridges to obtain Brownian bridges with larger time horizons. Therefore we study in this section large deviations of the empirical path measures for paths with unbounded time horizon. That allows us to put the Brownian bridges of time horizon $[0, k\beta]$ onto the path of unbounded time horizon. We conceive the empirical path measure as a random element in $\mathcal{M}_1(\Omega)$, hence, we need a convenient extension of any continuous path $[0,\beta] \to \mathbb{R}^d$ to a continuous path $[0,\infty) \to \mathbb{R}^d$ in the definition of the empirical path measure. For any $x \in \mathbb{R}^d$ we denote by P^x the Brownian probability measure on Ω, i.e., the canonical Wiener measure with deterministic start in $x \in \mathbb{R}^d$ (Chung and Zhao (1995)). In the following we write alternatively ω_t or $\omega(t)$ for any point of a path ω. Given a path $\omega \in \Omega_\beta$ with time horizon $[0,\beta]$ define

$$P_\omega^{(\beta)} = \delta_\omega \otimes_\beta P^{\omega_\beta(\beta)} \in \mathcal{M}_1(\Omega, \mathcal{B}),$$

where the product $\otimes_\beta$ is defined for the "splice" of two paths, i.e., for $\omega \in \Omega_\beta$ and $\widetilde{\omega} \in \Omega$ define $\overline{\omega} \in \Omega$ by $\overline{\omega}(t) = \omega(t \wedge \beta), t \in [0,\infty)$, and $\omega \otimes_\beta \widetilde{\omega} \in \Omega$ such that $\omega \otimes_\beta \widetilde{\omega} = \overline{\omega}$ if $\widetilde{\omega}(0) \neq \omega(\beta)$ and

$$\omega \otimes_\beta \widetilde{\omega}(t) = \begin{cases} \omega(t) & \text{for } t \in [0,\beta] \\ \widetilde{\omega}(t-\beta) & \text{for } t \in (\beta,\infty) \end{cases} \tag{19}$$

if $\widetilde{\omega}(0) = \omega(\beta)$. The mapping $\omega \in \Omega_\beta \mapsto P_\omega^{(\beta)} \in \mathcal{M}_1(\Omega, \mathcal{B})$ is measurable, and the family $\{P_\omega^{(\beta)} : \omega \in \Omega_\beta\}$ satisfies the Markov property, see (Deuschel and Stroock, 2001, Lemma 4.4.21). Hence, the empirical path measure

$$\widehat{L}_N : \Omega_\beta^N \to \mathcal{M}_1(\Omega), \omega \mapsto \widehat{L}_N(\omega) = \frac{1}{N} \sum_{i=1}^N \delta_{\omega^{(i)}} \otimes_\beta P^{\omega_\beta^{(i)}},$$

is $\Omega_\beta^{\otimes N}$ measurable. Here $\omega = (\omega^{(1)}, \dots, \omega^{(N)}) \in \Omega_\beta^N$. Our main result concerns a large deviations principle for the distributions of $\widehat{L}_N$ under the symmetrised measure $\mathbb{P}_N^{(\mathrm{sym})}$. Recall that $\mathbb{P}_N^{(\mathrm{sym})}$ is a probability measure on $\Omega_\beta^{\otimes N}$.

The rate function is given by

$$I^{(\mathrm{sym})}(\mu) = \inf_{Q\in\mathcal{M}} \Big\{ S^{(\mathrm{Bose})}(Q) + I^{(Q)}(\mu) \Big\} - \chi(\beta,\varrho) \qquad \mu \in \mathcal{M}_1(\Omega),$$

where

$$I^{(Q)}(\mu) = \sup_{F\in\mathcal{C}_{\mathrm{b}}(\Omega)} \Big\{ \langle F,\mu\rangle - \sum_{k\in\mathbb{N}} \widehat{Q}(k) \log \mathbb{E}_{0,0}^{k\beta}\Big(\mathrm{e}^{F(B)}\Big) \Big\} \qquad \mu \in \mathcal{M}_1(\Omega),$$

and where the function $\chi(\beta,\varrho) := \inf_{Q\in\mathcal{M}}\{\mathcal{S}^{(\mathrm{Bose})}(Q)\}$ is given as the negative logarithmic limit of the partition function $Z_N^{(\mathrm{sym})}(\beta)$, see Theorem 2.4, and where $\mathcal{C}_{\mathrm{b}}(\Omega)$ is the space of continuous bounded functions of the paths in Ω. $\mathbb{E}_{0,0}^{k\beta}$ denotes the expectation with respect to the Brownian bridge probability measure $\mathbb{P}_{0,0}^{k\beta}$ extended to a probability measure in $\mathcal{M}_1(\Omega)$. Here, $I^{(Q)}$ is a Fenchel-Legendre transform, but *not* the one of a logarithmic moment generating function of any random variable. In particular, $I^{(Q)}$, and therefore also $I^{(\mathrm{sym})}$, are nonnegative, and $I^{(Q)}$ is convex as a supremum of linear functions. There seems to be no way to represent $I^{(Q)}(\mu)$ as the relative entropy of μ with respect to any measure.

