MA3B8 - Complex Analysis

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Contents

Introduction

In this course we are going to investigate the properties of functions $f\colon \mathbb{C} \to \mathbb{C}$ that are *complex differentiable*. Here and in the sequel we are always going to denote by $\mathbb{C}$ the field of complex numbers. The theory covered in this course was mostly discovered by Cauchy, Lagrange, Riemann, Weierstrass and many other authors in the first half of the 19th century. It is thus quite old. The connections that they discovered remain until today among the most beautiful pieces of mathematics that I am aware of, full of small miracles and statements that (at least for someone who is accustomed to the technical difficulties of *real analysis*) seem to be too good to be true: every differentiable function is smooth, locally uniform limits of smooth functions are smooth etc.

Although this theory is relatively old there are many links to modern day research in diverse areas of mathematics. Let me just give two examples:

Example 0.1 For any $z \in \mathbb{C}$ with $\Re(z) > 1$ the *Riemann ζ-function* can be defined through the series

$$\zeta(z) := \sum_{n=1}^{\infty} \frac{1}{n^z}.$$

We will see later in the course that there is a unique way to extend ζ to a complex differentiable function on all of $\mathbb{C} \setminus \{1\}$. It is relatively easy to show that $\zeta(-2n) = 0$ for any $n \in \mathbb{N}$. In 1859 Riemann conjectured that all other zeros of ζ must lie on the line $\{z \colon \Re(z) = \frac{1}{2}\}$. Until today it has remained one of the most famous unsolved problems in mathematics to establish (or disprove) this *Riemann hypothesis*. It is one of the seven Clay Math Millennium problems (see http://www.claymath.org/millennium) and whoever solves it, will win a prize of $ 1 million.

This conjecture has close connections with the distribution of prime numbers. Towards the end of this course we will discuss this connection.

Example 0.2 Our second example concerns a recent development in statistical physics. Many stochastic processes that appear naturally as limits of (two dimensional) models in statistical physics are invariant under *conformal transformations* of the space. Here a mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ (or defined on some subset of $\mathbb{R}^2$) is called *conformal* if it preserves all angles. More precisely, f is conformal, if for any two smooth curves γ_1 and γ_2 that intersect at x at an angle α the image curves $f(\gamma_1)$ and $f(\gamma_2)$ intersect at $f(x)$ with the same angle α (and the same orientation). We will see below that if we identify $\mathbb{R}^2$ with $\mathbb{C}$ in the usual way, then f is conformal if and only if it is complex differentiable and the derivatives do not vanish. Hence, the tools from Complex Analysis are naturally very useful when investigating the properties of these processes.

The Riemann mapping theorem, which we will discuss in detail, is at the heart of one of the most remarkable constructions in this field, the *Schramm-Loewner-Evolution (SLE)*. Two recent Fields Medals (Wendelin Werner 2006, Stanislav Smirnov 2010) have been awarded for research in this area.

Conventions

We will assume that the reader is familiar with the field of complex numbers $\mathbb{C}$ and its basic arithmetic and topology. For a complex number $z \in \mathbb{C}$ we will often write $z = x+iy$ where $x = \Re(z) \in \mathbb{R}$ and $y = \Im(z) \in \mathbb{R}$ denote the real and imaginary part of z. In this way we will often identify $\mathbb{C}$ with $\mathbb{R}^2$ simply by identifying z with the vector $\begin{pmatrix} x \\ y \end{pmatrix}$. We will denote by $\bar{z} = x - iy$ the complex conjugate. Sometimes polar coordinates will be useful and for $z \neq 0$ we will write $z = |z|\,\mathrm{e}^{i\theta}$ where $|z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}$ denotes the absolute value of z and θ denotes the *argument* of z. Note that the argument is only defined uniquely up to a multiple of 2π.

Literature

1 The Algebra of the complex plane - a review

Definition 1.1 A complex number is an ordered pair (x, y) of real numbers $x, y \in \mathbb{R}$. Addition and multiplication are defined by

$$(x_1, y_1) + (x_2, y_2) := (x_1 + x_2, y_1 + y_2),$$
$$(x_1, y_1) \cdot (x_2, y_2) := (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1).$$

In the following, we drop the $\cdot$ symbol for multiplication. The set of complex numbers is denoted $\mathbb{C}$.

Theorem 1.2 *The set of complex numbers, with the operations defined in Definition 1.1, is a field. That is, the following axioms hold for any complex numbers* $z_i = (x_i, y_i), i = 1, 2, 3$ *:*

(a) Addition and multiplication are commutative:

$$z_1 + z_2 = z_2 + z_1$$
$$z_1z_2 = z_2z_1$$

(b) Addition and multiplication are associative:

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$$
$$(z_1z_2)z_3 = z_1(z_2z_3)$$

(c) There is an additive identity $(0, 0)$*:*

$$z_1 + (0, 0) = z_1$$

(d) There is a multiplicative identity $(1, 0)$*:*

$$z_1(1, 0) = z_1$$

(e) Each element has an additive inverse:

$$(x, y) + (-x, -y) = (0, 0)$$

(f) Each element other than $(0, 0)$ *has a multiplicative inverse:*

$$(x, y)\Big(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2}\Big) = (1, 0)$$

(g) Multiplication distributes over addition:

$$z_1(z_2 + z_3) = z_1z_2 + z_1z_3$$

Proof. All assertions (a)-(g) are direct consequences of the addition and multiplication defined in Definition 1.1 using only the field properties of the set $\mathbb{R}$ of real numbers. We only give the proof of some of the assertions and leave the remaining ones as an exercise for the reader. For example, (f) follows for each $z = (x, y) \neq (0, 0)$ as $z \neq (0, 0)$ implies $x^2 + y^2 \neq 0$ and by Definition 1.1,

$$(x, y)\Big(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2}\Big) = \Big(\frac{x^2}{x^2 + y^2} + \frac{y^2}{x^2 + y^2}, \frac{-xy}{x^2 + y^2} + \frac{yx}{x^2 + y^2}\Big) = (1, 0).$$

For example, (g) holds because

$$\begin{aligned} z_1(z_2 + z_3) &= (x_1, y_1)(x_2 + x_3, y_2, y_3) \\ &= (x_1(x_2 + x_3) - y_1(y_2 + y_3).x_1(y_2 + y_3) + y_1(x_2 + x_3)) \\ &= (x_1x_2 - y_1y_2 + x_1x_3 - y_1y_3, x_1y_2 + y_1x_2 + x_1y_3 + y_1x_3) \end{aligned}$$

and

$$z_1z_2 + z_1z_3 = (x_1x_2 - y_1y_2 + x_1x_3 - y_1y_3, x_1y_2 + y_1x_2 + x_1y_3 + y_1x_3)$$

$\square$

The symbol (x, y) is not commonly used for a complex number. Instead we write this number as $x + \mathrm{i}y$, a notation which goes back to Euler, who used i to denote $\sqrt{-1}$ in 1777. Note that the map $(x, 0) \mapsto x$ defines an isomorphism between the set of complex numbers of the form $(x, 0)$ and the field $\mathbb{R}$ of real numbers. We define $\mathrm{i} := (0, 1)$. Then, using Definition 1.1,

$$x + \mathrm{i}y = (x, 0) + (0, 1)(y, 0) = (x, y)$$

and

$$\mathrm{i}^2 = (0, 1)(0, 1) = (0 \cdot 0 - 1 \cdot 1, 0 \cdot 1 + 1 \cdot 0) = (-1, 0) = -1.$$

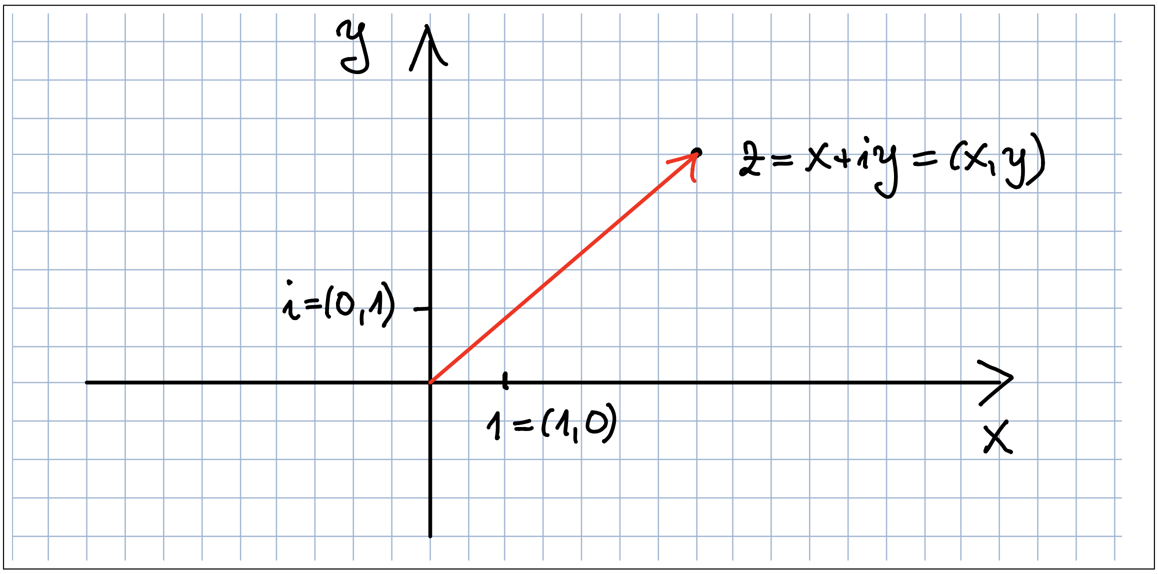

Figure 1:

Since ordered pairs (x, y) provide coordinates in the plane, we can visualise $\mathbb{C}$ as a plane, with the number $x + \mathrm{i}y$ corresponding to the point (x, y) as in Figure 1. The identification of $(x, 0)$ with $x \in \mathbb{R}$ then amounts to considering the real numbers as forming the *real axis*, the x-axis, as in Figure 1. The y-axis, at right angles to this, is the *imaginary axis*. For $z = x + \mathrm{i}y \in \mathbb{C}$ we write $\Re(z) = x$ for the real part and $\Im(z) = y$ for the imaginary part. Recall the modulus for a real number $x \in \mathbb{R}$ is given

$$|x| = \begin{cases} x & , x \geq 0, \\ -x & , x < 0 \end{cases}.$$

For $z = x + \mathrm{i}y \in \mathbb{C}$ we define the as

$$|z| := \sqrt{x^2 + y^2}. \tag{1.1}$$

Theorem 1.3 *The modulus defined in* (1.1) *has the following properties. For all $z_1, z_2 \in \mathbb{C}$,*

$$\begin{aligned} |z_1 + z_2| &\leq |z_1| + |z_2| \\ |z_1 z_2| &= |z_1||z_2| \\ ||z_1| - |z_2|| &\leq |z_1 - z_2|. \end{aligned}$$

Proof. Exercise (see similar proof in real analysis). □

Complex Conjugate: $z = x + \mathrm{i}y, z_1, z_2 \in \mathbb{C}$,

$$\begin{aligned} \overline{z} &:= x - \mathrm{i}y \\ \overline{z_1 + z_2} &= \overline{z_1} + \overline{z_2}, \quad \overline{z_1 z_2} = \overline{z_1 z_2} \\ \Re(z) &= \frac{1}{2}(z + \overline{z}) \\ \Im(z) &= \frac{1}{2\mathrm{i}}(z - \overline{z}) \\ |z|^2 &= z\overline{z} \\ \overline{z} \in \mathbb{R} &\Leftrightarrow z = \overline{z} \end{aligned}$$

Polar coordinates: $z = x + \mathrm{i}y$, see Figure 2,

$$\begin{aligned} & r := \sqrt{x^2 + y^2} \\ & \theta = \arg(z) \in [0, 2\pi) \text{ or } (-\pi, \pi] \text{ principal value} \\ & x = r\cos(\theta) \\ & y = r\sin(\theta) \end{aligned}$$

The unique value of $\theta \in (-\pi, \pi]$ is known as the principal value of the argument arg.

$$z = x + \mathrm{i}y = r\big(\cos(\theta) + \mathrm{i}\sin(\theta)\big)$$

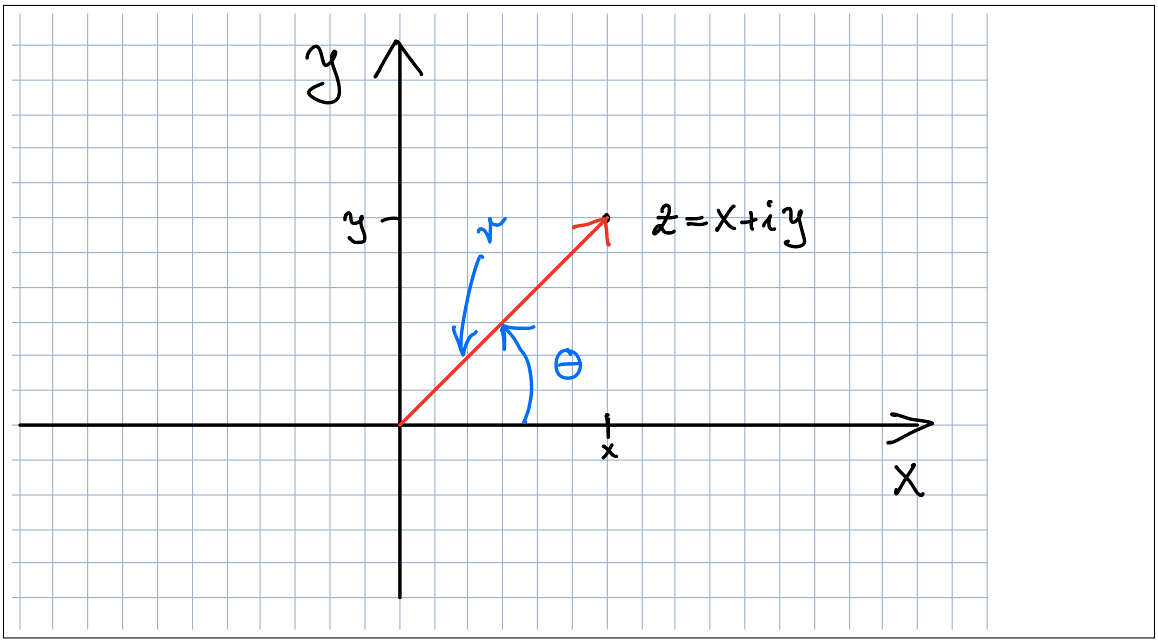

Figure 2:

For $z \in (-\infty, \infty)$ we have $\overline{z} = z$ and for $z \notin (-\infty, \infty)$ we have $\arg(\overline{z}) = -\arg(z)$.

The nth root of a complex number $w = x+\mathrm{i}y$**:** Write $w = x+\mathrm{i}y = r(\cos(\theta)+\mathrm{i}\sin(\theta))$. The n-th root of w is a complex number $z = \xi+\mathrm{i}\eta = \varrho(\cos(\alpha)+\mathrm{i}\sin(\alpha))$ such that $z^n = w$. There are exactly n roots for w, denoted $z_0, \dots, z_{n-1}$, which are constructed as follows. We need $\varrho^n = r$ and $n\alpha - \theta = 2k\pi$ for $k = 0, 1, \dots, n-1$. Hence, using the addition rules for trigonometric functions [1], we get $\varrho = \sqrt[n]{r}$ and $\alpha = \frac{\theta+2k\pi}{n}$, $k = 0, 1, \dots, n-1$.

$$z_k = \sqrt[n]{r}\Big(\cos\Big(\frac{\theta+2k\pi}{n}\Big) + \mathrm{i}\sin\Big(\frac{\theta+2k\pi}{n}\Big)\Big), \qquad k = 0, 1, \dots, n-1,$$

are the n roots which are corners of a regular polygon with n corners, see example in Figure 3 for the 6-th root of $-1 = \cos(\pi) + \mathrm{i}\sin(\pi)$.

Euler:

$$\mathrm{e}^{\mathrm{i}y} = \cos(y) + \mathrm{i}\sin(y), \qquad y \in \mathbb{R}. \tag{1.2}$$

Euler derived formula (1.2) in 1748, and nowadays (1.2) is called Euler's formula . Then, for any $z = x + \mathrm{i}y \in \mathbb{C}$,

$$\mathrm{e}^z = \mathrm{e}^{x+\mathrm{i}y} := \mathrm{e}^x\big(\cos(y) + \mathrm{i}\sin(y)\big)$$

Every $z \in \mathbb{C} \setminus \{0\}$ can be written as

$$z = \mathrm{e}^{\log(r)+\mathrm{i}\theta} = r\mathrm{e}^{\mathrm{i}\theta}, \qquad \theta = \arg(z), r = |z| > 0.$$

[1] $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$ and $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$

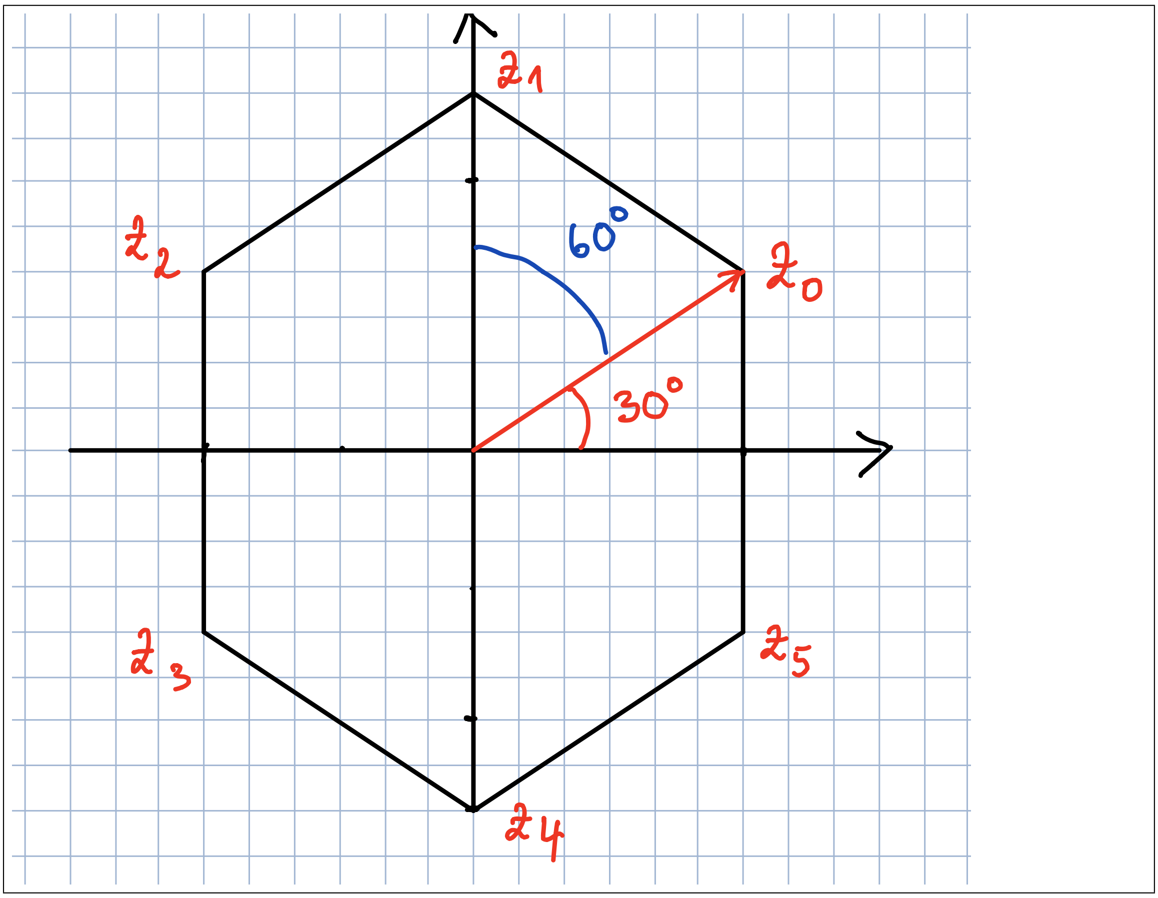

Figure 3:

What about the logarithm? The complex number $\xi + \mathrm{i}\eta$ is a logarithm of $z = x + \mathrm{i}y = r(\cos(\theta) + \mathrm{i}\sin(\theta))$ if

$$\mathrm{e}^{\xi+\mathrm{i}\eta} = x + \mathrm{i}y = \mathrm{e}^{\log(r)+\mathrm{i}\theta}.$$

Therefore

$$\xi = \log r \quad \text{and } \eta = \theta + 2n\pi \quad \text{with } n \in \mathbb{Z}.$$

Thus the *logarithm* of the complex number $z = x + \mathrm{i}y = r\mathrm{e}^{\mathrm{i}\theta}$ is the set

$$\log z := \{\log(r) + \mathrm{i}(\theta + 2n\pi) \colon n \in \mathbb{Z}\}. \tag{1.3}$$

Each element of the set on the right side of (1.3) is called element or *branch of the logarithm* . For example, the set $\{\mathrm{i}(\pi/2 + 2n\pi) \colon n \in \mathbb{Z}\} = \log \mathrm{i}$ is the logarithm of i. Having defined the logarithm, we can now define the *complex power*:

Suppose $z_0 = x_0 + \mathrm{i}y_0 = r\mathrm{e}^{\mathrm{i}\theta} \neq 0$ and $z_1 = x_1 + \mathrm{i}y_1 \in \mathbb{C}$. We say that $\xi + \mathrm{i}\eta$ is an element of general power set $z_0^{z_1}$ if $\xi + \mathrm{i}\eta$ is an element of $\mathrm{e}^{(x_1+\mathrm{i}y_1)\log(z_0)}$, that is, if there exists $n \in \mathbb{Z}$ such that

$$\xi + \mathrm{i}\eta = \mathrm{e}^{x_1\log(r) - y_1(\theta+2n\pi) + \mathrm{i}(y_1\log(r) + x_1(\theta+2n\pi))},$$

where we used the branch/element with $n \in \mathbb{Z}$ of the logarithm $\log z_0 = \log(r) + \mathrm{i}(\theta + 2n\pi)$. Therefore,

$$z_0^{z_1} := \{\mathrm{e}^{x_1\log(r) - y_1(\theta+2n\pi) + \mathrm{i}(y_1\log(r) + x_1(\theta+2n\pi))} \colon n \in \mathbb{Z}\}. \tag{1.4}$$

For example,

$$\mathrm{i}^{\mathrm{i}} = \{\mathrm{e}^{-(\pi/2+2n\pi)} \colon n \in \mathbb{Z}\} \subset \mathbb{R}$$

is the set of powers for i^{i} which is actually a subset of real numbers. Recall that $\arg(z) \in (-\pi, \pi]$ is called the *principal value of the argument* of $z \in \mathbb{C} \setminus \{0\}$. The function $\arg \colon \mathbb{C} \setminus \{0\} \to \mathbb{R}$ is not continuous on the negative real axis $\{z = x + \mathrm{i}y \colon y = 0, x \leq 0\}$. Define the *cut plane* $\mathbb{C}_\Pi := \mathbb{C} \setminus \{z = x + \mathrm{i}y \colon y = 0, x \leq 0\}$.

Proposition 1.4 *The function* arg *is continuous in the cut plane* $\mathbb{C}_\Pi$.

Proof. Exercise and TA class. □

This implies that the complex logarithm log is not continuous on the negative real axis but log is continuous in the cut plane $\mathbb{C}_\Pi$.

2 Some geometry of the complex plane: Stereographic projections and Möbius transforms

2.1 The Riemann sphere

Sometimes it is useful to add an extra point to $\mathbb{C}$ which we denote by ∞. This will be convenient, for example, because it allows us to define functions with poles unambiguously on all of $\mathbb{C}$. In this section, we discuss a construction that gives a natural interpretation to this *extended complex plane* $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$. It also defines a natural metric on this set. Why we need the extended complex plane? The mapping $z \mapsto \frac{1}{z}$ gives an inversion in the unit circle. All $z \in \mathbb{C}$ with $|z| > 1$ are mapped into the *unit disc* $\mathbf{\Delta} := \{z \in \mathbb{C} \colon |z| < 1\}$, whereas all $z \in \mathbf{\Delta} \setminus \{0\}$ are mapped to $\mathbf{\Delta}^{\mathrm{c}}$. The problem is that no image point is presently associated with $z = 0$, nor is 0 to be found among the image points. Note that as z moves further and further away from the origin, $1/z$ moves closer and closer to 0. Thus as z travels to ever greater distances (in any direction), it is a though it were approaching a single point at infinity, written ∞, whose image is 0. We define

$$\frac{1}{\infty} := 0 \ \text{ and } \frac{1}{0} := \infty.$$

With that convention we can show that the *inversion*

$$\mathrm{inv} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}, z \mapsto \frac{1}{z}$$

is a one-to-one mapping of the extended complex plane. Problem is that we are accustomed to using the symbol ∞ only in conjunction with a limiting process, not as a thing in its own right. How can we construct the element ∞ as a definite point that is infinitely far away? Riemann's idea was to interpret complex numbers as points on a sphere in the three-dimensional Euclidean space. We identify the complex plane with $\mathbb{R}^2$ which we imagine embedded into $\mathbb{R}^3$. Throughout the lecture, imagine the complex plane (i.e., $\mathbb{R}^2$) positioned horizontally in the space $\mathbb{R}^3$. The "equator" of the sphere coincides with the unit circle $\partial\mathbf{\Delta} := \{z \in \mathbb{C} \colon |z| = 1\}$. We denote the unit sphere in $\mathbb{R}^3$ by

$$\mathrm{S}^2 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \colon |x_1|^2 + |x_2|^2 + |x_3|^2 = 1\}$$

with the the north pole $N := (0,0,1)$ and the south pole $S := (0,0,-1)$. When we look down on $\mathbb{C}$ (from the upper half sphere) from above, a positive (counterclockwise) rotation of $\pi/2$ carries the point $1 = (1,0)$ to i. If we think of S^2 as the surface of the Earth, then this is the ancient problem of how to draw a geographical map. Ptolemy (AD 125) was the first to construct such a map, which he used to plot the positions of heavenly bodies on the "celestial sphere". His method is stereographic projection: From the north pole N of the sphere S^2, draw the line through the point $z \in \mathbb{C}$, the stereographic image of z on the sphere S^2 is the point $\widehat{z}$ where this line intersects S^2. This gives us a one-to-one correspondence between points in $\mathbb{C}$ and points on $\mathrm{S}^2 \setminus \{N\}$. Some immediate facts are in order:

1. The unit disc $\boldsymbol{\Delta}$ is mapped to the lower/southern hemisphere of S^2, and the origin 0 is mapped to the south pole S.

2. Each point on $\partial\boldsymbol{\Delta}$ is mapped to itself, now viewed as lying on the equator.

3. The complement $\boldsymbol{\Delta}^{\mathrm{c}}$ of $\boldsymbol{\Delta}$ is mapped to the upper/northern hemisphere, except that the north pole N is not the image of any point (with bounded modulus) in the complex plane. However, it is clear that as z moves further and further away from the origin 0 in any direction, the image point $\widehat{z}$ on the sphere moves closer and closer to N.

Definition 2.1 The *stereographic projection* $\pi\colon \mathrm{S}^2 \setminus \{N\} \to \mathbb{C}$ is defined by

$$\pi\colon (x_1,x_2,x_3) \mapsto z = \pi(x_1,x_2,x_3) = \left(\frac{x_1}{1-x_3}\right) + \mathrm{i}\left(\frac{x_2}{1-x_3}\right). \tag{2.1}$$

It is easy to see that π defines a bijection from $\mathrm{S}^2 \setminus \{N\}$ to $\mathbb{C}$. In particular, the map π maps the south pole $(0,0,-1)$ to 0, the lower/southern hemisphere into the unit disc, and the upper hemisphere outside of the unit disc. To see that

$$\pi(\{\text{southern hemisphere}\}) = \partial\Delta \cup \Delta =: \overline{\Delta} := \{z \in \mathbb{C}\colon |z| \le 1\},$$

note that any point on the southern hemisphere of S^2 satisfies $x_3 \le 0$, and thus $1 - x_3 \ge 1$ implying that

$$|z|^2 = |\pi(x_1,x_2,x_3)|^2 = \frac{(x_1)^2}{(1-x_3)^2} + \frac{(x_2)^2}{(1-x_3)^2} \le 1.$$

We will start by showing the following remarkable geometric fact about the stereographic projection.

Lemma 2.2 *A circle $\mathcal{C}$ on the sphere is the intersection of S^2 with a plane $\{(x_1,x_2,x_3) \in \mathbb{R}^2\colon ax_1 + bx_2 + cx_3 = d\}$ for suitable real coefficients $a,b,c,d \in \mathbb{R}$. Then the image of every (nonempty) circle on S^2 is either a line or a circle in $\mathbb{C}$.*

Proof. We begin by calculating the inverse to π. Assume we have $z = \pi(x_1, x_2, x_3)$. Then the formula (2.1) yields the following identities

$$z + \bar{z} = \frac{2x_1}{1 - x_3}, \qquad z - \bar{z} = \frac{2ix_2}{1 - x_3}, \qquad \text{and} \quad |z|^2 = \frac{1 - x_3^2}{(1 - x_3)^2} = \frac{1 + x_3}{1 - x_3}.$$

Solving for x_3 first and then for x_1, x_2 yields

$$x_3 = \frac{|z|^2 - 1}{1 + |z|^2}, \qquad x_1 = \frac{z + \bar{z}}{1 + |z|^2}, \qquad \text{and} \quad x_2 = \mathrm{i}\frac{\bar{z} - z}{1 + |z|^2}. \tag{2.2}$$

Now consider a point $z \in \mathbb{C}$ with $\pi^{-1}(z) \in \mathcal{C}$. Plugging (2.2) into the equation for the plane we get

$$a(\bar{z} + z) - ib(z - \bar{z}) + c(|z|^2 - 1) = d(1 + |z|^2).$$

Writing $z = x + \mathrm{i}y$ we get

$$(d - c)(x^2 + y^2) + (d + c) - 2ax - 2by = 0. \tag{2.3}$$

<u>Case 1:</u> If $d = c$ the identity (2.3) reduces to $(d + c) = 2ax + 2by$ which describes a line in $\mathbb{C}$. Note that this is the case if and only if $\mathcal{C}$ goes through the north pole.

<u>Case 2:</u> If $d \neq c$ we can divide the (2.3) by $d - c$ and complete the square to obtain

$$\left(x - \frac{a}{d - c}\right)^2 + \left(y - \frac{b}{d - c}\right)^2 = \frac{a^2 + b^2 + c^2 - d^2}{(d - c)^2}.$$

This equation describes a circle with midpoint $\frac{a}{d-c} + \mathrm{i}\frac{b}{d-c}$ if and only if the right hand side is non-negative (here we adopt the convention that a point is also a circle of radius 0). As may be expected, this is the case if the plane intersects S^2 (i.e. if $\mathcal{C}$ is non-empty). In that case there exists at least one point $(x_1, x_2, x_3) \in \mathrm{S}^2$ that satisfies $ax_1 + bx_2 + cx_3 = d$. This implies using the Cauchy-Schwarz inequality that

$$d^2 = (ax_1 + bx_2 + cx_3)^2 \leq (a^2 + b^2 + c^2)(x_1^2 + x_2^2 + x_3^2) = (a^2 + b^2 + c^2).$$

□

Actually, the converse is also true. We leave it for the reader to check that the preimage of any circle or line in $\mathbb{C}$ under the stereographic projection is a circle on S^2.

We note that (2.2) (or our geometric intuition) shows that $\pi^{-1}(z_n)$ converges to the north pole N for any sequence (z_n) with $|z_n| \to \infty$. This makes it natural to identify $\widehat{\mathbb{C}}$ with all of S^2 simply by setting $\pi(N) = \infty$. In particular, $\widehat{\mathbb{C}}$ obtains a natural metric simply by adopting the Euclidean metric from the sphere. More precisely, we set for $z, w \in \widehat{\mathbb{C}}$

$$\mathrm{d}(z, w) := \|\pi^{-1}(z) - \pi^{-1}(w)\|,$$

where $\|\cdot\|$ denotes the usual Euclidean norm $\|(x_1,x_2,x_3)\| = \sqrt{x_1^2+x_2^2+x_3^2}$. It is an exercise to check that in this way we obtain

$$\mathrm{d}(z,w) = \frac{2|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}}, \qquad \text{if } z,w \neq \infty \qquad \text{and} \quad \mathrm{d}(z,\infty) = \frac{2}{\sqrt{1+|z|^2}}. \tag{2.4}$$

Rules:

$$\begin{aligned} z+\infty &= \infty + z, \quad \forall z \in \widehat{\mathbb{C}} \\ z\cdot\infty &= \infty\cdot z = \infty, \quad \forall z \in \widehat{\mathbb{C}}\setminus\{0\} \\ \frac{z}{\infty} &:= 0, \quad \forall z \in \mathbb{C} \\ \frac{z}{0} &:= \infty, \quad \forall z \in \widehat{\mathbb{C}}\setminus\{0\} \end{aligned}$$

2.2 Möbius transformations

In this section we are going to study a very interesting class of transformations of $\widehat{\mathbb{C}}$, the *Möbius transformations* . We start by defining mappings of the complex plane which we extend to mappings of the extended complex plane in a second step.

Definition 2.3 A Möbius transformation is a mapping

$$f(z) = \frac{az+b}{cz+d} \tag{2.5}$$

defined on $\mathbb{C}\setminus\left\{-\frac{d}{c}\right\}$ for some complex coefficients $a,b,c,d \in \mathbb{C}$ that satisfy $ad-bc \neq 0$.

In Definition 2.3 we have excluded the case $ad-bc=0$, because in that case

$$f(z) = \frac{az+b}{cz+d} = \frac{\frac{a}{c}(cz+d)+b-\frac{ad}{c}}{cz+d} = \frac{a}{c} + \frac{bc-ad}{c}\frac{1}{cz+d}$$

maps all of $\mathbb{C}\setminus\{-\frac{d}{c}\}$ to $\frac{a}{c}=\frac{b}{d}$, or is not defined if $c=d=0$. However, we can extend f to a function on the extended complex plane,

$$\widehat{f}\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}},\ z \mapsto \widehat{f}(z) = \begin{cases} f(z) & , z \in \mathbb{C}\setminus\{-\frac{d}{c}\}, \\ \infty & , z = -\frac{d}{c}, \\ \frac{a}{c} & , z = \infty. \end{cases}$$

The function $\widehat{f}$ defined in this way is continuous with respect to the metric defined in (2.4). In the following, when we speak of a Möbius transformation, re refer to this extension and will denote it f as well.

Note that if we replace a,b,c,d by $\lambda a, \lambda b, \lambda c, \lambda d$ for some $\lambda \in \mathbb{C}\setminus\{0\}$, the mapping defined by (2.5) does not change. Therefore, it is always possible (but not always convenient) to assume that $ad-bc=1$. We will call a Möbius transform *normalised* if this is satisfied.

We start by deriving the following simple properties.

Lemma 2.4 *The Möbius transformation*

$$f(z) = \frac{az+b}{cz+d} \tag{2.6}$$

is invertible from $\widehat{\mathbb{C}}$ to $\widehat{\mathbb{C}}$. The inverse f^{-1} is again a Möbius transform and given by

$$f^{-1}(z) = \frac{dz-b}{-cz+a}. \tag{2.7}$$

Proof. It is sufficient to check that for every $z \in \widehat{\mathbb{C}}$ and for f and f^{-1} defined above, we have $z = f(f^{-1}(z)) = f^{-1}(f(z))$. We will only show the first equality and leave the second one to the reader. For $z \neq \frac{a}{c}, \infty$ we get

$$f(f^{-1}(z)) = \frac{a\left(\frac{dz-b}{-cz+a}\right)+b}{c\left(\frac{dz-b}{-cz+a}\right)+d} = \frac{adz-ab-bcz+ab}{cdz-cb-dcz+ad} = \frac{(ad-bc)z}{ad-bc} = z.$$

For $z = \frac{a}{c}$ we have $f^{-1}(z) = \infty$ which is mapped to $\frac{a}{c}$ by f, and for $z = \infty$ we get $f^{-1}(z) = -\frac{d}{c}$ which is mapped to ∞ under f. □

Note that this lemma, together with the observation that Möbius transformations are continuous from $\widehat{\mathbb{C}}$ to $\widehat{\mathbb{C}}$, implies that every Möbius transformation defines a homeomorphism from $\widehat{\mathbb{C}}$ to itself. As a next step, we consider the composition of Möbius transformations.

Lemma 2.5 *Let f_1 and f_2 be two Möbius transformations given by*

$$f_i = \frac{a_i z + b_i}{c_i z + d_i} \qquad i = 1, 2.$$

Then $f_1 \circ f_2$ is again a Möbius transformation and given by

$$f_1 \circ f_2(z) = \frac{(a_1a_2 + b_1c_2)z + (a_1b_2 + b_1d_2)}{(c_1a_2 + d_1c_2)z + (c_1b_2 + d_1d_2)}. \tag{2.8}$$

Proof. It is an easy exercise, that we leave to the reader to establish (2.8). The only thing one might worry about, is the question if the non-degeneracy condition $ad - bc \neq 0$ is preserved under composition of Möbius transformations. To see that this is indeed the case, one can simply argue that by Lemma 2.4 the Möbius transformations f_1 and f_2 are homeomorphisms of $\widehat{\mathbb{C}}$ and hence so is their composition. As mappings that do not satisfy the non-degeneracy condition, i.e., $ad - bc \neq 0$, they do map all of $\mathbb{C} \setminus \{-\frac{d}{c}\}$ onto a single point, this case can be excluded. We will see a nicer argument in Remark 2.6 below. □

Remark 2.6 In the previous two lemmas we have actually shown that the set of Möbius transformations forms a group under composition, more precisely a subgroup of the group of homeomorphisms of $\widehat{\mathbb{C}}$. The formula (2.8) shows even more. Namely, if we associate with the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{C}^{2\times 2}$ the Möbius transformation $M_A := \frac{az+b}{cz+d}$ then (2.8) shows that for any matrices A, B we have

$$M_A \circ M_B = M_{AB}.$$

Also in this picture the non-degenaracy condition $ab - cd \neq 0$ has a simple interpretation as $\det(A) \neq 0$, and the fact that $M_A \circ M_B$ is non-degenerate follows immediately from the fact that determinants multiply under matrix multiplication. If we furthermore restrict ourselves to the normalised case with $\det(A) = 1$, i.e, to matrices in $SL(2, \mathbb{C})$ we see that $A \mapsto M_A$ is a homomorphism from $SL(2, \mathbb{C})$ to the group of Möbius transformations.

The simplest non-trivial example of a Möbius transformation is the (complex) *inversion*. It is the map $\mathrm{inv} : z \mapsto \mathrm{inv}\,(z) = \frac{1}{z}$. Its action is best understood in polar coordinates. The point $z = r\mathrm{e}^{\mathrm{i}\theta}$ is mapped to $\frac{1}{z} = \frac{1}{r}\mathrm{e}^{-\mathrm{i}\theta}$.

Lemma 2.7 *Let $\mathcal{C} \subseteq \mathbb{C}$ be a circle or a line. Then the image of $\mathcal{C}$ under the inversion* inv *is either a circle or a line.*

Proof. It is obvious, that lines through the origin are mapped to lines through the origin. Let us characterise the image of the set

$$\mathcal{C} := \{z \in \mathbb{C} \colon |z - a|^2 = r^2\},$$

for an $a \in \mathbb{C}$ and $r > 0$. Then a point $w \in \widehat{\mathbb{C}}$ lies in the image $\mathrm{inv}\,(\mathcal{C})$ if and only if it satisfies

$$r^2 = \frac{1 - aw - \bar{a}\bar{w} + |a|^2|w|^2}{|w|^2},$$

which is equivalent to

$$|w|^2(r^2 - |a|^2) = 1 - aw - \bar{a}\bar{w}. \tag{2.9}$$

This follows using $z = w^{-1} = \bar{w}/|w|^2$. If $r^2 - |a|^2 = 0$, this equation describes a line in $\mathbb{C}$. Note that this is the case, if and only if the original circle $\mathcal{C}$ touches the origin. In the same way, one can see that the image of the line

$$\ell = \{z \in \mathbb{C} \colon 0 = az + \bar{a}\bar{z} - 1\}$$

is the circle with radius $|a|$ around a, where, again one uses $z = w^{-1} = \bar{w}/|w|^2$ leading to $|w|^2 = a\bar{w} + \bar{a}w$.

Finally, if $r^2 - |a|^2 \neq 0$ equation (2.9) is equivalent to

$$\left| w + \frac{a}{r^2 - |a|^2} \right|^2 = \frac{r^2}{(r^2 - |a|^2)^2}$$

which describes a circle. □

Definition 2.8 We will call the following Möbius transformations *elementary*:

(i) *Inversion* Define the map $\mathrm{inv} : z \mapsto \mathrm{inv}\,(z) = \frac{1}{z}$.

(ii) *Translations*: Define $f \colon z \mapsto f(z) = \frac{1z+b}{0z+1} = z + b$. This map simply shifts any point z by the complex number b.

(iii) *Rotation*: For $a = \mathrm{e}^{\mathrm{i}\theta} \in \mathbb{C}$ define the map $f \colon z \mapsto f(z) = \frac{az+0}{0z+1} = az$. This map rotates every point around the origin by an angle θ.

(iv) *Expansion/Contraction*: For $\mathbb{R} \ni r > 0$ define the map $f \colon z \mapsto f(z) = \frac{rz+0}{0z+1} = rz$. This map acts as an expansion (if $r > 1$) or a contraction (if $r < 1$).

Suppose $z = re^{\mathrm{i}\alpha}$ and $a = e^{\mathrm{i}\theta}$. Then we identify z with the vector $(\Re(z), \Im(z)) = (r\cos(\alpha), r\sin(\alpha))$, and $f(z) = az$ with the vector

$$\begin{pmatrix} r\cos(\alpha+\theta) \\ r\sin(\alpha+\theta) \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} r\cos(\alpha) \\ r\sin(\alpha) \end{pmatrix}$$

using the addition theorems for sin and cos. The right hand side is written as a matrix applied (having determinant equal to one) to the vector $(r\cos(\alpha), r\sin(\alpha))$, this is a rotation by the angle θ.

We show in the next lemma that every Möbius transformation is a composition of elementary Möbius transformations.

Lemma 2.9 *Every Möbius transformation can be written as a composition of elementary Möbius transformations.*

Proof. For $c \neq 0$ we write $\frac{az+b}{cz+d} = \frac{a}{c} + \frac{b - \frac{ad}{c}}{cz+d}$. Then this map is obtained by

$$z \mapsto cz \mapsto cz + d \mapsto \frac{1}{cz+d} \mapsto \frac{b - \frac{ad}{c}}{cz+d} \mapsto \frac{a}{c} + \frac{b - \frac{ad}{c}}{cz+d}. \tag{2.10}$$

The case where $c = 0$ is even easier. □

The following is a fundamental property of the Möbius transformations.

Theorem 2.10 *The image of a circle or a line in $\widehat{\mathbb{C}}$ under a Möbius transform is a circle or a line.*

Remark 2.11 (a) The Möbius transformations are actually the only complex differentiable mappings from $\widehat{\mathbb{C}}$ to $\widehat{\mathbb{C}}$ with this property (see below for the definition of complex differentiability).

(b) Recall from Lemma 2.2 above that circles and lines correspond to circles on the Riemann sphere. Recall furthermore that lines correspond precisely to those circles on S^2 that touch the north pole. They can thus be interpreted as circles with infinite radius. The content of Theorem 2.10 can thus be summarised in the more catchy phrase:

Möbius transformations map circles to circles.

◇

Proof of Theorem 2.10. It is sufficient to show this property for elementary Möbius transformations. For inversions it is the content of Lemma 2.7. It is very easy to see that lines are mapped to lines and circles are mapped to circles for translations, rotations, and expansions/contractions. □

The following result concerns the flexibility of the Möbius transformations.

Theorem 2.12 *Given three distinct points $z_1, z_2, z_3 \in \widehat{\mathbb{C}}$ and three distinct points $w_1, w_2, w_3 \in \widehat{\mathbb{C}}$, there exists a unique Möbius transformation f that satisfies $f(z_i) = w_i$ for $i = 1, 2, 3$.*

This property makes precise our intuition that the group of Möbius transformations has three complex degrees of freedom.

Proof. Existence: Suppose for the moment that $z_i \neq \infty$ for $i = 1, 2, 3$. Then we define the map

$$S(z) := \frac{(z - z_2)(z_1 - z_3)}{(z - z_3)(z_1 - z_2)}.$$

This map has the property that

$$S(z_1) = 1, \quad S(z_2) = 0, \quad \text{and } S(z_3) = \infty. \tag{2.11}$$

In the case where one of the z_i is equal to ∞ we just drop the corresponding terms, i.e., for example we set

$$S(z) = \frac{z - z_2}{z - z_3}$$

if $z_1 = \infty$. In either case, (2.11) is still satisfied.

In the same way we define the mapping

$$T(z) := \frac{(z - w_2)(w_1 - w_3)}{(z - w_3)(w_1 - w_2)} \text{ and thus } T^{-1}(z) = \frac{-w_3(w_1 - w_2)z + w_2(w_1 - w_3)}{(w_2 - w_1)z + w_1 - w_3}.$$

Then we simply set $f = T^{-1}S$. This is a Möbius transformation according to Lemma 2.4 and Lemma 2.5, and it has the desired property by construction. For example,

$$f(z_1) = T^{-1}(S(z_1)) = T^{-1}(1) = \frac{w_1(w_2 - w_3)}{w_2 - w_3} = w_1.$$

Uniqueness: It is sufficient to consider the case $w_1 = 1, w_2 = 0$, and $w_3 = \infty$. Actually, suppose we have established uniqueness in this case. Then from two distinct transformations f_1 and f_2 map z_1, z_2, z_3 to arbitrary distinct w_1, w_2, w_3 we could always manufacture two different transformations that map z_1, z_2, z_3 to $1, 0, \infty$ by composing with a map from the existence argument.

Hence, let us assume that f_1 and f_2 are two Möbius transformations that map z_1, z_2, z_3 to $1, 0, \infty$. We need to show that $g := f_1 \circ f_2^{-1}$ is the identity. Note that

$$g(z) = \frac{az + b}{cz + d}$$

maps 0 to 0, 1 to 1 and ∞ to ∞. The fact that $g(\infty) = \infty$ implies $c = 0$. As we always have one parameter to choose freely, we will set $d = 1$. Then $g(0) = 0$ implies that $b = 0$ and finally $g(1) = 1$ implies $a = 1$. Hence, g is the identity and the claim is proved. □

Definition 2.13 The cross-ratio of pairwise distinct complex numbers z_1, z_2, z_3, z_4, is the continuous mapping $\mathbf{CR}\colon \mathbb{C}^4 \setminus \mathrm{Diag}(\mathbb{C}^4), (z_1, \ldots, z_4) \mapsto \mathbf{CR}(z_1, \ldots, z_4)$ given as

$$\mathbf{CR}(z_1, z_2, z_3, z_4) := \frac{(z_3 - z_1)}{(z_3 - z_2)} \Big/ \frac{(z_4 - z_1)}{(z_4 - z_2)} = \frac{(z_3 - z_1)(z_4 - z_2)}{(z_3 - z_2)(z_4 - z_1)}.$$

$$\mathrm{Diag}(\mathbb{C}^4) = \{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4\colon\ \exists i, j \in \{1, \ldots, 4\}, i \neq j,\ \text{with } z_i = z_j\}.$$

Proposition 2.14 *The cross ration* **CR** *is invariant under Möbius transformations.*

Proof. Exercise (use the earlier proofs for linear mappings and inversions). □

Example 2.15 We are looking for a Möbius transformation f that maps the half space $\mathrm{H_R} := \{z \in \mathbb{C}\colon \Re(z) > 0\}$ to the unit disc $\mathbf{\Delta} := \{z \in \mathbb{C}\colon |z| < 1\}$. Certainly the imaginary axis $\mathrm{I} := \{z \in \mathbb{C}\colon \Re(z) = 0\}$ should be mapped to $\mathrm{S}^1 := \{z \in \mathbb{C}\colon |z| = 1\} = \partial\mathbf{\Delta}$ under f. According to Theorem 2.12 there exists a unique Möbius transformation f with $f(0) = -1$, $f(\mathrm{i}) = \mathrm{i}$, and $f(-\mathrm{i}) = -\mathrm{i}$ and this seems to be a good candidate. Plugging this in

$$f(z) = \frac{az + b}{cz + d}$$

gives $\frac{b}{d} = -1$, $\frac{a\mathrm{i}+b}{c\mathrm{i}+d} = \mathrm{i}$, and $\frac{-a\mathrm{i}+b}{-c\mathrm{i}+d} = -\mathrm{i}$ which implies

$$f(z) = \frac{z - 1}{z + 1}.$$

This map does indeed map the imaginary axis onto the S^1: By Theorem 2.10 the image of the imaginary axis has to be line or a circle. But, as the points -1, i, and $-\mathrm{i}$ lie in this image, it can only be S^1. As f is a homeomorphism of $\widehat{\mathbb{C}}$, the image of $\mathrm{H_R}$ must be either $\mathbf{\Delta}$ or $\widehat{\mathbb{C}} \setminus \overline{\mathbf{\Delta}}$, where $\overline{\mathbf{\Delta}} = \mathbf{\Delta} \cup \partial\mathbf{\Delta} = \{z \in \mathbb{C}\colon |z| \leq 1\}$ denotes the closure of $\mathbf{\Delta}$. But as $f(1) = 0$ it can only be $\mathbf{\Delta}$.

In a similar way one can see that f maps the real axis to itself and all parallel lines to circles that intersect with the unit disc in 1. All lines that are parallel to the imaginary axis are mapped onto circles that touch S^1 in 1 and that are otherwise contained in $\mathbf{\Delta}$.

This Möbius transformation is used in electrical engineering to visualise complex resistors in *Smith diagrams*. ♣

Example 2.16 Let us try to find the image of the unit disc $\mathbf{\Delta} = \{z \in \mathbb{C}\colon |z| < 1\}$ under the Möbius transformation

$$f(z) = \frac{\mathrm{i}z + 3}{\mathrm{i}z - 1} = 1 + \frac{4}{\mathrm{i}z - 1}.$$

The function f can be written a $f = f_5 \circ f_4 \circ f_3 \circ f_2 \circ f_1$, where

$$f_1(z) = \mathrm{i}z\ , \quad f_2(z) = z - 1\ , \quad f_3(z) = \frac{1}{z}\ , \quad f_4(z) = 4z\ , \quad f_5(z) = z + 1\ .$$

The mapping f_1 maps the unit disc onto itself, f_2 shifts it to “the left” by one. This shifted disc is mapped onto the half space $\{z \in \mathbb{C}\colon \Re(z) < -\frac{1}{2}\}$ under the inversion. Indeed the shifted boundary circle contains the origin and hence the inversion maps it to a line. This line has to contain $-\frac{1}{2}$ as the image of -2 under the inversion. Finally, it is readily seen to be invariant under reflection at the imaginary axis. This implies that the boundary curve is mapped onto the line $\{z \in \mathbb{C}\Re(z) = -\frac{1}{2}\}$. As -1 is invariant under the inversion the image of the disc $f_2(\boldsymbol{\Delta})$ has to be the half space claimed above.

The image of this half space under $f_5 \circ f_4$ and hence the image of $\boldsymbol{\Delta}$ under f is the half space $\{z \in \mathbb{C}\colon \Re(z) < -1\}$.