Theorem 3.5 (Large deviations for $\widehat{\mathbf{L}}_{\mathbf{N}}$, Adams (2007)) *Let $\varrho \in (0,\infty)$ and $\Lambda_N \subset \mathbb{R}^d$ centred boxes with $\Lambda_N \uparrow \mathbb{R}^d$ and $N/|\Lambda_N| \to \varrho$ as $N \to \infty$.*

Under the symmetrised measure $\mathbb{P}_N^{(\mathrm{sym})}$ the empirical path measures $(\widehat{L}_N)_{N\in\mathbb{N}}$ satisfy a large deviations principle on $\mathcal{M}_1(\Omega)$ with speed N and rate function $I^{(\mathrm{sym})}$.

Remark 3.6 *To be more precise we have a large deviations principle for $\widehat{L}_N$ under the symmetrised distribution such that the initial distribution is subtracted, i.e., all motions are considered to start at the origin. This is a technical detail, and we refer to Adams (2008a) and Adams et al. (2008), where our analysis is combined with marked point processes in $\mathbb{R}^d$. However, as we focus here solely on the non-interacting case, we can relax the abstraction and let the motions start at the origin.*

We give a brief informal interpretation of the shape of the rate functions in $I^{(\mathrm{sym})}$ and $I^{(Q)}, Q \in \mathcal{M}$. As remarked earlier, the symmetrised measure $\mathbb{P}_N^{(\mathrm{sym})}$ arises from a two-step probability mechanism. This is reflected in the representation of the rate function $I^{(\mathrm{sym})}$: in a peculiar way, the term $S(Q) - \chi(\beta,\varrho)$ describes the large deviations of the discrete empirical shape measure for integer partitions. The discrete empirical shape measures Q_N governs a particular distribution of N independent, but not identically distributed, Brownian bridges. Under this distribution, $\widehat{L}_N$ satisfies a large deviations principle with rate function $I^{(Q)}$, which can also be guessed from the Gärtner-Ellis theorem (Dembo and Zeitouni, 1998, Th. 4.5.20). The presence of a two-step mechanism makes it impossible to apply this theorem directly to $\mathbb{P}_N^{(\mathrm{sym})}$.

Let us contrast this to the case of i.i.d. Brownian bridges $B^{(1)}, \dots, B^{(N)}$, starting in the origin, i.e., we replace $\mathbb{P}_N^{(\mathrm{sym})}$ by $(\mathbb{P}_{0,0}^{\beta})^{\otimes N}$. Here the empirical path measure $\widehat{L}_N$ satisfies a large deviations principle with rate function

$$I(\mu) = \sup_{F \in \mathcal{C}_{\mathrm{b}}(\Omega)} \Big\{ \langle F, \mu \rangle - \log \mathbb{E}_{0,0}^{\beta}\big(\mathrm{e}^{F(B)}\big) \Big\},$$

as follows from an application of Cramér's theorem (Dembo and Zeitouni, 1998, Theorem 6.1.3). Note that $I(\mu)$ is the relative entropy of μ with respect to $\mathbb{P}_{0,0}^{\beta} \circ B^{-1}$. Although there is apparently no reason to expect a direct comparison between the distributions of L_N under $\mathbb{P}_N^{(\mathrm{sym})}$ and under $(\mathbb{P}_{0,0}^{\beta})^{\otimes N}$, the rate functions admit a simple relation: it is easy to see that $I^{(Q)} \geq I$ for the measure $Q \in \mathcal{M}$ with $\widehat{Q}(k) = \delta_1$, since

$$-\sum_{k=1}^{\infty} \widehat{Q}(k) \log \mathbb{E}_{0,0}^{k\beta}\big(\mathrm{e}^{F(B)}\big) \geq -\log \mathbb{E}_{0,0}^{\beta}\big(\mathrm{e}^{F(B)}\big).$$

In particular, $I^{(\mathrm{sym})} \geq I$.