Example 2.17 Let us try to find the unique Möbius transformation

$$f(z) = \frac{az+b}{cz+d}$$

with $f(\pm 1) = \pm 1$ and $f(0) = \frac{1}{2}$. Actually, $f(0) = \frac{1}{2}$ implies that $\frac{b}{d} = \frac{1}{2}$. As we can always choose one of the parameters freely, we can set $b = 1$ and $d = 2$. The other two conditions yield

$$a + 1 = c + 2 \qquad \text{and} \qquad -a + 1 = (-1)(-c + 2),$$

which implies that $a = 2$ and $c = 1$. Summarising, the function f is given by

$$f(z) = \frac{2z+1}{z+2}.$$

Note that it was convenient, not to give f in the normalised form. We also observe that all the coefficients in this Möbius transformation can be chosen as real numbers which implies immediately that f maps all real numbers onto real numbers (or ∞). Finally we calculate that

$$f(i) = \frac{1+2i}{2+\mathrm{i}} = \frac{(1+2\mathrm{i})(2-\mathrm{i})}{(2+\mathrm{i})(2-\mathrm{i})} = \frac{4+3\mathrm{i}}{5},$$

which lies again on the unit circle $\mathrm{S}^1 = \{z \in \mathbb{C}\colon |z| = 1\}$. We argue that this already implies that f maps *every* point in S^1 to S^1. To see, recall that Theorem 2.10 implies that the image of S^1 under f is *some* circle or line. But as the points $1, -1$, and $\frac{4+3\mathrm{i}}{5}$ lie in $f(\mathrm{S}^1)$ it must be S^1.

Example 2.18 In this example we argue that all Möbius transformations of the form

$$f(z) = \frac{a-z}{1-\bar{a}z} \qquad \text{for} \quad a \in \mathbb{C} \quad |a| < 1, \tag{2.12}$$

are bijections from the unit disc $\boldsymbol{\Delta} := \{z \in \mathbb{C}\colon |z| < 1\}$ into itself (and hence by continuity they are also bijections from S^1 to itself). Before we show that, note that $f(a) = 0$ and hence the condition $|a| < 1$ is clearly necessary for this to be true. Also note that the transformation discussed in Example 2.17 above is “almost” of this form. Actually, we can write

$$\frac{2z+1}{z+2} = \frac{z+\frac{1}{2}}{1+\frac{1}{2}z} = -\frac{(-\frac{1}{2})-z}{1-(-\frac{1}{2})z},$$

which is a mapping of the form (2.12) (for $a = \bar{a} = -\frac{1}{2}$) composed with a rotation around the origin by the angle $-\pi$.

In the general case (2.12), for any z we can calculate

$$|f(z)|^2 = \frac{a\bar{a} + z\bar{z} - 2\Re(a\bar{z})}{1 + |\bar{a}z|^2 - 2\Re(a\bar{z})}.$$

It is easy to see (exercise !) that this expression is $= 1$ if $|z| = 1$ and < 1 if $|z| < 1$.

Transformations of the form (2.12) are almost the most general Möbius transformations that are bijections of the unit disc. The most general transformation is of the form

$$f(z) = \mathrm{e}^{\mathrm{i}\theta}\frac{a - z}{1 - \bar{a}z} \qquad \text{for } |a| < 1 \quad \text{and} \quad \theta \in [-\pi, \pi). \tag{2.13}$$

We will see below in Corollary 5.27 the remarkable fact that this already includes all complex diffeomorphisms of the unit disc. If a complex function f maps $\mathbf{\Delta}$ bijectively and in a (complex) differentiable way into itself, it must be a Möbius transform!

The following statement is a fundamental property of continuous functions (recalling basic training in analysis):

Theorem 2.19 *If f is a continuous mapping of a metric space (X, d_X) into a metric space (Y, d_Y), and $E \subset X$ is a connected subset of X, then the image $f(E)$ is connected. Here, a set D is connected if it cannot be expressed as the union of non-empty open sets D_1 and D_2 with $D_1 \cap D_2 = \varnothing$ (compare with Definition 4.16 below).*

Lemma 2.20 *Every Möbius transformation $f \neq \mathrm{id}_{\widehat{\mathbb{C}}}$ has at most two fixed points* [2].

Proof. First note that

$$f = \mathrm{id}_{\widehat{\mathbb{C}}} \Leftrightarrow c = b = 0 \text{ and } \frac{a}{d} = 1.$$

Thus

$$\begin{aligned} f(z) = \frac{az + b}{cz + d} = z &\Leftrightarrow (az + b)(\bar{c}\bar{z} + \bar{d}) = |cz + d|^2 z \\ &\Leftrightarrow 0 = cz^2 + (d - a)z - b\,. \end{aligned}$$

Now the quadratic equation has at most two solutions, and we conclude with our statement.

□

We continue with some very useful properties of Möbius transformations.

[2] A fixed point of a function f is an element z_0 with $f(z_0) = z_0$.

Remark 2.21 (Möbius transformations) (a) Lemma 2.20 provides an easy proof idea of the uniqueness in Theorem 2.12 above. That theorems states that, given two triples z_1, z_2, z_3, and w_1, w_2, w_3, of pairwise distinct complex numbers in $\widehat{\mathbb{C}}$, there exists a unique Möbius transformation f with $f(z_i) = w_i, i = 1, 2, 3$. Suppose now that there exist another Möbius transformation g, $g \neq f$, with $g(z_i) = w_i, i = 1, 2, 3$. Then the composition $(g^{-1} \circ f)$ is a Möbius transformation and has three distinct fixed points, and thus, according to Lemma 2.20, $(g^{-1} \circ f) = \mathrm{id}_{\widehat{\mathbb{C}}}$ and therefore $f = g$.

(b) $\mathrm{H}^+ := \{z \in \mathbb{C} \colon \Im(z) > 0\}$; $\mathrm{H}^- := \{z \in \mathbb{C} \colon \Im(z) < 0\}$. If a Möbius transformation f maps $\mathbb{R} \cup \{\infty\}$ onto $\mathbb{R} \cup \{\infty\}$, then, due to Theorem 2.19, either $f(\mathrm{H}^+) = \mathrm{H}^+$ and $f(\mathrm{H}^-) = \mathrm{H}^-$, or $f(\mathrm{H}^+) = \mathrm{H}^-$ and $f(\mathrm{H}^-) = \mathrm{H}^+$.

(c) Recall that a Möbius transformation f maps a line ℓ or a circle $\mathcal{C}$ onto a line ℓ_1 or a circle $\mathcal{C}_1$. Lines and circles split the complex plane into two open disjoint components, that is $\widehat{\mathbb{C}} \backslash \{\ell\}$ and $\widehat{\mathbb{C}} \backslash \{\mathcal{C}\}$ both have two components, called D', D'' with $D' \cap D'' = \varnothing$. Likewise, the image planes $\widehat{\mathbb{C}} \setminus \{\ell_1\}$ and $\widehat{\mathbb{C}} \setminus \{\mathcal{C}_1\}$ are disjoint unions of open sets D_1' and D_1'' respectively. Furthermore, because f is a homeomorphism,

$$
\begin{aligned}
f(D') \cap \ell_1 &= \varnothing = f(D') \cap \mathcal{C}_1 \\
f(D'') \cap \ell_1 &= \varnothing = f(D'') \cap \mathcal{C}_1,
\end{aligned}
$$

and $f(D')$ (respectively, $f(D'')$) is open and connected in $\widehat{\mathbb{C}} \setminus \{\ell_1\}$ or $\widehat{\mathbb{C}} \setminus \{\mathcal{C}_1\}$. From

$$
f(D') = (f(D') \cap D_1') \cup (f(D') \cap D_1''))
$$

it follows that either $f(D') \cap D_1' = \varnothing$ or $f(D') \cap D_1'' = \varnothing$. Without loss of generality, let $f(D') \cap D_1' = \varnothing$, that is, $f(D') \subset D_1''$. Our reasoning below applies in an analogous way to the remaining component D'' with the cases $f(D'') \subset D_1'$ or $f(D'') \subset D_1''$, we leave these cases as an exercise for the reader. Back to our case $f(D') \subset D_1''$. The Möbius transformation is surjective and thus $f(D'') \subset D_1'$ as otherwise we would have $D_1' \cap f(\widehat{\mathbb{C}}) = \varnothing$ because of our assumption $D_1' \cap f(D') = \varnothing$. As f is bijective, we get

$$
f(D') = D_1'' \text{ and } f(D'') = D_1',
$$

and all remaining cases follow similarly.

(d) Suppose a Möbius transformation f maps the boundary of a disc onto the boundary of that disc, i.e. $f(\partial B_R(0)) = \partial B_R(0), R > 0$. Recall that $\partial B_R(0) = \{z \in \mathbb{C} \colon |z| = R\}$ and $B_R(0) = \{z \in \mathbb{C} \colon |z| < R\}$. Then either $f(B_R(0)) = B_R(0))$, or $f(B_R(0)) = B_R(0)^{\mathrm{c}} \cup \{\infty\}$, where the complement of the disc is $B_R(0)^{\mathrm{c}} = \{z \in \mathbb{C} \colon |z| > R\}$.

(e) Consider the following example for a Möbius transformation (compare with Example sheet 1):

$$
h(z) = -\mathrm{i}\frac{z-1}{z+1}.
$$

Recall $\mathbf{\Delta} = \{z \in \mathbb{C} \colon |z| < 1\}$ and $\partial\mathbf{\Delta} = \{z \in \mathbb{C} \colon |z| = 1\}$. Then $h(\partial\mathbf{\Delta}) = \mathbb{R} \cup \{\infty\}$, because

$$
h(z) = \frac{-\mathrm{i}(z-1)(\bar{z}+1)}{(x+1)^2 + y^2} = \frac{2y}{(x+1)^2 + y^2}, \qquad \text{where we used } x^2 + y^2 = 1 \text{ on } \partial\mathbf{\Delta}.
$$

Furthermore, we have $h(1) = 0$ and $h(-1) = \infty$, and, as $h(0) = \mathrm{i}$, we get that $h(\mathbf{\Delta}) = \mathrm{H}^+$. The inverse of h is given by $h^{-1}(w) = \frac{-w+\mathrm{i}}{w+\mathrm{i}}$. Note that $h^{-1}(\mathrm{H}^+) = \mathbf{\Delta}$. Suppose now that g is another Möbius transformation with $g(\mathbf{\Delta}) = \mathbf{\Delta}$, then

$$h \circ g \circ h^{-1}(\mathrm{H}^+) = \mathrm{H}^+.$$

Conversely, each Möbius transformation f with $f(\mathrm{H}^+) = \mathrm{H}^+$ leads to the Möbius transformation $h^{-1} \circ f \circ h$ with $h^{-1} \circ f \circ h(\mathbf{\Delta}) = \mathbf{\Delta}$.

$\diamond$

3 Complex differentiation

In the following we let $D \subset \mathbb{C}$ be an open set until specified otherwise.

3.1 Definitions and elementary properties

Definition 3.1 A function $f\colon D \to \mathbb{C}$, $D \subset \mathbb{C}$ open, is complex-differentiable at $z_0 \in D$ if the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} =: f'(z_0) \tag{3.1}$$

exists. Equivalently, f is complex-differentiable at $z_0 \in D$ if there exists a function $f_1\colon D \to \mathbb{C}$ which is continuous at z_0 such that

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} = f_1(z_0). \tag{3.2}$$

The complex number $f_1(z_0) \in \mathbb{C}$ is called the derivative of f at z_0, and we write

$$\frac{\mathrm{d}f}{\mathrm{d}z}(z_0) = f'(z_0) = f_1(z_0) = \mathrm{d}f(z_0).$$

Remark 3.2 Equations (3.1) and (3.4) can be restated as: For every $\varepsilon > 0$ there exists a $\delta > 0$ such that for $|z - z_0| < \delta$ we have

$$|f(z) - f(z_0) - f'(z_0)(z - z_0)| \leq \varepsilon |z - z_0|. \tag{3.3}$$

Thus the complex-differentiable function looks locally (at z_0) like a complex affine function. $\diamond$

Proposition 3.3 *If $f\colon D \to \mathbb{C}$ is differentiable at $z_0 \in D$, then f is continuous at z_0.*

Proof. We easily get that

$$\lim_{z \to z_0} f(z) - f(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \cdot (z - z_0) = f'(z_0) \cdot 0 = 0$$

which shows continuity of f at z_0. $\square$

In the next proposition we summarise elementary properties of complex-differentiable functions.

Proposition 3.4 *If the functions $f, g\colon D \to \mathbb{C}$ are complex-differentiable at z_0, the so are the functions $(f+g)$, $f(f-g)$, $f \cdot g$, and f/g (provided that $g(z_0) \neq 0$). The derivatives are*

(i) $(f+g)'(z_0) = f'(z_0) + g'(z_0)$

(ii) $(f-g)'(z_0) = f'(z_0) - g'(z_0)$

(iii) $(f \cdot g)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0)$

(iv)
$$(f/g)'(z_0) = \frac{(f'(z_0)g(z_0) - f(z_0)g'(z_0))}{(g(z_0))^2}.$$

Proof. Exercise. □

The definition of the derivative (3.1) looks absolutely identical to the definition in the real case and on the first glance one might think that there is not much difference. As we will see, this first impression is misleading.

Example 3.5 Let us consider the function $f\colon \mathbb{C} \to \mathbb{C}$ given by $f(z) = z^2$. We know that f maps $z = re^{i\theta}$ to $r^2e^{i2\theta}$, i.e., the absolute value is squared and the argument is doubled. To see that f "locally looks like a complex affine function" at a point z_0 we write

$$z^2 - z_0^2 = (z+z_0)(z-z_0) = 2z_0(z-z_0) + (z-z_0)^2.$$

In particular, setting $f'(z_0) = 2z_0$ we get

$$|f(z) - f(z_0) - f'(z_0)(z-z_0)| = |z-z_0|^2,$$

which shows that (3.3) is satisfied for the choice $\delta = \varepsilon$. ♣

Example 3.6 Consider the mapping $f: \mathbb{C} \to \mathbb{C}$ given by $f(z) = \bar{z}$. We argue that this function f is not complex differentiable at any point $z_0 \in \mathbb{C}$. Actually, condition (3.1) implies, in particular, that for *every* sequence z_n that converges to z_0 the limits

$$\lim_{n\to\infty} \frac{f(z_n) - f(z_0)}{z_n - z_0} \tag{3.4}$$

must exist and they must all be the same. To see that this is not the case for $f(z) = \bar{z}$, we take the sequences $z_n = z_0 + \frac{1}{n}$ and $\hat{z}_n = z_0 + \frac{i}{n}$. Both sequences converge to z_0 and we have

$$\frac{f(z_n) - f(z)}{z_n - z_0} = \frac{\bar{z}_0 + \frac{1}{n} - \bar{z}_0}{\frac{1}{n}} = 1 \qquad \frac{f(\hat{z}_n) - f(z)}{\hat{z}_n - z_0} = \frac{\bar{z}_0 - \frac{i}{n} - \bar{z}_0}{\frac{i}{n}} = -1.$$

In particular, the limits do not coincide. Hence f is not complex differentiable. ♣

Similarly, the functions $\Re(z)$, $\Im(z)$, and $|z|$ are nowhere complex-differentiable in $\mathbb{C}$.

3.2 The Cauchy-Riemann equations

In order to understand the concept of complex differentiability better, we identify $\mathbb{C}$ with $\mathbb{R}^2$ by setting $z = x + \mathrm{i}y \cong \binom{x}{y}$ in the usual way, and comparing it to the concept of differentiability of functions from $\mathbb{R}^2$ to $\mathbb{R}^2$. We start with the following observations:

- For any complex number $\lambda = \lambda_1 + \mathrm{i}\lambda_2$ consider the $\mathbb{C}$-linear mapping $\mathbb{C} \ni z \mapsto \lambda z$. In the "$\mathbb{R}^2$-picture" this mapping is given by
$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \lambda_1 & -\lambda_2 \\ \lambda_2 & \lambda_1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{3.5}$$
In particular, this mapping is (real) linear from $\mathbb{R}^2$ to $\mathbb{R}^2$.

- Conversely, the $\mathbb{R}$-real linear map
$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$
can be written as complex multiplication $z \mapsto az$ for an $a \in \mathbb{C}$ if and only if $a_{11} = a_{22}$ and $a_{12} = -a_{21}$. In this case we have $a_{11} = \Re(a)$ and $a_{21} = \Im(a)$.

- If we write $\lambda = r\mathrm{e}^{\mathrm{i}\theta}$ we can rewrite the matrix in (3.5) as
$$\begin{pmatrix} \lambda_1 & -\lambda_2 \\ \lambda_2 & \lambda_1 \end{pmatrix} = r \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix},$$
which confirms once more our intuition that multiplication by the complex number $r\mathrm{e}^{\mathrm{i}\theta}$ consists of an amplification by r and a rotation around the origin by the angle θ. In particular, complex multiplication preserves angles and orientation.

Finally, recall that a mapping $f\colon D \to \mathbb{R}^2$, $\binom{x}{y} \mapsto \binom{u(x,y)}{v(x,y)}$ is differentiable in $\binom{x_0}{y_0}$ if there exists a matrix $\mathrm{D}f\binom{x_0}{y_0}$ such that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for $\left|\binom{x}{y} - \binom{x_0}{y_0}\right| < \delta$ we have
$$\left| f\begin{pmatrix} x \\ y \end{pmatrix} - f\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} - \mathrm{D}f\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \left[\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \right] \right| \le \varepsilon \left| \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \right|. \tag{3.6}$$
Furthermore, $\mathrm{D}f\colon D \to \mathbb{R}^{2\times 2}, (x_0, y_0) \mapsto \mathrm{D}f(x_0, y_0)$ is given by the *Jacobi-matrix*
$$\mathrm{D}f = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}.$$

Now we observe, comparing the definition of *real* differentiability in (3.6) with the definition of *complex* differentiability in (3.3), that they coincide up to the fact that for *real* differentiability the function f locally has to look like a *real* linear function, whereas for *complex* differentiability it locally has to look like a *complex* linear function. Then including the observations above the following important theorem reduces to a mere tautology.

Theorem 3.7 (Cauchy-Riemann equations) *Let $D \subset \mathbb{C}$ be open and $f\colon D \to \mathbb{C}$, $x + \mathrm{i}y \mapsto u(x,y) + \mathrm{i}v(x,y)$ be a function. If f is* complex-differentiable *at $z_0 = x_0 + \mathrm{i}y_0 \in D$, then the partial derivatives of the real functions u and v, $\partial u/\partial x, \partial u/\partial y, \partial v/\partial x, \partial v/\partial y$, all exist at (x_0, y_0) and the partial derivatives satisfy the* Cauchy-Riemann equations

$$\frac{\partial u}{\partial x}(x_0,y_0) = \frac{\partial v}{\partial y}(x_0,y_0) \qquad \textit{and} \qquad \frac{\partial u}{\partial y}(x_0,y_0) = -\frac{\partial v}{\partial x}(x_0,y_0). \tag{3.7}$$

Proof. We shall calculate $f'(z_0)$ in two different ways to show (3.7) (the existence of the partial derivatives follows immediately from our definitions). Let $h \in \mathbb{R}$ and consider first $z_0 + h = (x_0 + h) + \mathrm{i}y_0$ to obtain

$$f'(z_0) = \lim_{h\to 0} \frac{f(z_0+h) - f(z_0)}{h} = \frac{\partial u}{\partial x}(x_0,y_0) + \mathrm{i}\frac{\partial v}{\partial x}.$$

If we take $z_0 + h = x_0 + \mathrm{i}(y_0 + h)$, then

$$f'(z_0) = \lim_{h\to 0} \frac{f(z_0+h) - f(z_0)}{\mathrm{i}h} = \frac{\partial v}{\partial y}(x_0,y_0) - \mathrm{i}\frac{\partial u}{\partial y}(x_0,y_0).$$

□

$\mathbb{R}$-linear mappings $\mathbb{C} \to \mathbb{C}$ are given by real (2×2)-matrices, e.g., the identity $\mathrm{id}\colon \mathbb{C} \to \mathbb{C}$ by $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and the complex conjugation $f\colon \mathbb{C} \to \mathbb{C}, z \mapsto f(z) = \overline{z}$ by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and the multiplication with a complex number $a = r\mathrm{e}^{\mathrm{i}\theta}$, $f(z) = az$ by the matrix

$$\begin{pmatrix} r\cos(\theta) & -r\sin(\theta) \\ r\sin(\theta) & r\cos(\theta) \end{pmatrix} \in \mathrm{SO}(2).$$

Recall the following definition of real-differentiability of a mapping $\mathbb{R}^2 \to \mathbb{R}^2$.

Definition 3.8 (Real-differentiability) Let $U \subset \mathbb{R}^2$ be open. The function $f\colon U \to \mathbb{R}^2$ is real-differentiable at $p_0 = (x_0, y_0) \in U$ if and only if there exists a $\mathbb{R}$-linear mapping $A\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$f(p_0 + v) = f(p_0) + Av + |v|R(v), \qquad v = (v_1, v_2) \in \mathbb{R}^2, |v| = \sqrt{v_1^2 + v_2^2},$$

with

$$\lim_{v\to 0} R(v) = 0.$$

The mapping (matrix) $A = f'(p_0) = \mathrm{d}f(p_0)$ is the derivative (differential) of f at p_0 (the best linear approximation of the function f at p_0). Note that we identify here the mapping A with its matrix.

How can we characterise complex-differentiable functions? We know that $f\colon D \to \mathbb{C}$ complex-differentiable at z_0 implies that

$$\lim_{h\to 0} \frac{f(z_0+h) - f(z_0) - f'(z_0)h}{h} = 0$$

which implies in turn that f is real-differentiable (considered as a mapping in $\mathbb{R}^2$ as discussed above) having $\mathbb{C}$-linear differentials. This $\mathbb{C}$-linearity is significant for the complex-differentiability, and is the deeper reason why the Cauchy-Riemann equations (3.7) hold.

Theorem 3.9 (Complex-differentiability) *The following statements about a function $f\colon D \to \mathbb{C}$, $D \subset \mathbb{C}$ open, and $z_0 = x_0 + \mathrm{i}y_0 \in D$, are equivalent.*

(i) f is complex-differentiable at $z_0 \in D$

(ii) f is real-differentiable at (x_0, y_0) and the differential $\mathrm{d}f(z_0)\colon \mathbb{C} \to \mathbb{C}$ is complex-linear ($\mathrm{d}f(z_0) = f'(z_0)$)

(iii) f is real-differentiable at (x_0, y_0) and the Cauchy-Riemann equations (3.7)

$$u_x(x_0, y_0) = v_y(x_0, y_0), u_y(x_0, y_0) = -v_x(x_0, y_0)$$

hold.

Proof. (i)$\Leftrightarrow$ (ii) follows from our discussion above. For (ii)$\Leftrightarrow$ (iii) we denote the differential matrix

$$M := \mathrm{d}f(z_0) = \begin{pmatrix} u_x(x_0, y_0) & u_y(x_0, y_0) \\ v_x(x_0, y_0) & v_y(x_0, y_0) \end{pmatrix}.$$

Clearly, the differential is $\mathbb{R}$-linear and we are left to show that M is $\mathbb{C}$-linear if and only if the Cauchy-Riemann equations are satisfied at z_0. We shall apply M to some complex $z = x + \mathrm{i}y \in D$, and we employ the matrix notation (note that we drop the arguments from the partial derivatives at the point (x_0, y_0)),

$$\begin{aligned} M(\mathrm{i}z) &= M\begin{pmatrix} -y \\ x \end{pmatrix} = \begin{pmatrix} -u_x y & u_y x \\ -v_x y & v_y x \end{pmatrix} = (-u_x y + u_y x) + \mathrm{i}(-v_x y + v_y x) \\ &= \mathrm{i}M(z) = -(v_x x + v_y y) + \mathrm{i}(u_x x + u_y y) \\ &\Leftrightarrow u_x = v_y \text{ and } u_y = -v_x. \end{aligned}$$

□

Recall that a function $f\colon U \to \mathbb{R}^2, U \subset \mathbb{R}^2$, is real-differentiable at some point if all partial derivatives exist at that point and are continuous at that point. Thus we obtain a sufficient criterion for complex-differentiability :

Let $U \subset \mathbb{R}^2$ be the set isomorphic to $D \subset \mathbb{C}$. If $u, v\colon U \to \mathbb{R}$ are continuously differentiable real-valued functions in U, then the complex function $f\colon D \to \mathbb{C}, z \mapsto f(z) = u(x,y) + \mathrm{i}v(x,y)$ is real-differentiable. If furthermore the Cauchy-Riemann equations (3.7) hold ($u_x = v_y; u_y = -v_x$), then f is complex-differentiable.

We shall introduce some notations.

Definition 3.10 Let $D \subset \mathbb{C}$ be open. A function $f\colon D \to \mathbb{C}$ that is complex-differentiable at every point of D is called *holomorphic* or *analytic* on D. The set of holomorphic function on D is denoted $\mathcal{H}(D)$. If $f\colon \mathbb{C} \to \mathbb{C}$ is complex-differentiable everywhere on $\mathbb{C}$ it is called *entire*.

Example 3.11 Recall that it is *not* sufficient for a function $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ to be differentiable in a point, to have partial derivatives. In the same way it is not sufficient for a function to be *complex-differentiable* at a point to have partial derivatives that satisfy the Cauchy-Riemann equations. Set for example,

$$f(z) = \exp\left(- z^{-4}\right)$$

for $z \neq 0$ and $f(0) = 0$. If we restrict f to the real axis, it is perfectly continuous and we have

$$\lim_{\mathbb{R}\ni h\to 0} \frac{f(h)-f(0)}{h} = \lim_{\mathbb{R}\ni h\to 0} \frac{1}{h}\exp\left(- h^{-4}\right) = 0.$$

Similarly, if we restrict f to the purely imaginary axis we get

$$\lim_{\mathrm{i}\mathbb{R}\ni h\to 0} \frac{f(h)-f(0)}{h} = \lim_{\mathbb{R}\ni h\to 0} \frac{1}{\mathrm{i}h}\exp\left(- \frac{1}{(\mathrm{i}h)^4}\right) = \lim_{\mathbb{R}\ni h\to 0} \frac{1}{\mathrm{i}h}\exp\left(- \frac{1}{(h)^4}\right) = 0.$$

In particular, the Cauchy-Riemann equations are satisfied in 0 and it looks as if f were complex-differentiable at 0. But in fact f is not even *continuous* at 0. For example, setting $h_n = \frac{1}{n}\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}$, we have $(h_n)^4 = -\frac{1}{n^4}$ and hence $\lim_{n\to\infty} f(h_n) = \infty$.

This is not a contradiction to Theorem 3.7, because f is also not differentiable viewed as a function from $\mathbb{R}^2 \to \mathbb{R}^2$. ♣

Let f be a holomorphic (analytic) function on D with real and imaginary part u and v. For the moment let us also make the additional assumption that f is $\mathcal{C}^2$ (i.e. twice continuously differentiable). Later we will see that this is automatically the case. One immediate consequence of the Cauchy-Riemann equations is the following

$$\frac{\partial^2 u(x,y)}{\partial x^2} = \frac{\partial^2 v(x,y)}{\partial y \partial x} = -\frac{\partial^2 u(x,y)}{\partial y^2}.$$

In particular we get

$$\Delta u(x,y) := \frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0.$$

A similar calculation shows that $\Delta v(x,y) = 0$. We then say that both u and v satisfy Laplace's equation.

The real and imaginary parts of holomorphic functions are harmonic.

Suppose we have a solution $\varphi\colon U \to \mathbb{R}, U \subset \mathbb{R}^2$ open, of the Laplace equation, i.e., $\Delta\varphi(x,y) = 0$ for all $(x,y) \in U$, and suppose that the function φ is given as a real part $\Re(f)$ of some holomorphic function $f \in \mathcal{H}(D)$ where D is isomorphic to U. Then $\Im(f)$ also satisfies the Laplace equation. This leads to the following notation.

Definition 3.12 Let $U \subset \mathbb{R}^2$ be open and identify it with $D \subset \mathbb{C}$. Suppose φ is harmonic on U. The harmonic conjugate ψ od φ is a harmonic function (over U) which is given as the imaginary part $\Im(f)$ of some $f \in \mathcal{H}(D)$ such that $\varphi = \Re(f)$.

Clearly, by the Cauchy-Riemann equations the imaginary part $v = \Re(f)$ is determined uniquely up to a constant. Such a v can often be found – the *Poincaré lemma* implies, in particular, that this is always the case locally. Still the following (important) example shows that the answer is not always positive.

Example 3.13 (Complex Logarithm) The so called Newton potential

$$u(x,y) = \log\left(\sqrt{x^2+y^2}\right) \qquad \text{for} \quad \begin{pmatrix} x \\ y \end{pmatrix} \neq 0,$$

is an important example of a two-dimensional harmonic function, so it is a natural question to ask if it is the real part of an holomorphic function f on all of $\mathbb{C} \setminus \{0\}$. Given the behaviour of the complex exponential function, it is natural to suspect that the imaginary part of f should be related to argument $\arg\colon \mathbb{C} \to (-\pi,\pi)$, $z = r\mathrm{e}^{\mathrm{i}\theta} \mapsto \arg(z) = \theta$. Recall that the argument of θ is only defined uniquely up to adding integer multiples of 2π. To define arg as a (single-valued) function we restrict ourselves to $\mathbb{C}\backslash\mathbb{R}_- := \mathbb{C}\backslash\{r \in \mathbb{R}\colon r \leq 0\}$ and we fix the argument by imposing $\arg(z) \in (-\pi,\pi)$. In order to check the Cauchy-Riemann equations we write

$$\arg(x+iy) = \begin{cases} \arctan\left(\frac{y}{x}\right) & \text{if } x > 0, \\ \arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right) & \text{if } y > 0, \\ -\arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right) & \text{if } y < 0. \end{cases}$$

Then we check that

$$\frac{\partial \arg}{\partial x} = \frac{-y}{x^2+y^2} = \frac{-\partial u}{\partial y} \qquad \text{and} \qquad \frac{\partial \arg}{\partial y} = \frac{x}{x^2+y^2} = \frac{\partial u}{\partial x}.$$

Hence the Cauchy-Riemann equations are satisfied and

$$\log(x+\mathrm{i}y) := \log\left(\sqrt{x^2+y^2}\right) + \mathrm{i}\arg(x,y) \tag{3.8}$$

is indeed holomorphic on $\mathbb{C} \setminus \mathbb{R}_-$.

But the arg function cannot be extended continuously to a continuous function on all of $\mathbb{C} \setminus \{0\}$. For example, the sequences $(\mathrm{e}^{-\mathrm{i}(\pi-\frac{1}{n})})_n$ and $(\mathrm{e}^{\mathrm{i}(\pi-\frac{1}{n})})_n$ both converge to -1 but the arg function as defined above converges to π and $-\pi$ along these sequences. Of course, we could have chosen a different convention for the argument but this problem would always occur. Hence there is no holomorphic function on all of $\mathbb{C} \setminus \{0\}$ that has the Newton potential as its real part. ♣

3.3 Power series criterion

We will now prove that elementary properties of real power series carry over to the complex case.

Theorem 3.14 *For a sequence of complex numbers (a_k) consider the power series*

$$\sum_{k=0}^{\infty} a_k z^k. \tag{3.9}$$

(a) There exists a $r \in [0, \infty]$, *called the* radius of convergence, *such that for any* $z \in \mathbb{C}$ *with* $|z| < r$ *the series (3.9) converges absolutely (and locally uniformly) and for every* $r' > r$ *there exists* $z \in \mathbb{C}$ *with* $|z| = r'$ *such that (3.9) does not converge absolutely.*

(b) The series that one obtains by formally differentiating (3.9), i.e.,

$$\sum_{k=0}^{\infty} k a_k z^{k-1} \tag{3.10}$$

has the same radius of convergence.

(c) The function $f(z) = \sum_{k=0}^{\infty} a_k z^k$ *is holomorphic on*

$$B_r(0) := \{z \in \mathbb{C}\colon |z| < r\},$$

where r *denotes the radius of convergence, and the derivative is given by (3.10).*

Here and below we use the convention to set $0^0 = 1$. In particular, the power series (3.9) evaluated at 0 yields a_0. Note that this theorem does not say anything about the convergence on the circle $\{z \in \mathbb{C}\colon |z| = r\}$.

Proof. (a) We can assume that there exists a $z_0 \neq 0$ for which (3.9) converges (otherwise $r = 0$). This implies that the sequence $a_n z_0^n$ converges to 0 and in particular $|a_n z_0^n|$ is bounded, say by a constant $C > 0$. We conclude that for every z with $|z| < |z_0|$ we have

$$\sum_{k=0}^{\infty} |a_k z^k| \leq \sum_{k=0}^{\infty} |a_k z_0^k| \frac{|z|^k}{|z_0|^k} \leq C \frac{|z_0|}{|z_0| - |z|},$$

which shows the absolute convergence. The same bound actually shows that the convergence is uniform in $B_{|z|}(0)$.

This already finishes the argument, because if we define

$$r := \sup\{r \geq 0\colon \text{there exists a } z \text{ with } |z| = r \text{ such that (3.10) converges}\},$$

then the calculation above shows that (3.9) converges whenever $|z| < r$ and by definition it cannot converge when $|z| > r$.

(b) Denote by r the radius of convergence of (3.9). We can assume that $r \neq 0$. We start by showing that the radius of convergence of (3.10) is at least r, i.e we need to show that (3.10) converges absolutely if $|z| < r$.

To this end let us introduce an auxiliary $\hat{r}$ that satisfies $|z| < \hat{r} < r$. Then we get

$$\sum_{k=1}^{\infty} |k a_k z^{k-1}| \leq \frac{1}{\hat{r}} \sum_{k=1}^{\infty} k \frac{|z|^{k-1}}{\hat{r}^{k-1}} |a_k \hat{r}^k|.$$

On the one hand, by assumption the series $\sum_{k=1}^{\infty} |a_k \hat{r}^k|$ converges. On the other hand the series $k \frac{|z|^k}{\hat{r}^k}$ converges to 0, and hence it is bounded. This shows the convergence.

To show that (3.10) cannot converge absolutely if (3.9) does not converge absolutely, it suffices to write

$$\sum_{k=1}^{\infty}|ka_k z^{k-1}| \geq \frac{1}{|z|}\sum_{k=1}^{\infty}|a_k z^k|.$$

(c) To show the differentiability of f at some point $z \in B_r(0)$ we start by observing that for h with $|h| < \frac{r-|z|}{2}$, the point $z+h$ is still contained in $B_r(0)$. Furthermore, by applying (b) once more, we see that the formal second derivative

$$\sum_{k=2}^{\infty} k(k-1)a_k z^k$$

has the same radius of convergence r. Then, we estimate

$$\left|\frac{\sum_{k=0}^{\infty} a_k(z+h)^k - \sum_{k=0}^{\infty} a_k z^k}{h} - \sum_{k=1}^{\infty} k a_k z^{k-1}\right| \leq \sum_{k=2}^{\infty}\left|\frac{a_k}{h}\Big((z+h)^k - z^k - kz^{k-1}h\Big)\right|. \tag{3.11}$$

Using the binomial theorem, we can write

$$\left|(z+h)^k - z^k - kz^{k-1}h\right| = \left|h^2\sum_{j=0}^{k-2}\binom{k}{j+2}z^{k-2-j}h^j\right| \leq |h|^2 k(k-1)(|z|+|h|)^{k-2}.$$

Using this we can bound the right hand side of (3.11) by

$$|h|\sum_{k=2}^{\infty} k(k-1)|a_k|(|z|+|h|)^{k-2} \leq |h|\sum_{k=2}^{\infty} k(k-1)|a_k|\Big(|z| + \frac{r-|z|}{2}\Big)^{k-2}.$$

The last sum is finite and hence we can conclude that the whole expression converges to 0 as h goes to 0. □

Corollary 3.15 *The function $f(z) = \sum_{k=0}^{\infty} a_k(z-z_0)^k$ is infinitely often complex - differentiable on the ball $B_r(z_0)$, where r denotes the radius of convergence. The n-th derivative is given by*

$$f^{(n)}(z_0) = n!\, a_n. \tag{3.12}$$

In particular, the power series development of f is unique.

Proof. Differentiating a power series yields another power series, which can again be differentiated. This shows recursively, that f can be differentiated arbitrarily often. To see (3.12) we write

$$f^{(n)}(z) = \sum_{k=n}^{\infty} k(k-1)\,\ldots(k-n+1)\,a_k(z-z_0)^{k-n}.$$

If we evaluate the expression for $z = z_0$, all terms except for the first one drop. □

4 Complex Integration

In this section we discuss the complex line integral and give a direct proof of Cauchy's theorem without making use of the theorems of Gauss and Stokes from Vector Analysis.

4.1 The complex integral

We start by introducing some vocabulary and let $D \subset \mathbb{C}$ be an open set until otherwise specified.

Definition 4.1 A *curve* in D is a continuous function $\gamma\colon [t_0, t_1] \to D, t_0 < t_1$, and we denote the image $\Gamma = \gamma([t_0, t_1])$ also a curve in D, and often the mapping γ is called a *path* . We will call the curve $\mathcal{C}^1$ (or $\mathcal{C}^k$) if γ is (k times) continuously differentiable (in the *real* sense). We call a curve *simple* if $\gamma(t) \neq \gamma(\hat{t})$ for $t \neq \hat{t}$, and *closed* if $\gamma(t_0) = \gamma(t_1)$.

It is clear from Definition 4.1 that a curve is the image $\Gamma(\gamma)$ of some path $\gamma\colon [t_0, t_1] \to \mathbb{C}$. For instance, a given curve will have different paths. A path $\gamma\colon [t_0, t_1] \to \mathbb{C}$ is called a *parametrisation* of the curve $\Gamma = \gamma([t_0, t_1])$,

Definition 4.2 (Complex line integral) Let γ be a $\mathcal{C}^1$ curve in D, i.e.,

$$\Gamma = \gamma([t_0, t_1]) \subset D\,,$$

and let $f\colon D \to \mathbb{C}$ be continuous. Then the complex line integral is defined as

$$\int_\Gamma f(z)\,\mathrm{d}z = \int_\gamma f(z)\,\mathrm{d}z = \int_{t_0}^{t_1} f(\gamma(t))\,\dot{\gamma}(t)\,\mathrm{d}t. \tag{4.1}$$

Remark 4.3 (i) This definition extends to *piecewise $\mathcal{C}^1$ curves* in the obvious way.

(ii) Here we use the convention to write a "time derivative" with a "dot", i.e. $\dot{\gamma}(t) = \frac{\mathrm{d}\gamma}{\mathrm{d}t}(t)$.

(iii) If $f(x + \mathrm{i}y) = u(x, y) + \mathrm{i}v(x, y)$ and $\gamma(t) = x(t) + \mathrm{i}y(t)$, we can write the complex integral in real coordinates as

$$\begin{aligned}\int_\gamma f(z)\,\mathrm{d}z &= \int_{t_0}^{t_1} \Big(u(x(t), y(t))\,\dot{x}(t)\, - v(x(t), y(t))\,\dot{y}(t)\Big)\mathrm{d}t \\ &\qquad + \mathrm{i}\int_{t_0}^{t_1} \Big(u(x(t), y(t))\,\dot{y}(t)\, + v(x(t), y(t))\,\dot{x}(t)\Big)\mathrm{d}t \\ &= \int_\gamma u\mathrm{d}x - v\mathrm{d}y + \mathrm{i}\int_\gamma v\mathrm{d}x + u\mathrm{d}y.\end{aligned}$$

Note in particular, the formal calculation

$$(u + \mathrm{i}v)(\mathrm{d}x + \mathrm{i}\mathrm{d}y) = (u\,\mathrm{d}x - v\,\mathrm{d}y) + \mathrm{i}(v\,\mathrm{d}x + u\,\mathrm{d}y).$$

(iv) For $N \in \mathbb{N}$ and for $0 \leq k \leq N$ we define $t_k^N := (1 - \frac{k}{N})t_0 + \frac{k}{N}t_1$. Then it is easy to check that the following "Riemann sums"

$$\sum_{k=0}^{N-1} f(\gamma(t_k^N)) \Big(\gamma(t_{k+1}^N) - \gamma(t_k^N)\Big)$$

converge to $\int_\gamma f(z)\,\mathrm{d}z$ as $N \to \infty$.

(v) If $\gamma(t) = t$ (i.e. if the curve is an interval on the real line), then the complex line integral $\int_\gamma f(z)\,\mathrm{d}z$ coincides with the "usual, real" integral $\int_{t_0}^{t_1} f(t)\,\mathrm{d}t$.

◇

Lemma 4.4 *The line integral is invariant under reparametrisations.*

Proof. To check this, let $\varphi : [\hat{t}_0, \hat{t}_1] \to [t_0, t_1]$ be strictly increasing and $\mathcal{C}^1$. Then if we set $\hat{\gamma} := \gamma \circ \varphi$ and we get

$$\int_{\hat{\gamma}} f(z)\,\mathrm{d}z = \int_{\hat{t}_0}^{\hat{t}_1} f(\gamma(\varphi(\hat{t})))\,\partial_{\hat{t}}\gamma(\varphi(\hat{t}))\,\mathrm{d}\hat{t} = \int_{t_0}^{t_1} f(\gamma(t))\,\dot{\gamma}(t)\,\mathrm{d}t.$$

But the orientation of γ matters. If we reverse it, i.e. if we set $\gamma_-(t) = \gamma(t_0 + t_1 - t)$, then we get

$$\int_{\gamma_-} f(z)\,\mathrm{d}z = -\int_\gamma f(z)\,\mathrm{d}z.$$

In the sequel we will freely use this fact to assume that curves are parametrised in a convenient way, e.g. on the interval $[0, 1]$. □

Proposition 4.5 *If f is bounded on the image Γ of γ by a constant C, then*

$$\Big|\int_\gamma f(z)\,\mathrm{d}z\Big| \leq CL(\gamma),$$

where the length $L(\gamma)$ is defined as

$$L(\gamma) := \int_{t_0}^{t_1} |\dot{\gamma}(t)|\,\mathrm{d}t.$$

Proof. Probably, the most direct proof of this fact is to show it for the approximations defined in Remark 4.3 and to pass to the limit. But one can also use the following trick: Denote $J = \int_\gamma f(z)\,\mathrm{d}z \neq 0$ (if the intergal is zero there is nothing left to show) and write

$$\begin{aligned} 1 =& \frac{\int_\gamma f(z)\,\mathrm{d}z}{J} = \Re\bigg(\frac{\int_\gamma f(z)\,\mathrm{d}z}{J}\bigg) = \Re\bigg(\int_{t_0}^{t_1} \frac{f(\gamma(t))\,\dot{\gamma}(t)}{J}\mathrm{d}t\bigg) \\ =& \int_{t_0}^{t_1} \Re\bigg(\frac{f(\gamma(t))\,\dot{\gamma}(t)}{J}\bigg)\mathrm{d}t \leq \int_{t_0}^{t_1} \frac{C\,|\dot{\gamma}(t)|}{|J|}\,\mathrm{d}t = \frac{CL(\gamma)}{|J|}. \end{aligned}$$

□

Definition 4.6 Two $\mathcal{C}^1$ curves/paths $\gamma\colon [a,b] \to \mathbb{C}$ and $\lambda\colon [c,d] \to \mathbb{C}$ with the same image $\Gamma = \gamma([a,b]) = \lambda([c,d])$ are smoothly equivalent parametrisations of Γ if there is a smooth bijective function $\varrho\colon [a,b] \to [c,d]$ (i.e. $\varrho \in \mathcal{C}^1$) with $\varrho'(t) > 0$ for all $t \in [a,b]$ and $\varrho(a) = c, \varrho(b) = d$, and $\gamma = \lambda \circ \varrho$. This establishes an equivalence relation as the inverse function theorem implies that ϱ^{-1} exists with $\varrho^{-1} \in \mathcal{C}^1$ and positive derivative.

Proposition 4.7 *If two $\mathcal{C}^1$ curves/paths $\gamma\colon [a,b] \to \mathbb{C}$ and $\lambda\colon [c,d] \to \mathbb{C}$ are smoothly equivalent parametrisations of the same curve Γ, then $L(\gamma) = L(\lambda)$.*

Proof. We have that $\gamma = \lambda \circ \varrho$ for $\varrho\colon [a,b] \to [c,d]$ with positive derivative, i.e. $\varrho'(t) = |\varrho'(t)|$. We change the integration variable according to $s = \varrho(t)$, and obtain $\mathrm{d}s = \varrho'(t)\mathrm{d}t$:

$$\begin{aligned} L(\gamma) &= \int_a^b |\gamma'(t)|\,\mathrm{d}t = \int_a^b |(\lambda \circ \varrho)'(t)|\,\mathrm{d}t = \int_a^b |\lambda'(\varrho(t))||\varrho'(t)|\,\mathrm{d}t \\ &= \int_a^b |\lambda'(\varrho(t))|\varrho'(t)\,\mathrm{d}t = \int_c^d |\lambda'(s)|\,\mathrm{d}s. \end{aligned}$$

□

Example 4.8 For two points $w_1 \neq w_2$ let γ be the straight line from w_1 to w_2. Of course, there are different ways to parametrise this line, but according to Lemma 4.4 the integral along γ does not depend on this choice. Thus we can choose

$$\gamma(t) = (1-t)w_1 + tw_2 \quad \text{for} \quad t \in [0,1].$$

Then the integral of a function f along γ is given by

$$\int_\gamma f(z)\,\mathrm{d}z = \int_0^1 f((1-t)w_1 + tw_2)\,(w_2 - w_1)\,\mathrm{d}t.$$

In the particular case where $f(z) = z^n$ for $n \in \mathbb{N}$ this yields

$$\int_\gamma z^n\,\mathrm{d}z = \int_0^1 \left((1-t)w_1 + tw_2\right)^n (w_2 - w_1)\,\mathrm{d}t = \frac{w_2^{n+1} - w_1^{n+1}}{n+1}.$$

♣

The previous example already suggests that a chain rule should hold for complex line integrals. This is indeed the case.

Lemma 4.9 *Suppose that $f\colon D \to \mathbb{C}$ is holomorphic on D and a $\mathcal{C}^1$ curve γ lies in D, i.e., $\Gamma = \gamma([a,b]) \subset D$. Then*

$$\int_\gamma f'(z)\,\mathrm{d}z = \int_a^b f'(\gamma(t))\,\dot\gamma(t)\,\mathrm{d}t = \int_a^b \frac{\mathrm{d}}{\mathrm{d}t} f(\gamma(t))\,\mathrm{d}t = f(b) - f(a).$$

In particular, if γ is a closed curve, we get that $\int_\gamma f'(z)\,\mathrm{d}z = 0$.

Example 4.10 (Arc intergals) We will now calculate integrals along the arc $\gamma\colon [0,\hat{\theta}] \to \mathbb{C}$, $\gamma(\theta) = r\,\mathrm{e}^{\mathrm{i}\theta}$ for some $r > 0$. According to the Definition 4.2 of the complex line integral, we get for every continuous function f that

$$\int_\gamma f(z)\,\mathrm{d}z = \int_0^{\hat{\theta}} f(r\mathrm{e}^{\mathrm{i}\theta})\,(\mathrm{i}r)\,\mathrm{e}^{\mathrm{i}\theta}\,d\theta.$$

Let us evaluate this in the special case $f(z) = z^n$ for $n \in \mathbb{Z}$. If $n \neq -1$ we get

$$\int_\gamma z^n\,\mathrm{d}z = \int_0^{\hat{\theta}} r^n \mathrm{e}^{\mathrm{i}n\theta}\,(\mathrm{i}r)\,\mathrm{e}^{\mathrm{i}\theta}\,d\theta = \mathrm{i}\,r^{n+1}\int_0^{\hat{\theta}} \mathrm{e}^{\mathrm{i}(n+1)\theta}\,\mathrm{d}\theta = \frac{r^{n+1}}{n+1}(\mathrm{e}^{\mathrm{i}(n+1)\hat{\theta}} - 1),$$

which is of course consistent with the chain rule from the previous example. In the special case $n = -1$ we get

$$\int_\gamma \frac{1}{z}\,\mathrm{d}z = \int_0^{\hat{\theta}} \mathrm{i}\,d\theta = \mathrm{i}\hat{\theta}.$$

In particular, if $\hat{\theta} = 2\pi$, i.e. if we integrate over the whole circle, we get

$$\int_\gamma \frac{1}{z}\,\mathrm{d}z = 2\pi\mathrm{i}. \tag{4.2}$$

This observation is closely related to the complex logarithm that we had defined above in Example 3.13. Actually, it can be checked like in the real case that the complex logarithm $\log\colon \mathbb{C}\setminus\mathbb{R}_- \to \mathbb{C}$ defined in (3.8) has derivative $\frac{1}{z}$. Then, if we consider for $\varepsilon > 0$ the curves $\gamma_\varepsilon\colon [-\pi+\varepsilon, \pi-\varepsilon]$, $\theta \mapsto \mathrm{e}^{\mathrm{i}\theta}$ and use the chain rule, we get

$$\int_{\gamma_\varepsilon} \frac{1}{z}\,\mathrm{d}z = \log\left(\mathrm{e}^{\mathrm{i}(\pi-\varepsilon)}\right) - \log\left(\mathrm{e}^{-\mathrm{i}(\pi-\varepsilon)}\right).$$

We see that the value $2\pi\mathrm{i}$ from (4.2) exactly corresponds to the discontinuity of the arg function on the half-line $\mathbb{R}_-$. ♣

Proposition 4.11 (Fundamental theorem of calculus) *Suppose that $\gamma\colon [a,b] \to \mathbb{C}$ is a $\mathcal{C}^1$ curve (path) with $\Gamma = \gamma([a,b]) \subset D$ for some open $D \subset \mathbb{C}$ and that a function $F\colon D \to \mathbb{C}$ is complex-differentiable on D with F' continuous at each point of Γ. Then*

$$\int_\gamma F'(z)\,\mathrm{d}z = F(\gamma(b)) - F(\gamma(a)).$$

Proof. $F \circ \gamma$ is differentiable on $[a,b]$ with derivative

$$(F\circ\gamma)'(t) = F'(\gamma(t))\dot{\gamma}(t)$$

by the chain rule. Then (applying the real Fundamental theorem of calculus for the real and imaginary parts respectively)

$$\begin{aligned}\int_\gamma F(z)\,\mathrm{d}z &= \int_a^b F'(\gamma(t))\dot{\gamma}(t)\,\mathrm{d}t = \int_a^b (F\circ\gamma)'(t)\,\mathrm{d}t \\ &= \int_a^b (\Re(F\circ\gamma)'(t) + \mathrm{i}\Im(F\circ\gamma)'(t))\,\mathrm{d}t \\ &= F(\gamma(b)) - F(\gamma(a)).\end{aligned}$$

□

4.2 Cauchy's theorem

Cauchy's theorem is the centrepiece of complex analysis. It states that $\int_\gamma f(z)\,\mathrm{d}z = 0$ under appropriate conditions on the function $f \in \mathcal{H}(D)$, $D \subset \mathbb{C}$, the closed curve (path) γ and the set D. We start with a simple version of Cauchy's theorem, and then gradually improve the statements in the course of the lecture. The following simple version is appealing as its proof is robust and easy to remember.