Remark 3.7 *The techniques of the proof of Theorem 3.5 apply also to a proof of a large deviations principle under the symmetrised measure $\mathbb{P}_N^{(\mathrm{sym})}$ for the empirical path measure $\widetilde{L}_N = 1/N \sum_{i=1}^{N} \delta_{B^{(i)}}$, which is a random element in $\mathcal{M}_1(\Omega_\beta)$. The rate function is*

$$\widetilde{I}^{(\mathrm{sym})}(\mu) =$$

$$\inf_{Q \in \mathcal{M}} \Big\{ \mathcal{S}^{(\mathrm{Bose})}(Q) - \sup_{F \in \mathcal{C}_{\mathrm{b}}(\Omega_\beta)} \Big\{ \langle F, \mu \rangle - \sum_{k=1}^{\infty} \widehat{Q}(k) \log \mathbb{E}_{0,0}^{k\beta}\Big(\mathrm{e}^{\sum_{l=0}^{k-1} F(B_{[l\beta,(l+1)\beta]})}\Big) \Big\} \Big\}.$$

Similar results hold for the mean Y_N of the occupation measures. However, these rate functions seem not to give enough information to derive the phase transition as in Theorem 3.8, and to obtain a probabilistic interpretation of Bose-Einstein condensation.

Our large deviations result is accompanied by an analysis of the variational formula for the rate function $I^{(\mathrm{sym})}$, i.e., the analysis for zeros of the rate function. This gives the proof of the phase transition for empirical path measures depending on the dimension and the density parameter in Theorem 3.8.

The result of Theorem 2.4 in Section 3.2 is an essential ingredient which leads to the analysis of the rate function $I^{(\mathrm{sym})}$. Let

$$A_k = \{\omega \otimes_{k\beta} \xi \colon \omega \in \Omega_k, \omega(0) = \omega(k\beta), \xi \in \Omega\} \subset \Omega, k \in \mathbb{N},$$

be the set of paths in Ω which result from the splice (19) of Brownian bridges paths of time horizon $[0, k\beta]$ with any path $\xi \in \Omega$.

Theorem 3.8 (Analysis of the rate function $I^{(\mathrm{sym})}$) Adams (2007).
Under the assumptions of Theorem 3.5 the following holds:

(i) $d = 1,2$. *A unique minimiser* $\mu^* \in \mathcal{M}_1(\Omega)$ *of the rate function* $\mu \mapsto I^{(\mathrm{sym})}(\mu)$ *exists with* $\sum_{k\in\mathbb{N}} k\mu^*(A_k) = 1$.

(ii) $d \geq 3$ *and* $\varrho < \varrho_c$. *A unique minimiser* $\mu^* \in \mathcal{M}_1(\Omega)$ *of the rate function* $\mu \mapsto I^{(\mathrm{sym})}(\mu)$ *is given with* $\sum_{k\in\mathbb{N}} k\mu^*(A_k) = 1$.

For $\varrho > \varrho_c$ *there is no unique minimiser be given, but there exist minimising sequences* $(\mu_n)_{n\geq 1}, \mu_n \in \mathcal{M}_1(\Omega)$, *with* $\sum_{n=1}^\infty k\mu_n(A_k) = 1$ *for any* $n \in \mathbb{N}$ *such that* $\mu_n \to \mu^0 \in \mathcal{M}_1(\Omega)$ *weakly as* $n \to \infty$ *with* $\sum_{n=1}^\infty k\mu^0(A_k) < 1$.

Let us draw an easy corollary from this theorem.

Corollary 3.9 (Law of large numbers, Adams (2007).) *Under the assumptions of Theorem 3.8 the following holds.*

(i) *For* $d = 1,2$, *and any density* $\varrho < \infty$, *there is a law of large numbers. Under the probability measure* $\mathbb{P}_N^{(\mathrm{sym})}$, *the sequence* $(\widehat{L}_N)_{N\in\mathbb{N}}$ *converges in distribution to the measure* $\mu^* \in \mathcal{M}_1(\Omega)$.

(ii) *For* $d \geq 3$ *and* $\varrho < \varrho_c$ *there is a law of large numbers. Under the probability measure* $\mathbb{P}_N^{(\mathrm{sym})}$, *the sequence* $(\widehat{L}_N)_{N\in\mathbb{N}}$ *converges in distribution to the measure* $\mu^* \in \mathcal{M}_1(\Omega)$.