Theorem 4.12 (Goursat's theorem - Cauchy's theorem) *Let $D \subset \mathbb{C}$ be an open set and $Q \subset D$ be a rectangle such that $Q \cup \partial Q \subset D$ where ∂Q denotes the boundary line of the rectangle, and let γ be a piecewise C^1 parametrisation of ∂Q, i.e. $\partial Q = \gamma([a,b])$, which surrounds the rectangle Q in mathematical positive direction (counterclockwise), see Figure 4. Then, for every $f \in \mathcal{H}(D)$ we have*

$$\int_\gamma f(z)\,\mathrm{d}z = 0.$$

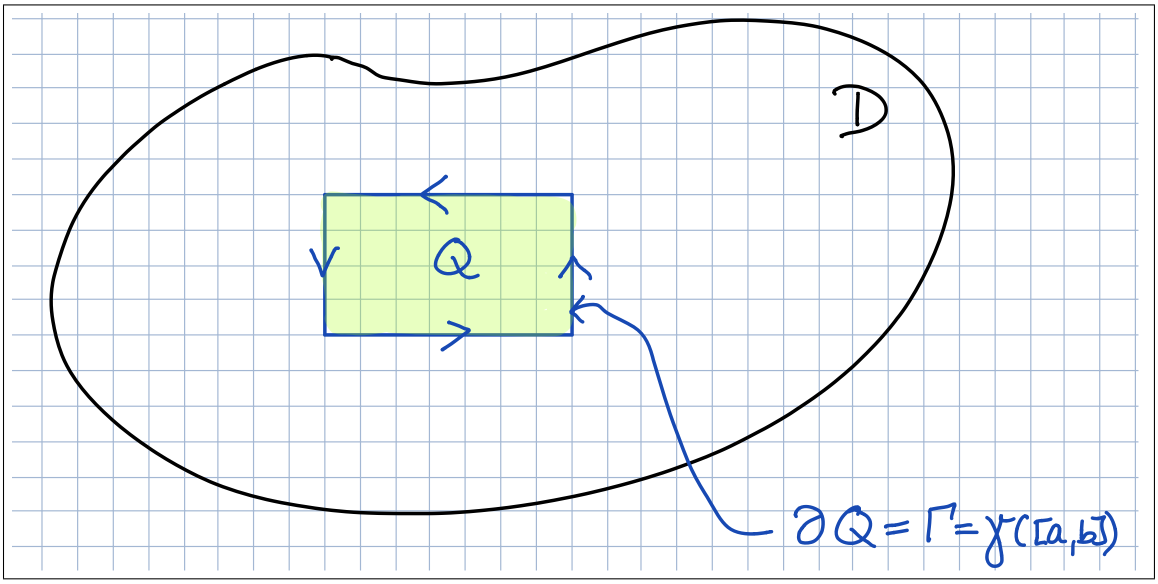

Figure 4:

Proof. Strategy: we first prove the statement for two special cases of the function f; then in a third step we use the fact that f is holomorphic on D to obtain the statement for every $f \in \mathcal{H}(D)$.

Step 1: Let $f(z) = 1$ for all $z \in D$. Then for any C^1 curve $\gamma\colon [a,b] \to D$,

$$\int_\gamma \mathrm{d}z = \int_a^b \dot\gamma(t)\,\mathrm{d}t = \gamma(b) - \gamma(a).$$

Suppose now that $\gamma\colon [a,b] \to D$ is a parametrisation of the boundary of the rectangle, that is, $\gamma([a,b]) = \partial Q$, and denote the four corners of the rectangle Q in counterclockwise direction by $z_1, \ldots, z_4$, respectively. Then γ is actually a joint of four straight lines γ_i

connecting the corners, e.g., $\gamma_1(t) = (1-t)z_1 - tz_2, t \in [0,1]$, connecting z_1 to z_2 with $\dot{\gamma}_1(t) = z_2 - z_1$.

$$\int_\gamma \mathrm{d}z = \sum_{i=1}^4 \int_{\gamma_i} \mathrm{d}z = z_2 - z_1 + z_3 - z_2 + z_4 - z_3 + z_1 - z_4 = 0,$$

see Figure 5.

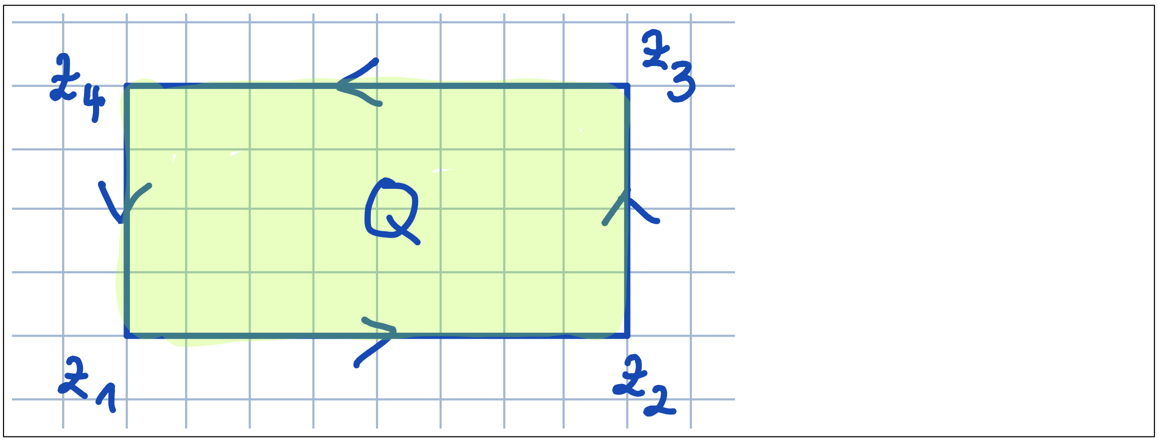

Figure 5:

Step 2: Let $f(z) = z$ for all $z \in D$. For any $\mathcal{C}^1$ curve $\gamma\colon [a,b] \to D$,

$$\int_a^b \gamma(t)\dot{\gamma}(t)\,\mathrm{d}t = \frac{1}{2}\int_a^b \frac{\mathrm{d}}{\mathrm{d}t}(\gamma(t))^2\,\mathrm{d}t = \frac{1}{2}[\gamma(b)^2 - \gamma(a)^2],$$

and thus we obtain

$$\int_\gamma f(z)\,\mathrm{d}z = 0.$$

We have thus shown the statement in Theorem 4.12 for all linear functions $f \in \mathcal{H}(D)$ of the form $f(z) = c_0 + c_1 z, z \in D, c_0, c_1 \in \mathbb{C}$.

Step 3: Let $f \in \mathcal{H}(D)$. Divide the rectangle Q into four equal size rectangles $Q_1, \ldots, Q_4$ (see Figure 6), and denote Q_1 the one of the four rectangles for which the integral

$$\int_{\partial Q_i} f(z)\,\mathrm{d}z$$

takes its maximum value. Let γ_1 the $\mathcal{C}^1$ curve with $\partial Q_1 = \gamma_1([a_1, b_1])$. Then

$$|\int_\gamma f(z)\,\mathrm{d}z| \leq 4|\int_{\gamma_1} f(z)\,\mathrm{d}z|.$$

Continue splitting the rectangle which maximises the integral, e.g., split the chosen Q_1 into another batch of four equal size rectangles and choose the one among these four which maximises the integral, and denote the chosen one Q_2, etc. We thus obtain a sequence of nested rectangles (e.g., see Figure 7 for the one in the fourth generation)

$$Q \supset Q_1 \supset Q_2 \supset Q_3 \cdots$$

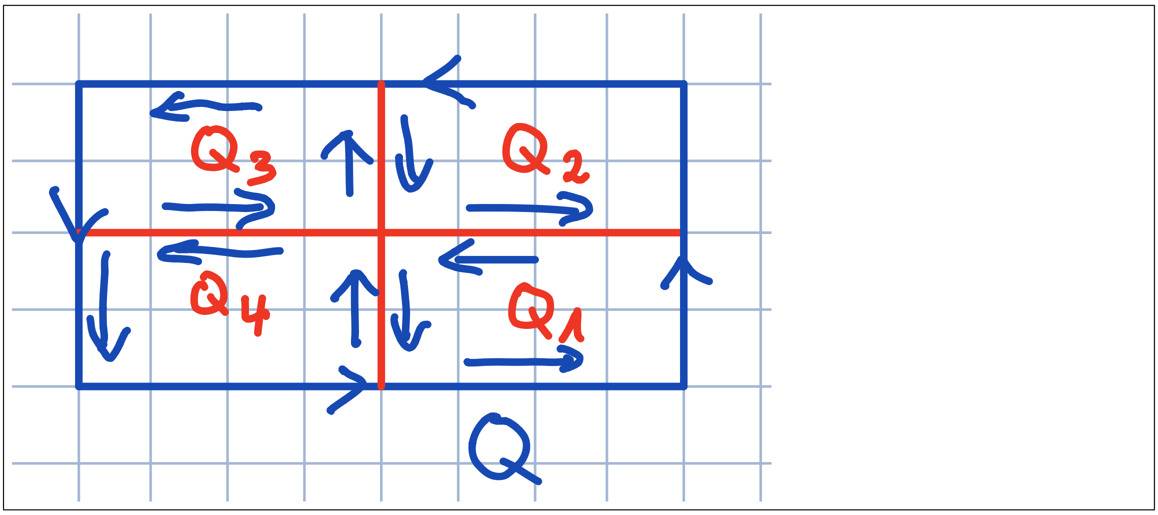

Figure 6:

each boundary curve ∂Q_i with a $\mathcal{C}^1$ parameterisation γ_i. We obtain the following estimate for our sequence of rectangles:

$$|\int_\gamma f(z)\,\mathrm{d}z| \leq 4^n |\int_{\gamma_n} f(z)\,\mathrm{d}z|,$$

and

$$\{z_0\} = \bigcap_{n\in\mathbb{N}} Q_n, \qquad z_0 \in D.$$

We now use the assumption that f is holomorphic on D. Namely, for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(z) - f(z_0) - (z - z_0)f'(z_0)| \leq \varepsilon |z - z_0|$$

for all $|z - z_0| < \delta$. Suppose now that ϱ is the diameter of the rectangle Q and that ℓ is the circumference of the rectangle Q. Then $2^{-n}\varrho$ is the diameter of Q_n and $2^{-n}\ell$ is the circumference of Q_n. For every $z \in Q_n \cup \partial Q_n$ we have the estimate $|z - z_0| \leq \varepsilon 2^{-n}\varrho$ and thus

$$|f(z) - f(z_0) - f'(z_0)(z - z_0)| \leq \varepsilon 2^{-n}\varrho.$$

Now we use Step 1 and Step 2 above to see that

$$\int_{\gamma_n} \big(- f(z_0) - f'(z_0)(z - z_0)\big)\,\mathrm{d}z = 0.$$

Henceforth we obtain the estimate

$$\Big|\int_\gamma f(z)\,\mathrm{d}z\Big| \leq 4^n \Big|\int_{\gamma_n} f(z)\,\mathrm{d}z\Big| = \Big|\int_{\gamma_n} \big(f(z) - f(z_0) - f'(z_0)(z - z_0)\big)\,\mathrm{d}z\Big|$$
$$\leq 4^n 2^{-n} 2^{-n} \varepsilon\varrho\ell = \varepsilon\varrho\ell.$$

As $\varepsilon > 0$ is arbitrary, we finally obtain $\int_\gamma f(z)\,\mathrm{d}z = 0$.

□

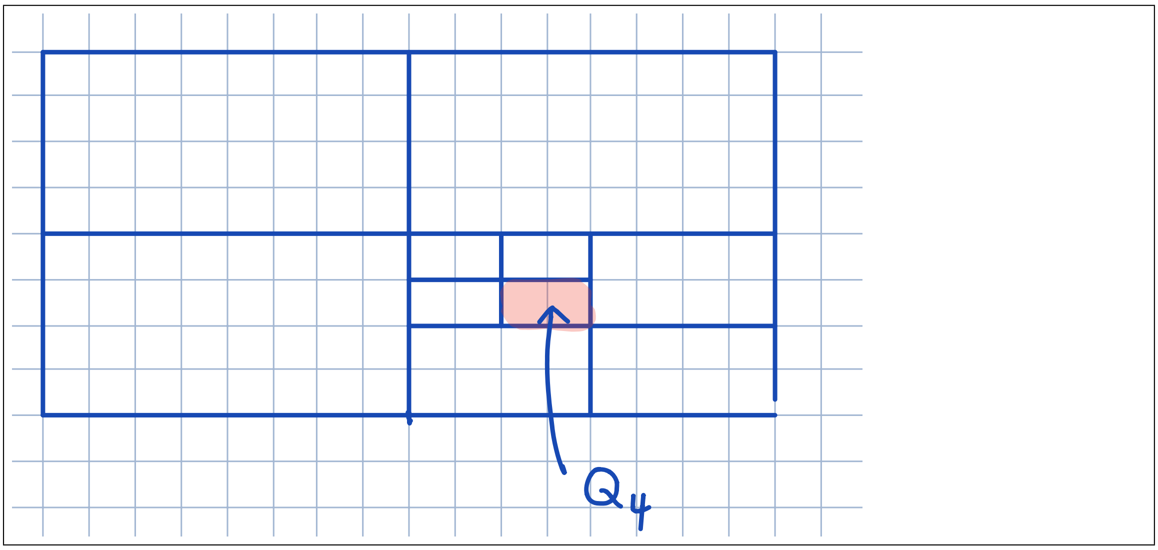

Figure 7:

Corollary 4.13 (Cauchy's theorem for images of rectangles) *Let $D \subset \mathbb{C}$ be open, $f \in \mathcal{H}(D)$, and Q be a rectangle with $\overline{Q} := Q \cup \partial Q \subset D$ and a $\mathcal{C}^1$ mapping $\varphi \colon \overline{Q} \to D$ (image $\varphi(\overline{Q}) \subset D$), see Figure 8. Let γ be a piecewise $\mathcal{C}^1$ curve which surrounds Q exactly once. Then*

$$\int_{\varphi \circ \gamma} f(z)\,\mathrm{d}z = 0.$$

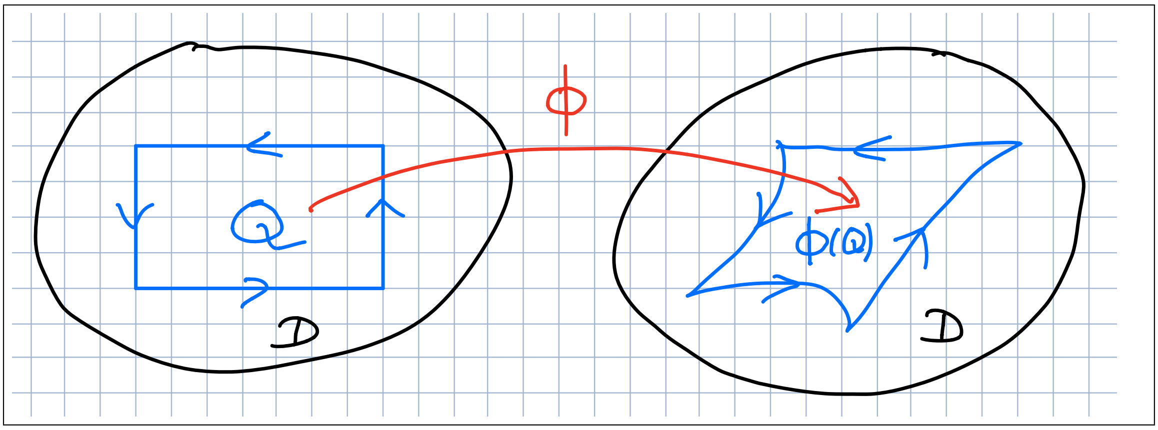

Figure 8:

Proof. This is a straightforward generalisation of our proof of Theorem 4.12 above. As φ is $\mathcal{C}^1$ and $\overline{Q}$ is a compact set, we know that $\dot{\varphi}(\overline{Q})$ is also compact and that there exists a constant $C > 0$ such that $|\dot{\varphi}(\overline{Q})| \le C$. Therefore we only get a different estimate for the diameter (and circumference) of $\varphi(Q_n)$ which is bounded by $\varrho C 2^{-n}$ as in Step 3 of the proof above. The remaining part is left as an exercise for the reader. □

Example 4.14 Suppose that $\alpha, \beta \colon [a, b] \to D$ are two $\mathcal{C}^1$ curves (paths) with distinct initial and terminal point (i.e. $\alpha(a) \neq \beta(a)$ and $\alpha(b) \neq \beta(b)$) such that all straight lines connecting any two points on $\alpha([a, b])$ and $\beta([a, b])$ are contained in D, i.e.

$$\{(1-\tau)\alpha(t) + \tau\beta(t) \colon \tau \in [0, 1], t \in [a, b]\} \subset D.$$

We define two straight lines, h_a, h_b connecting the initial and terminal points, i.e.,

$$h_a \colon [0, 1] \to D, \tau \mapsto h_a(\tau) = (1-\tau)\alpha(a) + \tau\beta(a)$$

and

$$h_b \colon [0, 1] \to D, \tau \mapsto h_b(\tau) = (1-\tau)\alpha(b) + \tau\beta(b).$$

Then

$$\int_{h_a} f(z)\,\mathrm{d}z + \int_{\beta} f(z)\,\mathrm{d}z - \int_{h_b} f(z)\,\mathrm{d}z - \int_{\alpha} f(z)\,\mathrm{d}z = 0.$$

This follows with Corollary 4.13 and the mapping

$$\varphi \colon [a, b] \times [0, 1] \to D, \varphi(t, \tau) = (1-\tau)\alpha(t) + \tau\beta(t).$$

♣

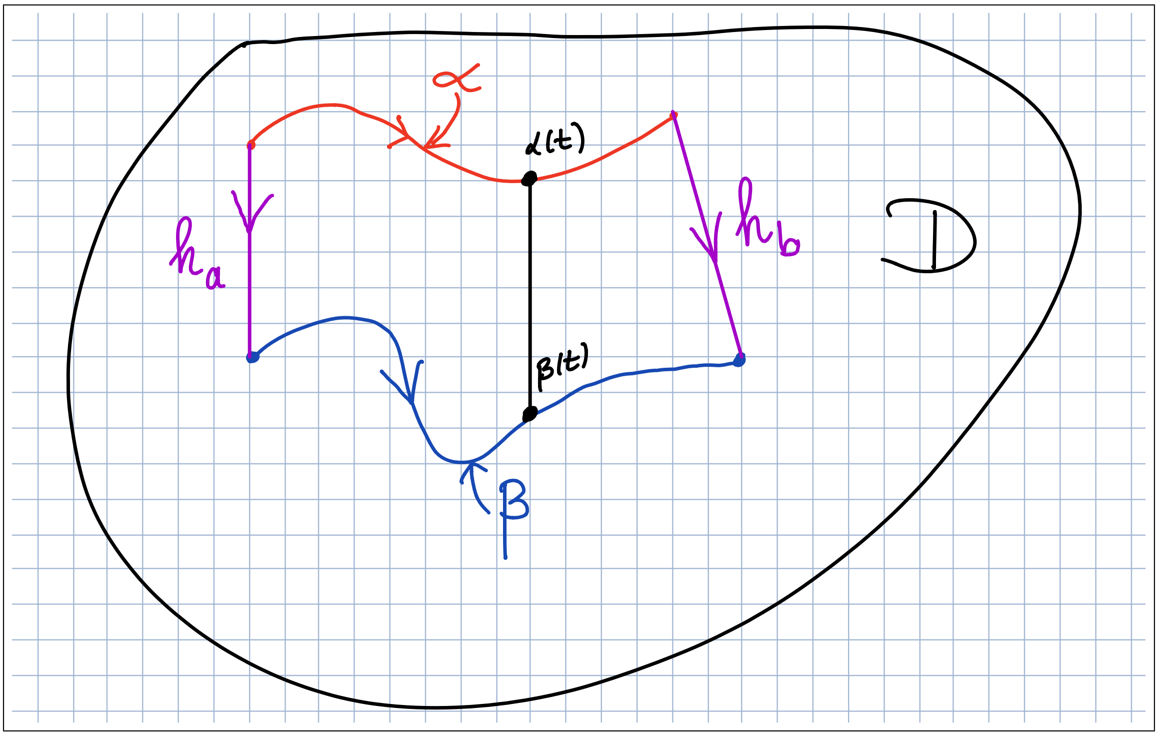

Figure 9:

We discuss a few more examples.

Example 4.15 (i) Let $\tau \subset D$ be a triangle wholly contained in D. Suppose that γ is a $\mathcal{C}^1$ parametrisation of $\partial\tau$ which surrounds the triangle once, see Figure 10. Then

$$\int_{\tau} f(z)\,\mathrm{d}z = 0.$$

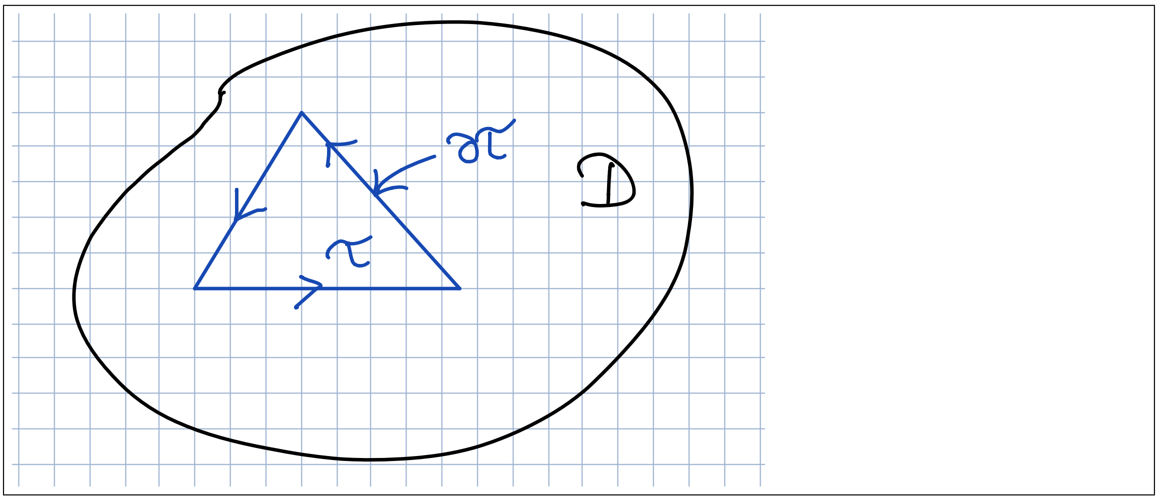

Figure 10:

(ii) Suppose that $\alpha, \beta \colon [a,b] \to D$ are two $\mathcal{C}^1$ curves (paths) with $\alpha(a) = \alpha(b)$ and $\beta(a) = \beta)b)$, see Figure 11. Denote $h \colon [0,1] \to D$ the straight line connecting $\alpha(a)$ and $\beta(a)$. Then

$$\int_\alpha f(z)\,\mathrm{d}z + \int_h f(z)\,\mathrm{d}z - \int_\beta f(z)\,\mathrm{d}z - \int_h f(z)\,\mathrm{d}z = 0.$$

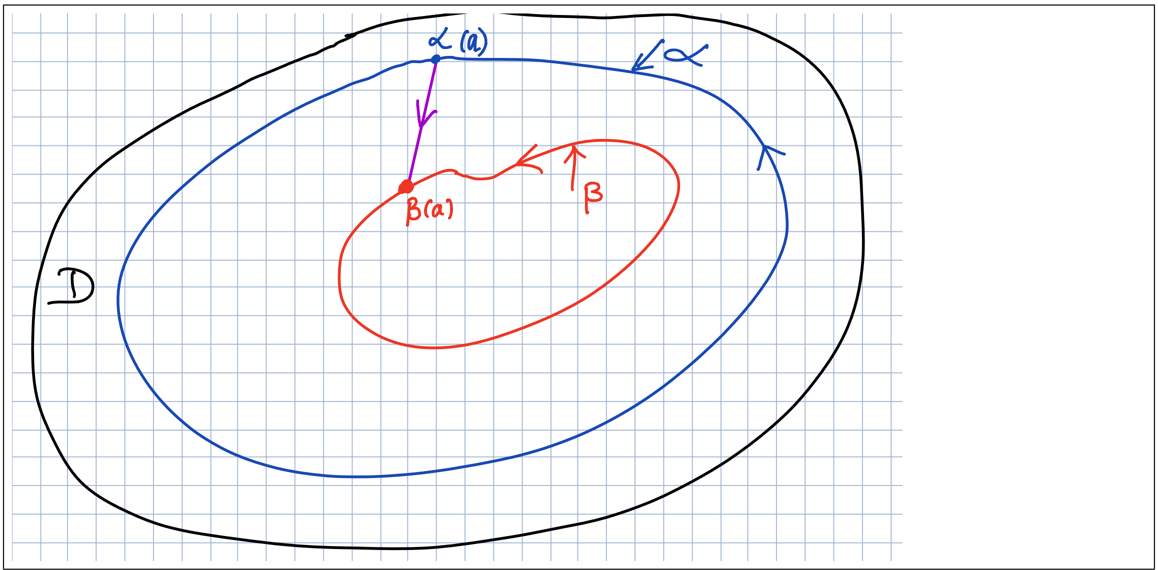

Figure 11:

(iii) Suppose that $f \colon D \to \mathbb{C}$ is holomorphic on

$$D \supset \{z \in \mathbb{C} \colon r \leq |z - z_0| \leq R\}, 0 < r < R.$$

Then

$$\int_{\partial B_r(z_0)} f(z)\,\mathrm{d}z = \int_{\partial B_R(z_0)} f(z)\,\mathrm{d}z,$$

where $B_r(z_0) := \{z \in \mathbb{C}\colon |z - z_0| < r\}$ and $\partial B_r(z_0) = \{z \in \mathbb{C}\colon |z - z_0| = r\}$. For $r = 0$ we obtain Cauchy's theorem for discs.

(iv) Let $\alpha, \beta\colon [a, b] \to D$ be two $\mathcal{C}^1$ curves (paths) with $\alpha(a) = \beta(a)$ and $\alpha(b) = \beta(b)$ with $\alpha(t) \neq \beta(t)$ for all $t \in (a, b)$. Then

$$\int_\alpha f(z)\,\mathrm{d}z = \int_\beta f(z)\,\mathrm{d}z.$$

♣

As a next step we want to allow for more general curves γ. We start by the following definitions.