The main conclusion of the large deviations principle in Theorem 3.5 and Theorem 3.8 is the following phase transition for the mean empirical path measure, which gives a path measure interpretation of Bose-Einstein condensation (BEC).

Path measures and their interpretation as Bose-Einstein condensation

Let N Brownian motions with time horizon $[0,\beta]$ confined in centred sets $\Lambda_N \subset \mathbb{R}^d$ given such that $\Lambda_N \uparrow \mathbb{R}^d$ and $N/|\Lambda_N| \to \varrho \in (0,\infty)$ as $N \to \infty$. Then the following holds:

(i) For $\beta > 0$ there is a $\varrho_c = \varrho_c(\beta, d)$ such that:

no BEC: Case $\varrho < \varrho_c$ for $d \geq 3$, $\varrho > 0$ for $d = 1,2$:
$\widehat{L}_N \to \mu^* \in \mathcal{M}_1(\Omega)$ under $\mathbb{P}_N^{(\mathrm{sym})}$ as $N \to \infty$ with $\sum_{k=1}^\infty k\mu^*(A_k) = 1$

BEC: Case $\varrho < \varrho_c$ and $d \geq 3$:
$\widehat{L}_N \to \mu^0 \in \mathcal{M}_1(\Omega)$ under $\mathbb{P}_N^{(\mathrm{sym})}$ as $N \to \infty$ with $\sum_{k=1}^\infty k\mu^0(A_k) < 1$.

(ii) For $\varrho \in (0,\infty)$ there exists a

$$\beta_c = \begin{cases} \frac{1}{4\pi}\Big(\frac{\varrho}{\zeta(d/2)}\Big)^{2/d}, & \text{for } d \geq 3 \\ +\infty, & \text{for } d = 1,2 \end{cases},$$

such that:

no BEC: Case $\beta < \beta_c$ for $d \geq 3$ and $\beta > 0$ for $d = 1,2$:
$\widehat{L}_N \to \mu^* \in \mathcal{M}_1(\Omega)$ under $\mathbb{P}_N^{(\mathrm{sym})}$ as $N \to \infty$ with $\sum_{k=1}^\infty k\mu^*(A_k) = 1$

BEC: Case $\beta > \beta_c$ and $d \geq 3$:
$\widehat{L}_N \to \mu^0 \in \mathcal{M}_1(\Omega)$ under $\mathbb{P}_N^{(\mathrm{sym})}$ as $N \to \infty$ with $\sum_{k=1}^\infty k\mu^0(A_k) < 1$.

If $d = 1, 2$, or $\varrho < \varrho_c$ for $d \geq 3$, the mean empirical path measure has support on those paths in which one can insert, starting from time origin, a concatenation of any finite number of Brownian motions with time horizon $[0, \beta]$, i.e., for any $k \in \mathbb{N}$ one can find in paths $\omega_k \in A_k$ exactly k Brownian motions concatenated to a Brownian bridge with horizon $[0, k\beta]$. This follows from the concatenation of the Brownian motions due to the cycle structure of the permutations and due to the Lebesgue integration of any initial position in the definition of the symmetrised measure $\mathbb{P}_N^{(\mathrm{sym})}$. If the density ϱ is high enough for $d \geq 3$, i.e., $\varrho > \varrho_c$ (or equivalently, if the inverse temperature is sufficiently large for given density, i.e., $\beta > \beta_c$, for $d \geq 3$), the mean path measure has positive weight for paths with an infinite time horizon, that is, concatenation of any finite number of Brownian motions with time horizon $[0, \beta]$, i.e., any finite cycle path in A_k, is not sufficient, because there is an excess density $(\varrho - \varrho_c)$ of Brownian motions with time horizon $[0, \beta]$. These motions concatenate to infinite long cycle, that is, these cycles grow with the system size in the thermodynamic limit. The fraction of these motions is

$$1 - \frac{\varrho_c}{\varrho} = 1 - \Big(\frac{\beta_c}{\beta}\Big)^{d/2}.$$

4 Appendix: Bose functions

These functions are defined by

$$g_s(\alpha) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{\mathrm{e}^{t+\alpha} - 1} \mathrm{d}t = \sum_{k=1}^\infty k^{-s} \mathrm{e}^{-\alpha k} \qquad \text{for all } \alpha > 0 \text{ and all } s > 0, \tag{20}$$

and also $\alpha = 0$ and $s > 1$. In the latter case,

$$g_s(0) = \sum_{k=1}^\infty k^{-s} = \zeta(s),$$

which is the zeta function of Riemann. The behaviour of the Bose functions about $\alpha = 0$ is given by