Definition 4.16 (i) Let $D \subset \mathbb{C}$. We say that D is *convex* if for any pair of points $a, b \in D$ the straight line $\overline{[a, b]} := \gamma_{a,b}([0, 1]) \subset D$ connecting a and b is wholly contained in D. Here, $\gamma_{a,b}\colon [0, 1] \to \mathbb{C}, t \mapsto \gamma_{a,b}(t) = (1 - t)a + tb$ is the $\mathcal{C}^1$ parametrisation of the straight line connecting a and b.

(ii) $D \subset \mathbb{C}$ is *polygonally connected* if, given any two points $a, b \in D$, there is a polygonal path (curve) from a to b lying wholly in D. A polygonal path connecting a and b means that there exists $n \in \mathbb{N}$ and $z_i \in D, i = 0, 1, \ldots, n$, such that $z_0 = a, z_n = b$ and the polygonal path is the union of the straight lines $\overline{[z_i, z_{i+1}]}, i = 0, 1, \ldots n - 1$ with $\overline{[z_i, z_{i+1}]} \subset D$ for $i = 0, 1, \ldots, n - 1$.

(iii) $D \subset \mathbb{C}$ is *connected* if it cannot be expressed as the union of non-empty open sets D_1 and D_2 with $D_1 \cap D_2 = \varnothing$. A region is a non-empty open connected subset of $\mathbb{C}$.

(iv) A set $D \subset \mathbb{C}$ is *star shaped* if there exists a $z_\star \in D$ such that for all $z \in D$ the whole line connecting $z_\star$ and z is fully contained in D. More precisely, $(1 - t)z + tz_\star \in D$ for all $t \in [0, 1]$. Every $z_\star \in D$ with this property is called a midpoint of D.

Remark 4.17 Every $z_\star \in D$ with this property is called a *midpoint* of D. Of course, the midpoint needs not be unique. If D is convex, then it is star shaped and *every* $z \in D$ is a midpoint. Every star shaped set is trivially polygonally connected. ◇

Theorem 4.18 *Let D be a non-empty open subset of $\mathbb{C}$. Then D is a region if and only if D is polygonally connected. In particular, any non-empty open convex set is a region.*

Proof. Support class in week 4. A detailed proof is in [Pri03], page 38-39. □

Theorem 4.19 (Cauchy's theorem for star shaped domains) *Let $D \subset \mathbb{C}$ be open and star shaped and let $f\colon D \to \mathbb{C}$ be holomorphic on D. Then, for every closed piecewise $\mathcal{C}^1$ curve $\gamma\colon [a, b] \to D$, we have*

$$\int_\gamma f(z)\,\mathrm{d}z = 0.$$

Proof. We will prove that f has an antiderivative F. This suffices by the chain rule.

Let $z_\star$ be a midpoint of D and for any $z \in D$ let α_z be the the line that connects $z_\star$ to z, i.e. we define

$$\alpha_z\colon [0,1] \to D \quad \alpha_z(t) = tz + (1-t)z_\star.$$

By assumption the image of α_z is fully contained in D. We set $F(z) := \int_{\alpha_z} f(\xi)\,\mathrm{d}\xi$. We claim that F is complex-differentiable and that for every z_0 in D we have $F'(z_0) = f(z_0)$.

In order to see that, fix some point $z_0 \in D$. As D is open, there exists an $r > 0$ such that the open ball $B_r(z_0) := \{z \in \mathbb{C}\colon |z - z_0| < r\}$ with its boundary is contained in D. Then for $z \neq z_0$ in $B_r(z_0)$ we can define the curve $\beta_z(t) = (1-t)z_0 + tz$ for $t \in [0,1]$. By assumption the image of the line β_z is fully contained in D. It is then easy to check, that the fact that D is star shaped implies that the whole triangle with corners z_0, $z_\star$, and z is contained in D.

We get

$$\frac{F(z) - F(z_0)}{z - z_0} = \frac{1}{z - z_0}\left(\int_{\alpha_z} f(\xi)\,\mathrm{d}\xi - \int_{\alpha_{z_0}} f(\xi)\mathrm{d}\xi\right) = \frac{1}{z - z_0}\int_{\beta_z} f(\xi)\,\mathrm{d}\xi.$$

In the second equality we have made use of Goursat's theorem, Theorem 4.12. Plugging into the definition of the complex line integral, we get

$$\frac{1}{z - z_0}\int_{\beta_z} f(\xi)\,\mathrm{d}\xi = \frac{1}{z - z_0}\int_0^1 f((1-t)z_0 + tz)\,(z - z_0)\,\mathrm{d}t = \int_0^1 f((1-t)z_0 + tz)\,\mathrm{d}t.$$

Using the continuity we can conclude that this expression converges to $f(z_0)$ as $z \to z_0$ which proves the claim. □

Definition 4.20 (i) A *circline* curve (path) is a finite join of straight lines (line segments) and circular arcs. A *circular arc* is the image of some $\gamma\colon [\theta_1, \theta_1] \to \mathbb{C}, \gamma(t) = a + r\mathrm{e}^{\mathrm{i}t}, a \in \mathbb{C}, r > 0, [\theta_1, \theta_2] \subset [0, 2\pi]$.

(ii) A *contour* is a simple closed circline path, i.e. there is a parametrisation $\gamma\colon [a,b] \to \mathbb{C}$ with $\gamma(a) = \gamma(b)$ and $\gamma(t) \neq \gamma(s)$ for all $s, t \in (a,b], t \neq s$ such that $\Gamma = \gamma([a,b])$ consists of finitely many line segments and circular arcs and does not cross itself.

(iii) A contour γ is *positively oriented* if, as t increases, $\gamma(t)$ moves anticlockwise round any point inside it (a more formal definition in terms of index, is given in Section 7).

Theorem 4.21 (Jordan curve theorem for contours) *Let γ be a contour. Then the complement of the image $\Gamma = \gamma([a,b])$ is of the form*

$$\mathrm{I}(\gamma) \cup \mathbf{O}(\gamma),$$

where $\mathrm{I}(\gamma)$ and $\mathbf{O}(\gamma)$ are disjoint connected open sets, $\mathrm{I}(\gamma)$ (the inside of γ) is bounded (i.e. there exists $R > 0$ such that $\mathrm{I}(\gamma) \subset B_R(0)$) and $\mathbf{O}(\gamma)$ (the outside of γ) is unbounded.

Proof. A detailed proof of this theorem goes beyond the remit of the module and we refer the interested reader to [Pri03], page 53, and references therein. □

Remark 4.22 The Jordan curve theorem implies immediately the following contour version of Cauchy's theorem: Suppose that f is holomorphic inside and on a contour γ. Then $\int_\gamma f(z)\,\mathrm{d}z = 0$. $\diamond$

Theorem 4.23 (Deformation theorem) *(a) Suppose that $\gamma\colon [a,b] \to \mathbb{C}$ is a positively oriented contour and that $\overline{B_r(z_0)} \subset \mathrm{I}(\gamma)$, $z_0 \in \mathbb{C}, r > 0$. Let f be holomorphic inside and on $\Gamma = \gamma([a,b])$, i.e. $f \in \mathcal{H}(\Gamma \cup \mathrm{I}(\gamma))$. Then*

$$\int_\gamma f(z)\,\mathrm{d}z = \int_{\partial B_r(z_0)} f(z)\,\mathrm{d}z.$$

(b) Suppose that $\gamma\colon [a,b] \to \mathbb{C}$ and $\widetilde{\gamma}\colon [\tilde{a},\tilde{b}] \to \mathbb{C}$ are positively oriented contours such that $\widetilde{\gamma}$ lies inside γ, that is, $\widetilde{\Gamma} \cup \mathrm{I}(\widetilde{\gamma}) \subset \mathrm{I}(\gamma)$, $\widetilde{\Gamma} = \widetilde{\gamma}([\tilde{a},\tilde{b}])$. Let $f \in \mathcal{H}(\mathrm{I}(\gamma) \cup \Gamma)$, $\Gamma = \gamma([a,b])$. Then

$$\int_\gamma f(z)\,\mathrm{d}z = \int_{\widetilde{\gamma}} f(z)\,\mathrm{d}z.$$

(c) Suppose that $\gamma_i\colon [a_i,b_i] \to \mathbb{C}, i = 1,2$, are circline paths with $\gamma_1(a_1) = \gamma_2(a_2)$ and $\gamma_1(b_1) = \gamma_2(b_2)$ (common initial and terminal point). Let $\gamma = \gamma_1 \cup (-\gamma_2)$ be the joint of γ_1 and the inverse of γ_2, and suppose that γ is a closed simple curve. Let f be holomorphic inside and on γ, i.e. $f \in \mathcal{H}(\mathrm{I}(\gamma) \cup \Gamma)$. Then

$$\int_{\gamma_1} f(z)\,\mathrm{d}z = \int_{\gamma_2} f(z)\,\mathrm{d}z.$$

Proof. (a) Denote $c := \gamma(a)$ the initial point. For some $\delta > 0$ we have $B_\delta(c) \cap \mathrm{I}(\gamma) \neq \varnothing$. Pick some $d \in \mathrm{I}(\gamma) \cap B_\delta(c)$. The interior $\mathrm{I}(\gamma)$ is by Theorem 4.21 open and connected, and thus by Theorem 4.18 polygonally connected, and thus there is a polygonal path $\gamma_1\colon [a_1,b_1] \to \mathbb{C}$ joining the point d with the centre point a of the disc $B_r(a)$, we assume without loss of generality that γ_1 is simple. There is a point $b := \gamma_1(T) \in \gamma_1([a_1,b_1])$ such that $|\gamma_1(t) - a| > r$ for all $t \in [a_1,T)$, see Figure 12. We denote $\tilde{\gamma}_1$ the polygonal path connecting d with b and define $\gamma_2 := \overline{[d,c]} \cup \tilde{\gamma}_1$ the joint of the straight line segment connecting c and d with $\tilde{\gamma}_1$. Furthermore, denote γ_r^- the clockwise parametrisation of the circline $\partial B_r(a)$ with $\gamma_r^-(0) = b$. Then the joint

$$\widehat{\gamma} := \gamma \cup \gamma_2 \cup \gamma_r^- \cup (-\gamma_2)$$

is a contour, and thus

$$\int_{\widehat{\gamma}} f(z)\,\mathrm{d}z = 0 = \int_\gamma f(z)\,\mathrm{d}z - \int_{\partial B_r(a)} f(z)\,\mathrm{d}z.$$

(b) Simply apply statement (a) for $B_r(a) \subset \mathrm{I}(\widetilde{\gamma})$.

(c) Combine Theorem 4.19 and Remark 4.22. $\square$

The next lemma is straightforward from our discussion above.

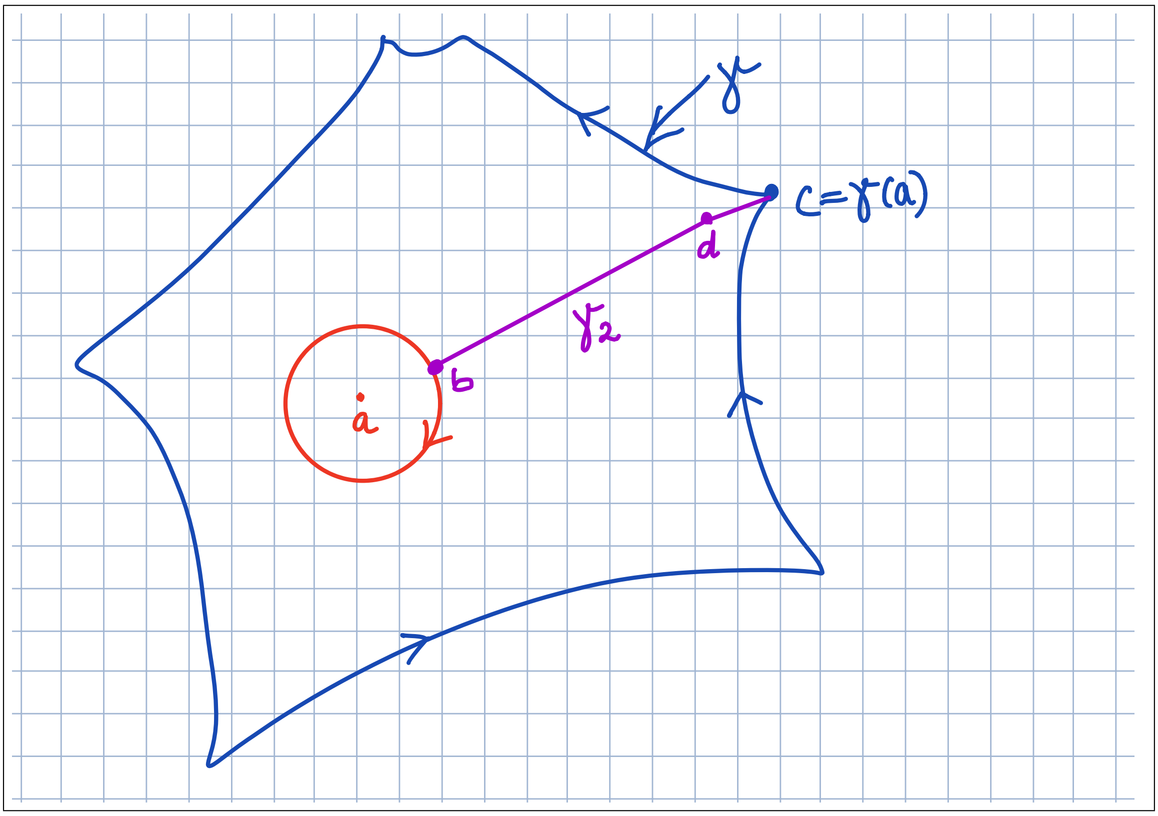

Figure 12:

Lemma 4.24 *Let γ be a positively oriented contour and $a \notin \Gamma = \gamma([a,b])$. Then for $n \in \mathbb{Z}$,*

$$\int_\gamma (z-a)^n \, \mathrm{d}z = \begin{cases} 0 & \text{if } a \in \mathrm{O}(\gamma), n \in \mathbb{Z}\,, \\ 0 & \text{if } a \in \mathrm{I}(\gamma), n \neq -1\,, \\ 2\pi \mathrm{i} & \text{if } a \in \mathrm{I}(\gamma), n = -1\,. \end{cases}$$

The following important theorem now follows easily:

Theorem 4.25 (Cauchy's integral formula) *Let $D \subset \mathbb{C}$ be open and connected and let $f\colon D \to \mathbb{C}$ be holomorphic on D. Suppose that for some $r > 0$ and $a \in \mathbb{C}$ the closed ball $\overline{B}_r(a) = \{z \in \mathbb{C}\colon |z-a| \leq r\}$ is contained in D. Then, for every $z_0 \in B_r(a)$, we have*

$$f(z_0) = \frac{1}{2\pi \mathrm{i}} \int_{\partial B_r(a)} \frac{f(\xi)}{\xi - z_0} \, \mathrm{d}\xi. \tag{4.3}$$

Remark 4.26 Again, this is not the most general assumption on the contour of the integration. We will see a much more general statement below. For the moment, the reader can check easily that nothing in the proof changes if the ball around $z_0 \in B_r(a)$ is replaced, for example, by a suitable square.

Proof. Fix any $z_0 \in B_r(a)$. For any $\delta > 0$ that is small enough to ensure that

$$B_\delta(z_0) \subseteq B_r(a)\,,$$

the function , defined as

$$g(\xi) := \xi \mapsto \frac{f(\xi)}{\xi - z_0},$$

is holomorphic on $B_r(a) \setminus B_\delta(z_0)$. We claim that the following holds:

$$\int_{\partial B_r(a)} \frac{f(\xi)}{\xi - z_0}\, d\xi = \int_{\partial B_\delta(z_0)} \frac{f(\xi)}{\xi - z_0}\, d\xi. \tag{4.4}$$

Indeed, we can write

$$\int_{\partial B_r(a)} \frac{f(\xi)}{\xi - z_0}\, d\xi - \int_{\partial B_\delta(z_0)} \frac{f(\xi)}{\xi - z_0}\, d\xi = \int_{\gamma_1} \frac{f(\xi)}{\xi - z_0}\, d\xi + \int_{\gamma_2} \frac{f(\xi)}{\xi - z_0}\, d\xi\,, \tag{4.5}$$

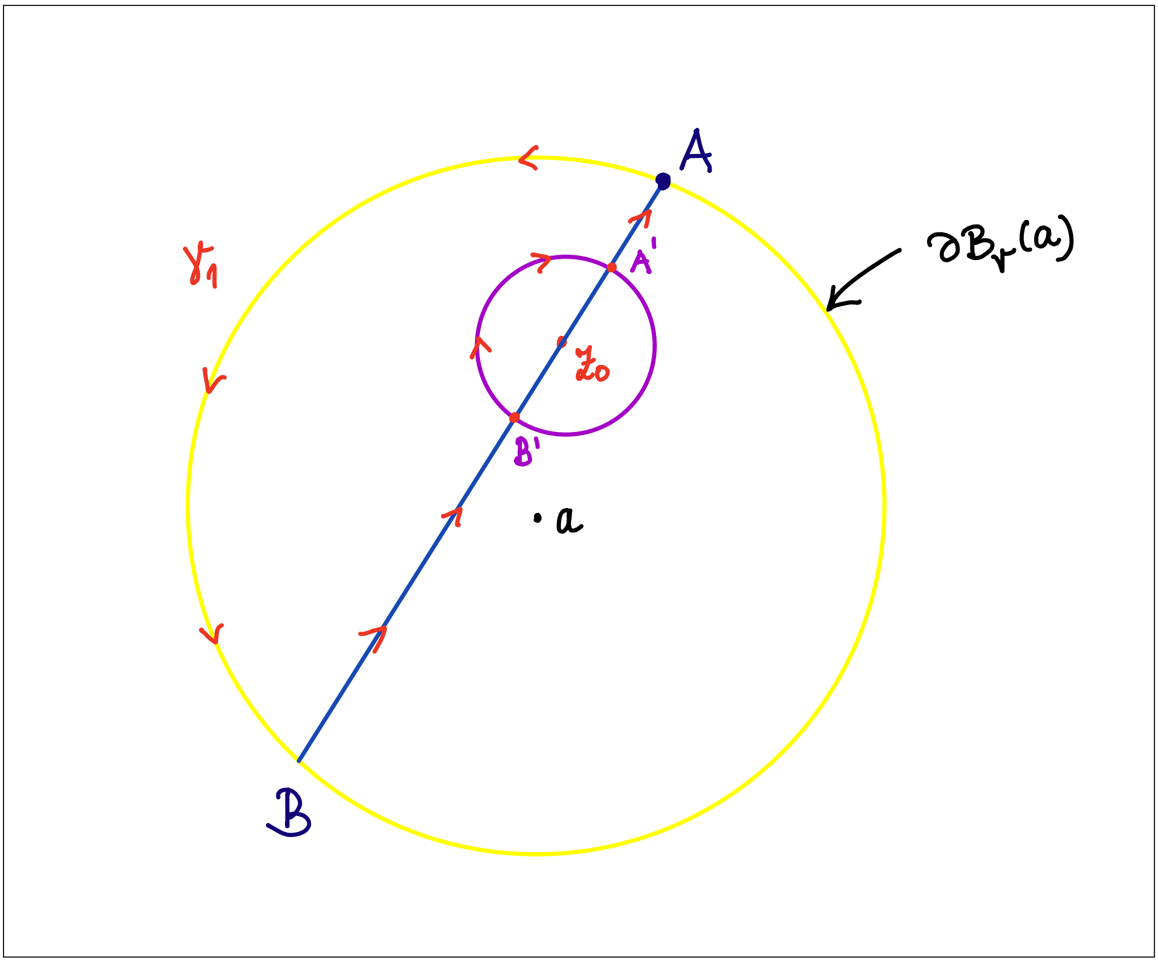

Figure 13:

where the paths γ_i are constructed as follows: the outer circle $\partial B_r(a)$ is connected to the inner circle by any line through z_0. Pick such an auxiliary line through z_0 (line from A to B in Figure 13), and denote the two intersections of it with the outer circle line $\partial B_r(z_0)$ by A and B. Then the curve γ_1 starts in A and follows the outer circle $\partial B_r(a)$ (in counterclockwise orientation) between the two intersections of this auxiliary line and the circle until it reaches the point B. Then it follows the chosen auxiliary line to the "inner circle" $\partial B_\delta(z_0)$, in this case up to point B' (see Figure 13, which it then follows in clockwise orientation until it hits again the line at the point A'. It then follows this line back to the point A on $\partial B_r(a)$. The curve γ_2 is constructed in the same way for the other halves of the circles $\partial B_r(a)$ and $\partial B_\delta(z_0)$. Equation (4.5) then follows, because the curves γ_i "patched together" cover once the outer circle with counterclockwise orientation and once the inner circle with clockwise orientation. The auxiliary lines in between these circles are covered once with both orientations, so they do not contribute to the total value.

Equation (4.4) then follows from (4.5) and Theorem 4.19. Indeed, the function

$$\xi \mapsto \frac{f(\xi)}{\xi - z_0}$$

is holomorphic on $D \setminus \{z_0\}$. It is easy to find open star shaped subsets D_1 and D_2 of $D \setminus \{z_0\}$ that contain the curves γ_1 and γ_2 which shows that the integrals over γ_1 and γ_2 vanish.

We now finish the proof of the theorem, based on equation (4.4). According to the differentiability of f in z_0, for every $\varepsilon > 0$ there exists a δ such that for $|\xi - z_0| < \delta$ we have

$$f(\xi) - f(z_0) = f'(z_0)(\xi - z_0) + R(z_0, \xi),$$

with $|R(z_0,\xi)| \le \varepsilon|\xi - z_0|$. Hence we get for such a δ

$$\int_{\partial B_\delta(z_0)} \frac{f(\xi)}{\xi - z_0}\,\mathrm{d}\xi = \int_{\partial B_\delta(z_0)} \frac{f(z_0)}{\xi - z_0}\,\mathrm{d}\xi + \int_{\partial B_\delta(z_0)} f'(z_0)\,\mathrm{d}\xi + \int_{\partial B_\delta(z_0)} \frac{R(z_0,\xi)}{\xi - z_0}\,\mathrm{d}\xi.$$

For the first integral we get

$$\int_{\partial B_\delta(z_0)} \frac{f(z_0)}{\xi - z_0}\,\mathrm{d}\xi = 2\pi\mathrm{i} f(z_0).$$

The second integral is 0, and for the third integral we get

$$\left| \int_{\partial B_\delta(z_0)} \frac{R(z_0,\xi)}{\xi - z_0}\,\mathrm{d}\xi \right| \le 2\pi\delta\varepsilon.$$

As ε can be chosen arbitrarily small, and as we can always assume that $\delta \le 1$, the desired conclusion follows. □

5 Applications of Cauchy's theorem

5.1 Immediate consequences

In this section we summarise a large number of properties of holomorphic functions. We start by showing that any holomorphic function can be expanded in a Taylor series. In particular, every holomorphic function is automatically $\mathcal{C}^\infty$.

Theorem 5.1 (Taylor's theorem) *Let $D \subset \mathbb{C}$ be open and connected and let $f\colon D \to \mathbb{C}$ be holomorphic on D. Suppose $z_0 \in D$ and $R > 0$ such that $\overline{B}_R(z_0) \subset D$. Then for all $z \in B_R(z_0)$ we can write*

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k,$$

with

$$a_k = \frac{1}{2\pi \mathrm{i}} \int_{\partial B_R(z_0)} \frac{f(\xi)}{(\xi - z_0)^{k+1}} \,\mathrm{d}\xi. \tag{5.1}$$

Proof. According to Cauchy's integral formula we have for all $z \in B_R(z_0)$

$$f(z) = \frac{1}{2\pi \mathrm{i}} \int_{\partial B_R(z_0)} \frac{f(\xi)}{(\xi - z)} \,\mathrm{d}\xi. \tag{5.2}$$

We write

$$\frac{1}{\xi - z} = \frac{1}{\xi - z_0} \frac{\xi - z_0}{\xi - z} = \frac{1}{\xi - z_0} \frac{1}{\frac{\xi - z}{\xi - z_0}} = \frac{1}{\xi - z_0} \frac{1}{\left(1 - \left(\frac{z - z_0}{\xi - z_0}\right)\right)}. \tag{5.3}$$

Now according to our assumption, we have $|z - z_0| < |\xi - z_0| = R$. Hence, the last expression on the right hand side of (5.3) can be written as a geometric series

$$\frac{1}{1 - \frac{z - z_0}{\xi - z_0}} = \sum_{k=0}^{\infty} \left(\frac{z - z_0}{\xi - z_0}\right)^k.$$

Plugging this back into (5.2) we get

$$\begin{aligned} f(z) =& \frac{1}{2\pi \mathrm{i}} \int_{\partial B_R(z_0)} \frac{f(\xi)}{(\xi - z_0)} \sum_{k=0}^{\infty} \left(\frac{z - z_0}{\xi - z_0}\right)^k \mathrm{d}\xi \\ =& \frac{1}{2\pi \mathrm{i}} \sum_{k=0}^{\infty} \Big(\int_{\partial B_R(z_0)} \frac{f(\xi)}{(\xi - z_0)^{k+1}} \,\mathrm{d}\xi \Big) \ (z - z_0)^k. \end{aligned}$$

The interchange of the summation and the integration is justified because the sum converges uniformly in the integration variable ξ. □

Remark 5.2 (a) Of course, the value of the integrals in (5.1) is independent of the choice of the radius R, as long as the ball of radius R around z_0 along with its boundary is contained in D.

(b) **Radius of convergence:** Theorem 5.1 has another non-trivial consequence, namely the fact that the radius of convergence of a Taylor series must be at least as large as the distance from the nearest point where f ceases to be holomorphic . This is particularly interesting, as it gives a natural explanation for radii of convergence, which cannot be seen if one restricts oneself to the real case. For example, the real function $f\colon x \mapsto \frac{1}{1+x^2}$ can be developed in a Taylor series around 0: For $|x| < 1$ we get

$$\frac{1}{1 + x^2} = \sum_{k=0}^{\infty} (-1)^k x^{2k}.$$

This series fails to converge for $|x| > 1$ although f is defined and smooth on all of $\mathbb{R}$. This observation can be explained easily, if one considers f as a function of a complex variable z and observes that f has a singularity at $\pm i$.

◇

Corollary 5.3 *Every holomorphic function is $\mathcal{C}^\infty$.*

Corollary 5.4 *Let $D \subset \mathbb{C}$ be open and connected and $f\colon D \to \mathbb{C}$. Then the following statements are equivalent:*

- *f is holomorphic in D.*
- *f is real-differentiable in every point, and satisfies the* Cauchy-Riemann equations *.*
- *f can be expanded in a Taylor series around every point in D.*

Remark 5.5 (a) Calculating the radius of convergence for a series $\sum_{n\in\mathbb{N}_0} a_n$, $a_n \in \mathbb{C}$:

(i) **d'Alembert's Ratio test**

Assume the sequence $(a_n)_{n\in\mathbb{N}_0}$ is such that

$$\ell := \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

exists. If $\ell < 1$, then $\sum_{n\in\mathbb{N}_0} |a_n|$ converges (series converges absolutely). If $\ell > 1$, then $\sum_{n\in\mathbb{N}_0} |a_n|$ diverges. If $\ell = 1$ then the test gives no information.

(ii) **Cauchy's n-th root test**

Assume the sequence $(a_n)_{n\in\mathbb{N}_0}$ is such that

$$\ell := \lim_{n\to\infty} \sqrt[n]{|a_n|}$$

exists. If $\ell < 1$, then $\sum_{n\in\mathbb{N}_0} |a_n|$ converges (series converges absolutely). If $\ell > 1$, then $\sum_{n\in\mathbb{N}_0} |a_n|$ diverges. If $\ell = 1$ then the test gives no information.

(b) The series

$$\sum_{n\in\mathbb{N}_0} \frac{z^n}{n!}$$

converges absolutely (and hence converges) for all $z \in \mathbb{C}$ as

$$\left|\frac{z^{n+1}/(n+1)!}{z^n/n!}\right| = \frac{|z|}{n+1} \to 0 \text{ as } n \to \infty.$$

(c) The last example (b) shows that we can define

$$\mathrm{e}^z := \sum_{n\in\mathbb{N}_0} \frac{z^n}{n!}, \qquad z \in \mathbb{C}.$$

The function e^z is holomorphic on $\mathbb{C}$ and one can easily show the following properties (Exercise!)

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}z}\mathrm{e}^z &= \mathrm{e}^z \\
\mathrm{e}^0 &= 1 \\
\mathrm{e}^{z+w} &= \mathrm{e}^z\mathrm{e}^w, \qquad z, w \in \mathbb{C} \\
\mathrm{e}^z &\neq 0 \text{ for all } z \in \mathbb{C} \\
|\mathrm{e}^z| &= \mathrm{e}^x \text{ when } z = z + \mathrm{i}y
\end{aligned}$$

(d) **Trigonometric and hyperbolic functions and their radius of convergence R**

$$
\begin{aligned}
\cos(z) &:= \sum_{n\in\mathbb{N}_0} (-1)^n \frac{z^{2n}}{(2n)!}, \qquad R=\infty,\\
\sin(z) &:= \sum_{n\in\mathbb{N}_0} (-1)^n \frac{z^{2n+1}}{(2n+1)!}. \qquad R=\infty,\\
\cosh(z) &:= \sum_{n\in\mathbb{N}_0} \frac{z^{2n}}{(2n)!}, \qquad R=\infty,\\
\sinh(z) &:= \sum_{n\in\mathbb{N}_0} \frac{z^{2n+1}}{(2n+1)!}, \qquad R=\infty,\\
\sin(z) &= \frac{1}{2\mathrm{i}}(\mathrm{e}^{\mathrm{i}z}-\mathrm{e}^{-\mathrm{i}z}),\\
\cos(z) &= \frac{1}{2}(\mathrm{e}^{\mathrm{i}z}+\mathrm{e}^{-\mathrm{i}z}),\\
\cosh(z) &= \frac{1}{2}(\mathrm{e}^{z}+\mathrm{e}^{-z}),\\
\sin(z) &= \frac{1}{2}(\mathrm{e}^{z}-\mathrm{e}^{-z}).
\end{aligned}
$$

◇

Corollary 5.6 *Suppose that $f(z)=\sum_{k=0}^{\infty} a_k z^k$ is holomorphic on $B_R(0)$ for some $R>0$ and that for all $z\in B_R(0)$ we have $|f(z)|\le M<\infty$. Then for all k the following estimate holds*

$$|a_k| \le \frac{M}{R^k}.$$

Proof. According to (5.1) we have for any $r<R$

$$|a_k| \le \frac{1}{2\pi}\left|\int_{\partial B_r(0)} \frac{f(\xi)}{\xi^{k+1}}\,\mathrm{d}\xi\right| \le \frac{1}{2\pi}2\pi r\frac{M}{r^{k+1}}.$$

Then let $r\to R$. □

Proposition 5.7 *Let $D\subset\mathbb{C}$ be open and polygonally connected, and let $f\in\mathcal{H}(D)$. If $f'(z)=0$ for all $z\in D$, or if either one of the functions $u=\mathfrak{R}(f), v=\mathfrak{I}(f)$, or $|f|$ is constant on D, then f is constant on D.*

Proof. The connectedness is crucial as a function can have different constant values on disjoint components of the set D. (1.) Suppose that $f'(z)=0$ for all $z\in D$. Then

$$f'(z)=\frac{\partial u}{\partial x}(x,y)+\mathrm{i}\frac{\partial v}{\partial x}(x,y)=\frac{\partial v}{\partial y}(x,y)-\mathrm{i}\frac{\partial u}{\partial y}(x,y)=0, \qquad z=x+\mathrm{i}y\in D$$

implying

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial y}=0=\frac{\partial v}{\partial x}=\frac{\partial v}{\partial y},$$

on D, and thus (real analysis) the real functions u and v are constant on D and so is f.

(2.) Suppose that u is constant on D (both partial derivatives vanish on D), then the Cauchy-Riemann equations (3.7) imply that

$$\frac{\partial v}{\partial x}(x,y) = \frac{\partial v}{\partial y}(x,y) = 0, \qquad z = x + \mathrm{i}y \in D,$$

and thus v is constant on D. (analogous proof when v is constant). If u and v are constant in D so is the function f.

(3.) Suppose that $|f(z)| = c$ for all $z \in D$. Then $u^2 + v^2 = C^2$ on D, and differentiating we obtain

$$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0 = 2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y}. \tag{5.4}$$

(5.4) is a linear equation in u and v with coefficient being the partial derivatives. If at least one of the functions u and v is not identical to zero, then the determinant of the system of equations (5.4) has to vanish:

$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y} = 0,$$

we plug in the Cauchy-Riemanm equations (3.7) and obtain

$$(\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y})^2 = 0 = (\frac{\partial v}{\partial x})^2 + (\frac{\partial v}{\partial y})^2,$$

and thus u and v are constant on D so is the function f. □

Corollary 5.8 (Liouville's theorem) *Any bounded entire function is constant.*

Proof. Pick any $z_0 \in \mathbb{C}$ and $M > 0$ such that $|f(z)| \le M$ for all $z \in \mathbb{C}$. For any $R > 0$ and $m(f,R,z_0) := \max_{z \in \partial B_R(z_0)} |f(z)|$ we have (using (4.3) and Taylor's theorem 5.1)

$$|f^{(n)}(z_0)| \le \frac{n!}{R^n} m(f,R,z_0) \le \frac{n!}{R^n},$$

and in particular,

$$|f' z_0| \le \frac{M}{R}.$$

As $R > 0$ is arbitrary, we get that $f'(z_0) = 0$, and thus (z_0 was an arbitrary choice) $f'(z) = 0$ for all $z \in \mathbb{C}$, and according to Proposition 5.7 we get that f is constant as $\mathbb{C}$ is polygonally connected. □

Proof. Suppose that

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

is entire and bounded by M on all of $\mathbb{C}$. Then according to Corollary 5.6 we have for any R and any $k \ge 1$ that

$$|a_k| \le \frac{M}{R^k},$$

and hence $a_k = 0$. □

Example 5.9 Note that $\sin(z) = \frac{1}{2\mathrm{i}}(\mathrm{e}^{\mathrm{i}z} - \mathrm{e}^{-\mathrm{i}z})$ is not bounded on $\mathbb{C}$. For example, if $z_n = \mathrm{i}n$ then

$$|\sin(z_n)| = \frac{1}{2}\Big|(\mathrm{e}^{-n} - \mathrm{e}^{n})\Big| \to \infty \text{ as } n \to \infty.$$

♣

Remark 5.10 The statement of Liouville's theorem is not valid for unbounded subsets of $\mathbb{C}$. For example, the Möbius transform $z \mapsto f(z) = \frac{z-\mathrm{i}}{z+\mathrm{i}}$ maps the upper half plane $\mathrm{H}^+ := \{z \in \mathbb{C} \colon \Im(z) > 0\}$ onto the unit disc $\mathbf{\Delta} = \{z \in \mathbb{C} \colon |z| < 1\}$ and in particular it is certainly bounded on H^+. Compare with Exercise 2(a) on Example Sheet 2 where $f = h^{-1}$ with $h(w) = -\mathrm{i}\frac{(w-1)}{(w+\mathrm{i})}$ and $h(\mathbf{\Delta}) = \mathrm{H}^+$ and $h(\mathrm{H}^+) = \mathbf{\Delta}$. Furthermore, $h(\partial\mathbf{\Delta}) = \mathbb{R} \cup \{\infty\}$ as $h(1) = 0, h(\mathrm{i}) = 1$, and $h(-1) = \infty$. ◇

Corollary 5.11 (Fundamental theorem of Algebra) *Every non-constant polynomial has at least one zero in* $\mathbb{C}$.

Proof Version 2019 - lecture. **Step 1:** Let $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots a_1 z + a_0$ with $a_n \neq 0$. Then, for all $\varepsilon > 0$ there exists $R = R(\varepsilon) > 0$, such that

$$(1-\varepsilon)|a_n||z|^n \le |P(z)| \le (1+\varepsilon)|a_n||z|^n \qquad \text{for all } |z| > R.$$

For $z \neq 0$ we have

$$\frac{P(z)}{a_n z^n} = 1 + \frac{a_{n-1}}{a_n}\frac{1}{z} + \cdots + \frac{a_0}{a_n}\frac{1}{z^n},$$

and thus

$$1 - \sum_{k=1}^n \Big|\frac{a_{n-k}}{a_n}\Big|\frac{1}{|z|^k} \le \Big|\frac{P(z)}{a_n z^n}\Big| \le 1 + \sum_{k=1}^n \Big|\frac{a_{n-k}}{a_n}\Big|\frac{1}{|z|^k}.$$

Now, as $\lim_{x\to\infty}\frac{1}{x^k} = 0$ for all $k \in \mathbb{N}$, there exists $R > 0$ such that

$$\sum_{k=1}^n \Big|\frac{a_{n-k}}{a_n}\Big|\frac{1}{R^k} < \varepsilon.$$

Step 2: Suppose the polynomial $P(z)$ has no zero in $\mathbb{C}$. Then $(P(z))^{-1}$ is complex-differentiable (holomorphic) on $\mathbb{C}$. According to Step 1, there exists $R > 0$ such that

$$\frac{1}{2}|a_n||z|^n \le |P(z)| \quad \text{ for } |z| > R,$$

and thus

$$\Big|\frac{1}{P(z)}\Big| \le \frac{2}{|a_n||z|^n} \le \frac{2}{|a_n|R^n} \quad \text{ for } |z| > R.$$

Furthermore, the image $P(\{z \in \mathbb{C} \colon |z| \le R\})$ is bounded as the closed disc $\overline{B_R(0)} = \{z \in \mathbb{C} \colon |z| \le R\}$ is compact and the polynomial P is continuous. Henceforth the mapping

$$z \mapsto \Big|\frac{1}{P(z)}\Big|$$

is bounded. Thus $1/P$ is bounded on the whole of $\mathbb{C}$ (follows from the estimate for $|z| > R$ and follows from the boundedness of the image for $|z| \leq R$). Therefore $1/P$ is constant and thus P is constant, and we obtain a contradiction, that is, P has at least one zero. $\square$

Proof Version from notes 2018. Suppose that $p = \sum_{k=0}^{n} a_k z^k$ with $a_n \neq 0$ and $n \geq 1$ does not have a zero in $\mathbb{C}$. Then $\frac{1}{p}$ is an entire function.

For any $z \in \mathbb{C}$ we can write

$$\left|\frac{p(z)}{z^n} - a_n\right| \leq \sum_{k=0}^{n-1} |a_k||z^{k-n}|.$$

The right hand side of this expression goes to 0 as $|z| \to \infty$ and in particular there exists an $R > 0$ such that for $|z| \geq R$ it can be bounded by $\frac{|a_n|}{2}$. This implies that for $|z| \geq R$ the right-hand side can be bounded by $\frac{|a_n|}{2}$. Thus,

$$\left|\frac{1}{p(z)}\right| \leq \frac{2}{|a_n||z|^n},$$

and in particular $\frac{1}{p}$ is bounded outside of $\overline{B}_R(0)$. On the other hand $\frac{1}{p}$ is continuous and the closed ball $\overline{B}_R(0)$ is compact. Hence $\frac{1}{p}$ is also bounded on $\overline{B}_R(0)$ and by Liouville's theorem, Corollary 5.8, it is constant. This is a contradiction. $\square$

The following is an inverse to Goursat's theorem.

Theorem 5.12 (Morera's theorem) *Let $D \subset \mathbb{C}$ be open and connected, and $f\colon D \to \mathbb{C}$ be a continuous function. Assume that for all triangles $\tau \subset D$ with boundary curve $\gamma = \partial\tau$ we have*

$$\int_\gamma f(z)\,\mathrm{d}z = 0. \tag{5.5}$$

Then f is holomorphic on D.

This condition can be interpreted as an *integral versions of the Cauchy-Riemann equations*. The reader may have encountered similar integral versions of different PDEs such as the integral version of the Maxwell equations.

Proof. Complex differentiability is a local property and hence, by making D smaller we can restrict ourselves without loss of generality to the case of a ball $D = \{z \in \mathbb{C}\colon |z| < r\}$. On this ball we can construct an antiderivative F of f in the same way as in the proof of Theorem 4.19.

But then the argument is finished, because if F is holomorphic , then so are all of its derivatives, and in particular f. $\square$

The following statement is a nice application of Morera's theorem.

Theorem 5.13 (Schwarz reflection principle) *Let $D \subset \mathbb{C}$ be open and connected and assume that D is invariant under complex conjugation (i.e. $z \in D \Leftrightarrow \bar{z} \in D$). Assume that $f\colon D \to \mathbb{C}$ is a continuous function, with the following properties:*

- *f is holomorphic on $D \cap \{z\colon \Im(z) > 0\}$.*
- *f only attains real values on $D \cap \mathbb{R}$.*
- *For any $z \in D$ we have $f(z) = \overline{f(\bar{z})}$.*

Then f is holomorphic on D.

Proof. The function f is holomorphic on $D \cap \{z\colon \Im(z) > 0\}$ by assumption. For any $z \in D \cap \{z\colon \Im(z) < 0\}$ the function f is differentiable in the sense of real analysis. In order to check that the Cauchy-Riemann equations hold we write $f(x+\mathrm{i}y) = u(x,y) + \mathrm{i}v(x,y)$. Then we get for $z = x + \mathrm{i}y$ with $y < 0$

$$\frac{\partial u}{\partial x}(x,y) = \frac{\partial u}{\partial x}(x,-y) = \frac{\partial v}{\partial y}(x,-y) = -\frac{\partial}{\partial y}(v(x,-y)) = \frac{\partial}{\partial y}(v(x,y)),$$

where we have used that the Cauchy-Riemann equations are satisfied in the upper half plane. In the same way we get

$$\frac{\partial u}{\partial y}(x,y) = -\frac{\partial u}{\partial y}(x,-y) = \frac{\partial v}{\partial x}(x,-y) = -\frac{\partial v}{\partial x}(x,y).$$

Hence, f is also holomorphic in $D \cap \{z\colon \Im(z) < 0\}$.

It remains to show that f is holomorphic in a neighbourhood of the real line. To this end let $z \in D \cap \mathbb{R}$ and let $R > 0$ be small enough to ensure that $B_R(z) \subseteq D$. It is sufficient to show that f is holomorphic on $B_R(z)$. To see this let Δ be an arbitrary triangle in $B_R(z)$. We need to show that

$$\int_{\partial\Delta} f(z)\,\mathrm{d}z = 0.$$

If Δ is fully contained in the upper half plane, or the lower half plane this follows immediately from Goursat's theorem for the images of squares, because we know already that f is holomorphic in these regions. Else, for any $\varepsilon > 0$ define the sets $\Delta_\varepsilon^\pm$ as

$$\Delta_\varepsilon^+ = \Delta \cap \{z\colon \Im(z) > \varepsilon\} \qquad \text{and} \qquad \Delta_\varepsilon^- = \Delta \cap \{z\colon \Im(z) < -\varepsilon\}.$$

Then on the one hand we have for any $\varepsilon > 0$

$$\int_{\partial\Delta_\varepsilon^\pm} f(z)\,\mathrm{d}z = 0$$

as f holomorphic and on the other hand

$$\lim_{\varepsilon\to 0} \int_{\partial\Delta_\varepsilon^+} f(z)\,\mathrm{d}z + \int_{\partial\Delta_\varepsilon^-} f(z)\,\mathrm{d}z = \int_{\partial\Delta} f(z)\,\mathrm{d}z.$$

This finishes the argument. □

5.2 Zeros of holomorphic functions

Definition 5.14 Let D be open and connected and let $f\colon D \to \mathbb{C}$ be a holomorphic function on D. If for $z_0 \in D$ we have $f(z_0) = 0$, then the *order* of the zero of f at z_0 is defined as

$$\operatorname{ord}(f, z_0) := \inf\{k \in \mathbb{N}\colon f^{(k)}(z_0) \neq 0\}.$$

Example 5.15 If $f(z) = (z-3)^2 \exp(z)$, then f has a zero of order 2 at $z_0 = 3$. ♣

Let us make a couple of observations.

Remark 5.16 (a) The set $\mathcal{O} = \{z \in \mathbb{C}\colon f \text{ has zero of order } \infty\}$ is open. To see this, note that if f has a zero of infinite order at z_0, then all the Taylor coefficients at z_0 vanish and f is identically to 0 on a whole neighbourhood of z_0.

(b) If f has a zero of order $n < \infty$ at z_0 we can write for all z in a neighbourhood of z_0

$$f(z) = \sum_{k=n}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z-z_0)^k = (z-z_0)^n \sum_{k=0}^{\infty} b_k (z-z_0)^k =: (z-z_0)^n g(z).$$

where $b_k = \frac{f^{(k+n)}(z_0)}{(k+n)!}$. Observe that g has no zeros in a neighbourhood of z_0. This implies that f does not have any further zeros in a whole neighbourhood of z_0. In other words: *All zeros of finite order are isolated.*

(c) If f has a zero of infinite order *at one point* z_0, then this implies already that f vanishes on all of D (here one uses the connectedness of D): To see that let $\mathcal{O}$ be as above. We have already seen that $\mathcal{O}$ is open and it is non-empty by assumption. But $\mathcal{O}$ is also relatively closed in D: Suppose that $z_n \in \mathcal{O}$ converge to $z \in D$. Then $f(z)$ must be zero by continuity. But z cannot be zero of finite order because then it would have to be isolated. Hence z is also in $\mathcal{O}$. Therefore, by the connectedness of D, we have $\mathcal{O} = D$.

◇

These observation have a very important consequence below, see Identity Theorem 5.21.

Definition 5.17 Let $D \subset \mathbb{C}$ be open.

(a) $f\colon D \to \mathbb{C}$ is called a *conformal mapping* if f is holomorphic on D with $f'(z) \neq 0$ for all $z \in D$.

(b) $f\colon D \to \mathbb{C}$ is called *bi-holomorphic* if f is a bijective conformal mapping such that the inverse f^{-1} is also a conformal mapping.

We will later see that every bijective holomorphic function is bi-holomorphic.

Let us make some more observations concerning the behaviour of an holomorphic function near its zeros.

If for any z_0 we have $f'(z_0) \neq 0$ (we do not assume that $f(z_0) = 0$ here), then it is locally bi-holomorphic. More precisely, there exists a neighbourhood V_1 of z_0 and a neighbourhood V_2 of $f(z_0)$ such that f is a bijection to V_2 when restricted to V_1, and such that f^{-1} is also holomorphic on V_2. In fact, the *inverse function theorem* from real analysis implies that f is locally bijective, and that its inverse is real differentiable. We also have for every $z \in V_1$ that

$$Df^{-1}(f(z)) = Df(z)^{-1},$$

where Df denotes the *Jacobi Matrix*. As Df satisfies the *Cauchy-Riemann equations* (i.e it acts as multiplication by a complex number $f'(z)$), so does $Df(z)^{-1}$ (it acts as multiplication by $\frac{1}{f'(z)}$).

The prototypical example of a zero of higher order is the mapping $f(z) = z^k$ for some $k \geq 1$. Recall the geometric interpretation of this function which can be explained easiest in polar coordinates: The point $z = r\mathrm{e}^{\mathrm{i}\theta}$ is mapped to $r^k\mathrm{e}^{\mathrm{i}k\theta}$, i.e. the function acts on the absolute value as a monomial and the *argument* is multiplied by k. In particular, for every $w = |w|\mathrm{e}^{\mathrm{i}\varphi}$ there exist exactly k distinct complex numbers, namely $z_1 = \sqrt[k]{|w|}\mathrm{e}^{\mathrm{i}\frac{\varphi}{k}}$, $z_2 = \sqrt[k]{|w|}\mathrm{e}^{\mathrm{i}\frac{2\pi+\varphi}{k}}, \ldots, z_n = \sqrt[k]{|w|}\mathrm{e}^{\mathrm{i}\frac{2(n-1)\pi+\varphi}{k}}$, i.e. the function is k to 1.

The following theorem states that every holomorphic function locally behaves in the same way near a zero of order k.

Theorem 5.18 *Let D be an open set and $f\colon D \to \mathbb{C}$ be holomorphic on D. Assume that f has a zero of order $k \geq 1$ at $z_0 \in D$. Then there exists a neighbourhood V_1 of z_0 and a neighbourhood V_2 of 0 and a bi-holomorphic function $h\colon V_1 \to V_2$ such that for every $z \in V_1$ we have*

$$f(z) = (h(z))^k.$$

In particular, f is locally k to one near z_0, that is, f takes every w exactly k times near z_0.

Remark 5.19 If f has a zero of order $k \in \mathbb{N}$ at z_0, then

$$\begin{aligned} f(z) &= \sum_{n=k}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n = (z-z_0)^k \sum_{n=0}^{\infty} b_n(z-z_0)^n \\ &=: (z-z_0)^k g(z), \end{aligned}$$

where

$$b_n := \frac{f^{(k+n)}(z_0)}{(k+n)!}.$$

The holomorphic function g has no zeros in a neighbourhood of z_0. Thus the function f doe snot have any further zeros in a whole neighbourhood of z_0.

All zeros of finite order are isolated

◇

Remark 5.20 If $f\colon D \to \mathbb{C}$ is holomorphic and bijective on D, then $f'(z_0) \neq 0$ for all $z_0 \in D$ as otherwise the function $g(z) := f(z) - f(z_0)$ would have a zero of order k for some $k > 1$. Theorem 5.18 then implies that the function g cannot be injective and therefore f cannot be injective. Henceforth, when f is bijective and holomorphic then it is bi-holomorphic (inverse function theorem). $\diamond$

Theorem 5.21 (Identity theorem) *Let D be open and connected and let $f_1, f_2\colon D \to \mathbb{C}$ be holomorphic on D. Assume that the set $\{z \in \mathbb{C}\colon f_1(z) = f_2(z)\}$ has at least one point of accumulation in D. Then $f_1 = f_2$ on all of D.*

Proof. Let $g = f_1 - f_2$. Let z be a point of accumulation of $\mathcal{O}$. Then g is holomorphic and it has a zero in z which is not isolated. Hence the zero is of infinite order and g vanishes everywhere. $\square$

Remark 5.22 (a) Recall that $z \in D$ is called a *point of accumulation* if there exists a sequence $z_n \in D \setminus \{z\}$ with $z_n \to z$.

(b) This implies in particular, that any function $f\colon \mathbb{R} \to \mathbb{C}$ has *at most* one holomorphic extension to a neighbourhood of the real line in the complex plane.

(c) Of course, the mapping h in Theorem 5.18 cannot be unique, because if h satisfies the desired properties, then so does $h\mathrm{e}^{\mathrm{i}\frac{2\pi}{k}}$.