$$g_s(\alpha) = \begin{cases} \Gamma(1-s)\alpha^{s-1} + \sum_{k=0}^\infty \zeta(s-k) \frac{(-\alpha)^k}{k!} & , s \neq 1, 2, \ldots \\ \frac{(-\alpha)^{s-1}}{(s-1)!} \Big[\log \frac{1}{\alpha} + \sum_{m=1}^{s-1} \frac{1}{m} \Big] + \sum_{\substack{k=0 \\ k \neq s-1}} \zeta(s-k) \frac{(-\alpha)^k}{k!} & , s = 1, 2, \ldots \end{cases}.$$

At $\alpha = 0$, $g_s(\alpha)$ diverges for $s \leq 1$; indeed for all s there is some kind of singularity at $\alpha = 0$, such as a branch point. For further details see Gram (1925).

References

Adams, S. (2001). Complete equivalence of the Gibbs ensembles for one-dimensional Markov systems. *Jour. Stat. Phys. 105*(5/6), 879–908.

Adams, S. (2007). Large deviations for empirical path measures in cycles of integer partitions. preprint arXiV:math.PR/0702052v2.

Adams, S. (2008a). Interacting Brownian bridges and probabilistic interpretation of Bose-Einstein condensation. Habilitation thesis, University Leipzig.

Adams, S. (2008b). Large deviations for shape measures of integer partitions under non-uniform distribution. in preparation.

Adams, S. and J.-B. Bru (2004a). Critical Analysis of the Bogoliubov Theory of Superfluidity. *Physica A 332*, 60–78.

Adams, S. and J.-B. Bru (2004b). Exact solution of the AVZ-Hamiltonian in the grand-canonical ensemble. *Ann. Henri Poincaré 5*, 405–434.

Adams, S. and J.-B. Bru (2004c). A New Microscopic Theory of Superfluidity at all Temperatures. *Ann. Henri Poincaré 5*, 435–476.

Adams, S., J.-B. Bru, and W. König (2006a). Large deviations for trapped interacting Brownian particles and paths. *Ann. Probab. 34*(4), 1340–1422.

Adams, S., J.-B. Bru, and W. König (2006b). Large systems of path-repellent Brownian motions in a trap at positive temperature. *Electronic Journal of Probability 11*, 460–485.

Adams, S. and T. Dorlas (2007). Asymptotic Feynman-Kac formulae for large symmetrised systems of random walks. *Annales de l'institut Henri Poincaré (B) Probabilités et Statistiques*. in press, preprint arXiV:math-ph/0610026.

Adams, S. and W. König (2007a). Interacting Brownian motions and the Gross-Pitaevskii formula. OUP.

Adams, S. and W. König (2007b, September). Large deviations for many Brownian bridges with symmetrised initial-terminal condition. *Prob. Theory Relat. Fields*. in press, published on line DOI 10.1007/s00440-007-0099-5.

Adams, S., W. König, and A. Collevecchio (2008). A variational formula for the free energy of non-dilute many-particle systems. in preparation.

Arratia, R., A. Barbour, and S. Tavaré (2003). *Logarithmic Combinatorial Structures: a probabilistic approach.* EMS Monographs in Mathematics. European Mathematical Society, Zürich.

Benfatto, G., M. Cassandro, I. Merola, and E. Presutti (2005). Limit theorems for statistics of combinatorial partitions with applications to mean field bose gas. *Journal of Math. Phys. 46*, 033303.

Bose, S. (1924). Plancks Gesetz und Lichtquantenhypothese. *Zeitschrift für Physik 26*, 178–181.

Bratteli, O. and D. Robinson (1997). *Operator Algebras and Quantum Statistical Mechanics II* (2nd ed. ed.). Springer Berlin.

Chung, K. and Z. Zhao (1995). *From Brownian motion to Schrödinger's equation*. Springer Berlin.

Dawson, D. and J. Gärtner (1994). Multilevel large deviations and interacting diffusions. *Probab. Theory Relat. Fields 98*, 423–487.

de Witt, C. and R. Storaeds (Eds.) (1970). *Some Applications of Functional Integration in Statistical Mechanics and Field Theory*. Cargèse Lectures in Theoretical Physics: Gordon and Breach Science Publ. New York.

Dembo, A. and O. Zeitouni (1998). *Large Deviations Techniques and Applications*. Springer New York.

Dembo, A., O. Zeitouni, and A. Vershik (2000). Large deviations for integer partitions. *Markov processes and related fields 6*(No.2), 147–179.

Deuschel, J. and D. Stroock (2001). *Large Deviations*. AMS Chelsea Publishing American Mathematical Society, Providence, Rhode Island.

Donsker, M. and S. Varadhan (1975-1983). Asymptotic evaluation of certain Markov process expectations for large time, I–IV. *Comm. Pure Appl. Math. 28, 29, 36*, 1–47, 279–301, 389–461, 183–212.

Dorlas, T., P. Martin, and J. Pulé (2005). Long cycles in a perturbed mean field model of a boson gas. *Journal of. Stat. Phys. 121*(Nos.3/4), 433–461.

Einstein, A. (1925). Quantentheorie des einatomigen idealen Gases. *Sitzber. Kgl. Preuss. Akad. Wiss. 3*, 3–14.

Ewens, W. (1972). The sampling theory of selectively neutral alleles. *Theor. Pop. Biol. 3*, 87–112.

Feynman, R. (1953). Atomic theory of the λ transition in Helium. *Phys. Rev. 91*, 1291–1301.

Fichtner, K. (1991). Random permutations of countable sets. *Probab. Theory Relat. Fields 89*, 35–60.

Föllmer, H. (1988). *Random fields and diffusion processes*, Volume 1362 of *Lecture Notes in Math.* Springer Berlin. Ecole d'Eté de Saint Flour XV-XVII.

Gärtner, J. (1977). On large deviations from the invariant measure. *Theory Probab. Appl. 22:1*, 24–39.

Goncharov, V. (1944). On the Field of Combinatory Analysis. *Translations of the American Mathematical Society 19*, 1–46.

Gram, J. (1925). Tafeln für die Riemannsche Zetafunktion. *Skrifter København (8) 9*, 311–325.

Griffin, A., D. Snoke, and S. Stringari (1995). *Bose-Einstein Condensation*. Cambridge University Press, Cambridge.

Huang, K. (1987). *Statistical Mechanics*. Wiley, New York.

Khinchin, A. (1960). *Mathematical Foundations of Quantum Satistics*. Dover Publications, New York.

Kingman, J. (1975). Random discrete distributions. *J.R. Statist. Soc. B 37*, 1–15.

Kingman, J. (1978a). Random partitions in population genetics. *Proc. Roy. Soc. 361*, 1–20.

Kingman, J. (1978b). The Representation of Partition Sructures. *J. London Math. Soc. 18*, 374–380.

Kolchin, V. (1986). *Random Mappings*. Optimization Software, Inc., Publications Division, New York.

Landau, L. (1941). The theory of superfluidity of Helium II. *J. Phys. USSR 5*, 71.

Lieb, E., R. Seiringer, J. Solovej, and J. Yngvason (2005). *The mathematics of the Bose gas and its condensation*. Birkhäuser Verlag Basel.

London, F. (1938). The λ-Phenomenon of Liquid Helium and the Bose-Einstein Degeneracy. *Science 141*, 643.

Onsager, L. and O. Penrose (1956). Bose-Einstein condensation and liquid Helium. *Phys. Rev. 104*, 576–584.

Penrose, O. (1951). On the quantum mechanics of Helium II. *Phil. Mag. 42*, 1373–1377.

Pitaevskii, L. and S. Stringari (2003). *Bose-Einstein Condensation*. Clarendon Press, Oxford.

Pitman, J. (2002). *Combinatorial Stochastic Processes*. Number 1875 in Lecture Notes in Mathematics. Springer Berlin. Ecole d'Eté de Probabilités.

Ruelle, D. (1969). *Statistical Mechanics: Rigorous Results*. Addison-Wesley, Reading, Massachusetts.

Schrödinger, E. (1931). Über die Umkehrung der Naturgesetze. *Sitzungsber. Preuß. Akad. Wiss., Phys.-Math. Kl. 1931 No. 8/9*, 144–153.

Sznitman, A. (1998). *Brownian Motion, Obstacles and Random Media*. Springer Berlin.

Thirring, W. (1980). *Quantenmechanik grosser Systeme*. Springer Verlag Wien. Lehrbuch der Mathematischen Physik 4.

Vershik, A. (1996). Statistical mechanics of combinatorial partitions, and their limit shapes. *Functional Analysis and Its Applications 30*(3), 90–105.

Watterson, G. (1976). The Stationary Distribution of the Infinitely-Many Neutral Alleles Diffusion Model. *Journal of Applied Probability 13*(4), 639–651.