$\diamond$

Proof of Theorem 5.18. We have seen above that we can write

$$f(z) = (z - z_0)^k g(z),$$

where g is holomorphic on D and g does not attain the value 0 on a whole neighbourhood of z_0. We want to locally define a holomorphic k-th root r in a neighbourhood of $g(z_0)$. Note, that in general we cannot expect to be able to find a holomorphic k-th root everywhere. But locally, there is no problem: We start by choosing r at a single value $g(z_0) =: w_0 \neq 0$. If $w_0 = |w_0|\mathrm{e}^{\mathrm{i}\theta}$ we set $r(w_0) = \sqrt[k]{|w_0)|}\mathrm{e}^{\mathrm{i}\frac{\theta}{k}} = r_0$. Then $r_0^k = w_0$. Furthermore, then the derivative of the function $p(z) = z^k$ in r_0 does not vanish and hence it locally has a bi-holomorphic inverse, which is our desired function r. Finally, we set $h(z) = (z - z_0)r(g(z))$. This function satisfies all the desired properties. $\square$

The following consequence of our discussion above is very important:

Theorem 5.23 (Open mapping theorem) *Let D be open and connected and assume that $f\colon D \to \mathbb{C}$ is holomorphic on D and not constant. Then the image $f(D)$ of D under f is also open and connected.*

Before we embark on the proof of this pivotal statement, note that the pre-image of *every* continuous function on any topological space is open (that is the definition of continuity). But the statement for forward images is specific to the case of holomorphic functions. To see that this property does not hold, for example in the case of smooth functions from $\mathbb{R}$ to $\mathbb{R}$, it is sufficient to consider the function $f(z) = z^2$ near 0.

Proof. The image of every connected set under a continuous mapping is connected.

In order to see that $f(D)$ is open, let us fix an z_0 in D with $f(z_0) = w_0$. We need to show that $f(D)$ contains a whole neighbourhood of w_0.

The function $z \mapsto f(z) - w_0$ has a zero at z_0. As f is not constant and as D is connected, it must be a zero of finite order, say k. But then, by Theorem 5.18, locally we have that $f(z) = w_0 + (h(z))^k$, where h maps any neighbourhood of z_0 onto a whole neighbourhood of 0. Also $z \mapsto z^k$ maps any neighbourhood of 0 onto a whole neighbourhood of 0. Hence f maps any neighbourhood of z_0 onto a neighbourhood of w_0. □

Corollary 5.24 (Maximum modulus principle) *Let D be open and connected and let $f\colon D \to \mathbb{C}$ be holomorphic on D and not constant. Then $|f|$ does not have any local maxima.*

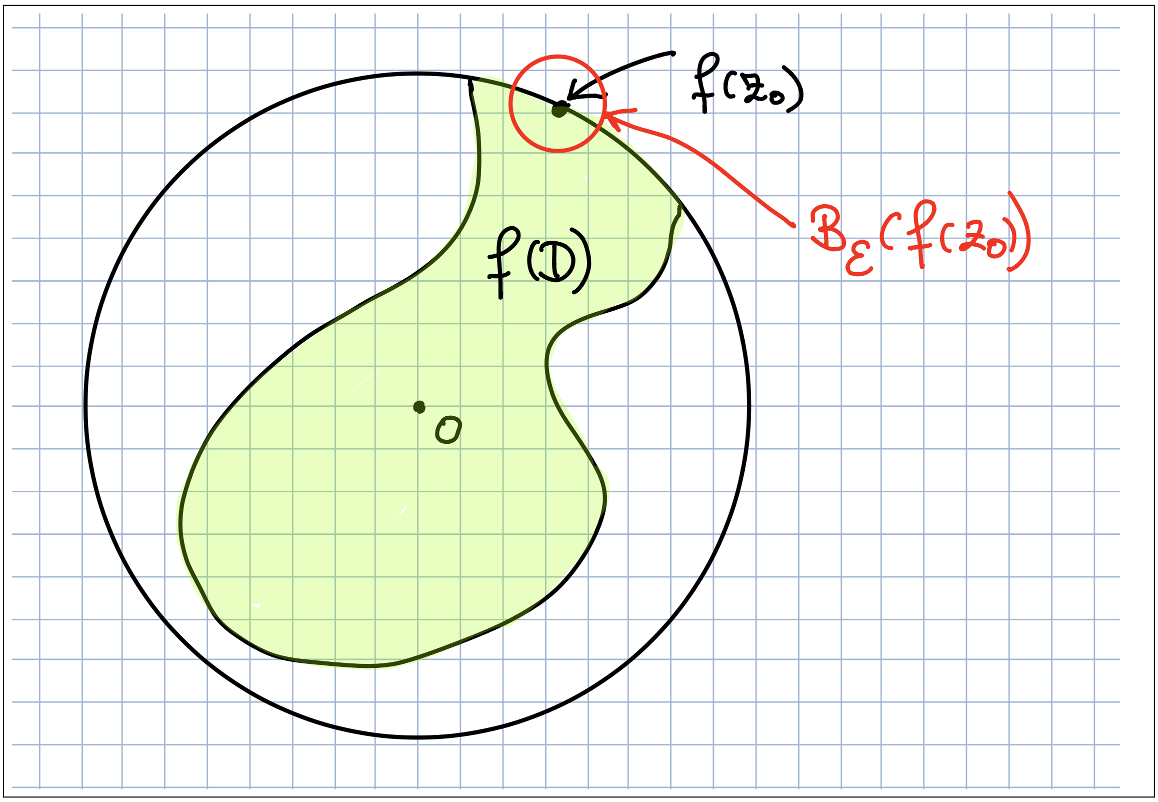

Figure 14:

Proof. Suppose that $|f|$ attains a local maximum in z_0. Then the image of any neighbourhood of z_0 is a full neighbourhood of $f(z_0)$, and in particular, it contains points with larger absolute value, see Figure 14. Hence, we have a contradiction. □

Remark 5.25 There is an alternative way to see the *maximum modulus principle*, directly based on Cauchy's integral formula. Actually, if f is holomorphic on a ball $B_r(z_0)$ we have

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi \mathrm{i}} \int_{\partial B_r(z_0)} \frac{f(z)}{z - z_0}\,\mathrm{d}z = \frac{1}{2\pi \mathrm{i}} \int_0^{2\pi} \frac{f(z_0 + r\mathrm{e}^{\mathrm{i}\theta})\mathrm{i}\mathrm{e}^{\mathrm{i}\theta}}{\mathrm{e}^{\mathrm{i}\theta}}\,\mathrm{d}\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + r\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta, \end{aligned}$$

i.e. $f(z_0)$ is the average value of the $f(z)$ evaluated on a circle around z_0. This property, called the *Mean value property*, is well known from real analysis. It characterises harmonic functions in arbitrary dimensions.

This property implies that f cannot attain a strict local maximum. To see this, assume that $|f(z_0)| \geq |f(z)|$ for all $z \in B_R(z_0)$ for some $R > 0$. But the mean value property implies that

$$|f(z_0)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + r\mathrm{e}^{\mathrm{i}\theta})| \,\mathrm{d}\theta,$$

for any radius $r < R$, which is impossible if we have the strict inequality $|f(z_0)| > |f(z)|$ for only one such z (and hence, by continuity on a small ball around that z). Hence, we can conclude that $|f(z_0)| \geq |f(z)|$ for all z in a ball around z_0, then in fact $|f(z_0)| = |f(z)|$ for all such z.

If at this point we use the *open mapping theorem* again, we can conclude that f is actually constant on the whole ball, because the image of the whole ball is contained in a circle and it cannot be open.

A similar maximum principle holds for solutions to many scalar second order partial differential equations. It is extremely useful, to prove uniqueness of solutions and to derive qualitative properties in many situations. $\diamond$

5.3 Schwarz lemma and its consequences

In this section, we are going to discuss properties of mappings from the unit disc into itself. We are always going to use the notation $\boldsymbol{\Delta} = \{z \colon |z| < 1\}$.

Theorem 5.26 (Schwarz lemma) *Let $f \colon \boldsymbol{\Delta} \to \boldsymbol{\Delta}$ be holomorphic on $\boldsymbol{\Delta}$ with $f(0) = 0$. Then*

$$(i)\ |f'(0)| \leq 1 \qquad \text{and} \qquad (ii)\ |f(z)| \leq |z| \quad \text{for all } z \in \boldsymbol{\Delta}.$$

Furthermore, if equality holds in (i), or in (ii) for only a single non-zero value of z, then f is a rotation, i.e. there exists an $\alpha \in \mathbb{C}$ with $|\alpha| = 1$, such that for all $z \in \boldsymbol{\Delta}$ we have $f(z) = \alpha z$.

Proof. As $f(0) = 0$, there exists a holomorphic function $g \colon \boldsymbol{\Delta} \to \mathbb{C}$ such that for all $z \in \boldsymbol{\Delta}$ we have $f(z) = zg(z)$. We have $f'(0) = g(0)$.

Let $r < 1$. Then by assumption, for all z with $|z| = r$

$$1 > |f(z)| = |z|\,|g(z)|,$$

and hence $|g(z)| < \frac{1}{r}$. By the maximum modulus principle, $|g|$ must attain its maximum on the ball $\bar{B}_r(0)$ on the boundary and we have $|g(z)| < \frac{1}{r}$ for all $|z| \leq r$. Then letting r tend to 1 we obtain $|g(z)| \leq 1$. This implies the estimates (i) and (ii).

Now, if $|g(z_0)| = 1$ for any $z_0 \in \boldsymbol{\Delta}$, then in particular, $|g|$ attains a local maximum in z_0. Using the maximum modulus principle again, we see that g must be constant. This implies the second statement. $\square$

The following application was already referred to before in Example 2.18. Indeed, in (2.13) it was shown that every Möbius transformation of the form

$$f(z) = \mathrm{e}^{\mathrm{i}\theta} \frac{a - z}{1 - \bar{a}z} \qquad \text{for } |a| < 1 \quad \text{and} \quad \theta \in [-\pi, \pi), \tag{5.6}$$

is a bi-holomorphic bijection from the unit disc into itself. We see now, that these mappings are not only the only Möbius transformations that map $\mathbf{\Delta}$ onto itself - there are the only bi-holomorphic mappings with this property.

Corollary 5.27 (Classification of bi-holomorphic mappings of the disc) *Let $f\colon \mathbf{\Delta} \to \mathbf{\Delta}$ be bi-holomorphic. Then f is a Möbius transformation as in* (5.6).

Proof. Suppose, at first that $f\colon \mathbf{\Delta} \to \mathbf{\Delta}$ is bi-holomorphic with $f(0) = 0$. Then by Schwarz lemma $|f(z)| \le |z|$ for all $z \in \mathbf{\Delta}$. On the other hand, by applying Schwarz lemma to f^{-1} we get $|z| = |f^{-1}(f(z))| \le |f(z)|$. Therefore, f is a rotation.

In the general case, set $a = f^{-1}(0)$. Then setting $\varphi(z) = \frac{a-z}{1-\bar{a}z}$, by the first part we know that $f \circ \varphi$ is a rotation and hence $f = f \circ \varphi \circ \varphi^{-1}$ is of the form (5.6). □

In the following statement, we are going to remove the assumption that f maps 0 to 0. We have not discussed this in class and the remainder of this chapter will not be part of the exam.

Theorem 5.28 (Schwarz-Pick lemma) *Let $f\colon \mathbf{\Delta} \to \mathbf{\Delta}$ be holomorphic. Then for all $z \in \mathbf{\Delta}$*

$$\frac{|f'(z)|}{1-|f(z)|^2} \le \frac{1}{1-|z|^2}. \tag{5.7}$$

If equality holds for only one $z \in \mathbf{\Delta}$, then f is a Möbius transform of the type

$$f(z) = \mathrm{e}^{\mathrm{i}\theta}\frac{z-a}{1-\bar{a}z} \qquad \text{for} \quad |a| < 1, \quad \theta \in \mathbb{R}.$$

Remark 5.29 If $f(0) = 0$, then for $z = 0$ (5.7) reduces to $|f'(0)| \le 1$, the first statement of the Schwarz lemma.

Proof. Step 1. Let us fix a $z_0 \in \mathbf{\Delta}$. We will show that (5.7) holds for this point z_0. In order to reduce the estimate (5.7) to the Schwarz lemma, we compose f with two mappings φ_1, φ_2 such that

$$0 \xrightarrow{\varphi_1} z_0 \xrightarrow{f} f(z_0) \xrightarrow{\varphi_2} 0.$$

For the φ_1, φ_2 we choose the Möbius transformations

$$\varphi_1(z) = \frac{z+z_0}{1+\overline{z_0}z} \qquad \text{and} \qquad \varphi_2(z) = \frac{z-f(z_0)}{1-\overline{f(z_0)}z}.$$

Then $F(z) = \varphi_2 \circ f \circ \varphi_1$ maps $\mathbf{\Delta}$ to $\mathbf{\Delta}$ and $F(0) = 0$. The Schwarz lemma then implies that

$$|F'(0)| = |\varphi_1'(0)|\,|f'(z_0)|\,|\varphi_2'(f(z))| \le 1. \tag{5.8}$$

Step 2. We claim that for any a with $|a| < 1$ (and in particular, for $a = -z_0$ or for $a = f(z_0)$) the Möbius transform

$$\varphi(z) = \frac{z-a}{1-\bar{a}z}$$

satisfies for all $|z| < 1$ that

$$|\varphi'(z)| = \frac{1 - |\varphi(z)|^2}{1 - |z|^2}. \tag{5.9}$$

To see (5.9) we start calculating for the left hand side:

$$\varphi'(z) = \frac{(1 - \bar{a}z) + (z - a)\,\bar{a}}{(1 + \bar{a}z)^2} = \frac{1 - |a|^2}{(1 - \bar{a}z)^2}. \tag{5.10}$$

For the right hand side we get

$$\begin{aligned} \frac{1 - |\varphi(z)|^2}{1 - |z|^2} &= \frac{1 - \left(\frac{z-a}{1-\bar{a}z}\,\frac{\bar{z}-\bar{a}}{1-a\bar{z}}\right)}{1 - |z|^2} = \frac{1 - \bar{a}z - a\bar{z} + |a|^2\,|z|^2 - (|z|^2 - \bar{a}z - a\bar{z} + |a|^2)}{(1 - |z|^2)\,|1 - az|^2} \\ &= \frac{1 - |z|^2 - |a|^2 + |a|^2\,|z|^2}{(1 - |z|^2)\,|1 - az|^2} = \frac{1 - |a|^2}{|1 - az|^2}. \end{aligned} \tag{5.11}$$

The absolute value of the right hand side of (5.10) and (5.11) are the same, which establishes (5.9).

Step 3. Now we are ready to conclude: Plugging (5.9) into (5.8) we obtain

$$1 \geq (1 - |z_0|^2)|f'(z_0)| \left|\frac{1}{1 - |f(z_0)|^2}\right|.$$

Dividing by $(1 - |z|^2)$ the desired estimate (5.7) follows. If we have equality in (5.7), then also in (5.9). Hence, the Schwarz lemma implies that F is a rotation, and in particular a Möbius transformation. But as $f = \varphi_1^{-1} \circ F \circ \varphi_2^{-1}$, this implies that f is a Möbius transformation as well. $\square$

The estimate (5.7) has a very nice interpretation in the *Hyperbolic space*. To explain this, we need to introduce some extra facts: As discussed above, the length of a curve in $\mathbb{R}^2$ or more generally in $\mathbb{R}^n$ is usually defined as

$$L(\gamma) = \int_{t_0}^{t_1} |\gamma'(t)|\,\mathrm{d}t.$$

But sometimes one wants to take into account that moving in certain regions of space may be more costly than in others. This can be captured by a function $\varrho\colon \mathbb{R}^n \to \mathbb{R}_+$. Then the ϱ-weighted length of a curve can be defined as

$$L_\varrho(\gamma) := \int_{t_0}^{t_1} \varrho(\gamma(t))|\gamma'(t)|\,\mathrm{d}t.$$

Actually, in *Riemannian geometry* one usually considers the even more general situation, where one has a mapping g that takes values in the positive definite matrices and one sets

$$L_\varrho(\gamma) := \int_{t_0}^{t_1} \sqrt{\langle g(\gamma(t))\gamma'(t),\, \gamma'(t)\rangle}\,\mathrm{d}t.$$

Once one has defined the lengths of curves, it is natural to define the distance between two points x_0 and x_1 as

$$d(x_0, x_1) = \inf_{\gamma} L_\varrho(\gamma),$$

where the inf is taken over all curves γ that connect x_0 and x_1.

The Schwarz-Pick lemma gets a very nice form, for a particular choice of ϱ, defined on $\mathbf{\Delta}$ namely for

$$\varrho(z) = \frac{1}{1 - |z|^2}.$$

If one, endows $\mathbf{\Delta}$ with d_ϱ for this choice of ϱ it is called *Poincaré plane* or *Hyperbolic space*. This space has many interesting properties: it is a space of constant negative curvature and it is one of the model spaces for non-Euclidian geometry. One gets the following:

Theorem 5.30 (Schwarz-Pick in the Poincaré plane) *Every holomorphic function $f\colon \mathbf{\Delta} \to \mathbf{\Delta}$ is a contraction on the Poincaré plane. The Möbius transformations that map $\mathbf{\Delta}$ into $\mathbf{\Delta}$ are isometries.*

Remark 5.31 Actually, these Möbius transformations are the only isometries of the Poincaré plane.

Proof. Let $f\colon \mathbf{\Delta} \to \mathbf{\Delta}$ be holomorphic. It is sufficient to show that for any (piecewise $\mathcal{C}^1$ curve) we have

$$L_\varrho(f \circ \gamma) \leq L_\varrho(\gamma).$$

To see this, note that for any $t \in [t_0, t_1]$ we get, using (5.7)

$$\varrho(f(\gamma(t)))\, |\partial_t f(\gamma(t))| = \frac{1}{1 - |f(\gamma(t))|^2} |f'(\gamma(t))|\, |\partial_t \gamma(t)| \leq \frac{|\partial_t \gamma(t)|}{1 - |\gamma(t)|^2} = \varrho(\gamma(t))\, |\partial_t \gamma(t)|. \tag{5.12}$$

Integrating this inequality over t, we see that f decreases the lengths of curves and therefore also distances between points. For Möbius transformations we have an equality in (5.12). □

6 Singularities

6.1 Some definitions

Definition 6.1 Suppose that D is open and connected, and that $f\colon D \to \mathbb{C}$ is holomorphic on D.

(i) The function f has an isolated singularity in a point $z_0 \in \mathbb{C} \setminus D$ if for some $\varepsilon > 0$ it is defined for all $z \in B_\varepsilon(z_0) \setminus \{z_0\}$.

(ii) We say $z_0 \in D$ is a *regular point* if f is complex-differentiable at z_0.

(iii) At point $z_0 \in D$ is a singularity if z_0 is a limit of regular points (i.e. z_n regular and $z_n \to z_0$ as $n \to \infty$) and is itself not regular.

(iv) A singularity at z_0 is an isolated singularity if f is holomorphic on $B_\varepsilon(z_0) \setminus \{z_0\}$.

Example 6.2 (a) (*Trival case:*) Let $D = \mathbf{\Delta} \setminus \{0\}$ and $f(z) = z$. Then f has a singularity in the sense of the definition at 0. Of course, this singularity is completely artificial and only stems from the strange choice of domain for f.

(b) (*More serious:*) Let $D = \mathbf{\Delta} \setminus \{0\}$ and $f(z) = \frac{1}{z}$. Then f has an isolated singularity at 0. For $|z| \to 0$ we have $|f(z)| \to \infty$ and hence f can be extended to a continuous function taking values in the extended complex plane $\widehat{\mathbb{C}}$.

(c) (*Even more serious:*) Let $D = \mathbf{\Delta} \setminus \{0\}$ and set $f(z) = \exp\left(\frac{1}{z}\right)$. Then $|f(z_n)|$ converges to ∞ along the sequence $z_n = \frac{1}{n}$, but it converges to 0 along the sequence $z_n = -\frac{1}{n}$. Hence f cannot be extended to a continuous function, even if we allow for values in $\widehat{\mathbb{C}}$.

♣

In this section we will see that these three cases capture all the possible behaviours of holomorphic functions near singularities. Before we do that let us extend the notion of *order* of an holomorphic function to negative values.

Definition 6.3 Assume that f has an isolated singularity at z_0. Then we define the order

$$\operatorname{ord}(f, z_0) = -\inf\{n \in \mathbb{Z} \colon \lim_{z \to z_0} (z - z_0)^n f(z) \quad \text{exists and is finite}\}.$$

We say that f has a

- *removable singularity* if $\operatorname{ord}(f, z_0) \geq 0$, i.e. if f is bounded in a neighbourhood of z_0,
- *pole of order* n at z_0 if $\operatorname{ord}(f, z_0) = -n \in (-\infty, -1]$,
- *essential singularity* at z_0 if $\operatorname{ord}(f, z_0) = -\infty$.

Remark 6.4 Note that the definition of $\operatorname{ord}(f, z_0)$ extends Definition 5.14. For $n \geq 1$ the function

$$\begin{aligned} f(z) &= (z - z_0)^n && \text{has a zero of order } n\text{, and we have } \operatorname{ord}(f, z_0) = n > 0 \quad \text{and} \\ f(z) &= (z - z_0)^{-n} && \text{has a pole of order } n\text{, and we have } \operatorname{ord}(f, z_0) = -n < 0. \end{aligned}$$

Definition 6.5 Let $D \subset \mathbb{C}$ be open and connected and let $S \subset \mathbb{C}$ be a discrete set (i.e. every point in S is isolated). A holomorphic function $f \colon D \setminus S \to \mathbb{C}$ is called *meromorphic* on D if none of the isolated singularities in $z \in S$ are essential. Equivalently, a meromorphic function $f \colon D \to \mathbb{C}$ is holomorphic on $D \setminus \mathcal{P}$ where $\mathcal{P} := \{z \in D \colon z \text{ pole of finite order}\}$.

Remark 6.6 (a) Let $f\colon D \to \mathbb{C}$ be holomorphic on D = and $f(z_0) \neq 0, z_0 \in D$. Then, for some $m \in \mathbb{N}$,

$$g(z) := \frac{f(z)}{(z - z_0)^m}$$

has a pole of order m at z_0, i.e. $\operatorname{ord}(g; z_0) = -m$. Conversely, suppose that $g\colon D \setminus \{z_0\} \to \mathbb{C}$ has a pole of order $m \in \mathbb{N}$ at z_0, then there exists a holomorphic function (on D) $f\colon D \to \mathbb{C}$ with $f(z_0) \neq 0$ and

$$g(z) = \frac{f(z)}{(z - z_0)^m},$$

because our assumptions imply that $(z - z_0)^m g(z)$ can be extended to a holomorphic function on D (from $D \setminus \{z_0\}$). Furthermore, if $f(z_0) = 0$, then the function $h(z) := (z - z_0)^{m-1} g(z) = \frac{f(z)}{z - z_0}$ can be extended to z_0. This contradicts our assumption that the pole is of order m.

(b) Suppose that $g, h\colon D \to \mathbb{C}$ are holomorphic on D, $D \subset \mathbb{C}$ open and connected, $h \not\equiv 0$ on D. Then

$$f(z) = \frac{g(z)}{h(z)}$$

is a meromorphic function after removing all removable singularities. Note that f is a priori only defined on $D \setminus \{z \in D\colon h(z) = 0\}$.

Proof. Pick $z_0 \in D$. If $h(z_0) \neq 0$, then f is holomorphic in z_0. Suppose now that $h(z_0) = 0$. Then there exists $m \in \mathbb{N}$ such that

$$h(z) = (z - z_0)^m \widetilde{h}(z) \quad \text{with } \widetilde{h}(z_0) \neq 0,$$

(note that h cannot have a zero of infinite order), to see that recall Remark 5.19:

$$h(z) = (z - z_0)^m \sum_{n=0}^{\infty} b_n (z - z_0)^n$$

with

$$b_n = \frac{h^{(m+n)}(z_0)}{(m+n)!},$$

and thus $\widetilde{h}(z_0) \neq 0$. Likewise, there exists $k \in \mathbb{N}$ such that $g(z) = (z - z_0)^k \widetilde{g}(z)$ with $\widetilde{g}(z_0) \neq 0$. Therefore, for $0 < |z - z_0| < \varepsilon$,

$$f(z) = (z - z_0)^{k-m} \frac{\widetilde{g}(z)}{\widetilde{h}(z)},$$

and z_0 is either a pole if $(k - m) < 0$, or f can be extended to z_0 for $(k - m) \geq 0$. □

◇

Definition 6.7 Let $D \subset \mathbb{C}$ be open and connected and $f\colon D \to \mathbb{C}$ meromorphic on D. The set of zeros and the set of poles are denoted by

$$\begin{aligned} \mathfrak{Z}_f &:= \{z \in D\colon f(z) = 0\}, \\ \mathfrak{P}_f &:= \{z \in D\colon z \text{ pole of } f \text{ at } z\}. \end{aligned}$$

Proposition 6.8 *Let $D \subset \mathbb{C}$ be open and connected. Suppose that $f\colon D \to \mathbb{C}$ is meromorphic on D and $f \not\equiv 0$ on D. Then both, $\mathfrak{Z}_f$ and $\mathfrak{P}_f$, do not have an accumulation point in D.*

Proof. A pole of f is an isolated singularity, and hence it cannot be an accumulation point of poles. Likewise, any point at which f is holomorphic cannot be an accumulation point of poles. Thus $\mathfrak{P}_f$ has no accumulation point in D. Suppose now that $\mathfrak{Z}_f$ has an accumulation point $z_0 \in D$, then z_0 cannot be a pole as otherwise

$$f(z) = \frac{g(z)}{(z - z_0)^m}\,, \qquad m \in \mathbb{N}, g(z_0) \neq 0,$$

and thus $f(z) \neq 0$ for all $0 < |z - z_0| < \varepsilon$ for $\varepsilon > 0$ small enough. However, this would contradict our assumption that z_0 is an accumulation point of $\mathfrak{Z}_f$. We are left to show that $D \setminus \mathfrak{P}_f$ is open and connected, because then the *Identity Theorem* 5.21 shows that $\mathfrak{Z}_f$ cannot have an accumulation point in $D \setminus \mathfrak{P}_f$. This follows immediately with the following Lemma 6.9. □

Lemma 6.9 *Let $D \subset \mathbb{C}$ be open and connected and $M \subset D$ a subset with no accumulation point in D. Then $D \setminus M$ is open and connected.*

Proof. Suppose $z_0 \in D \setminus M$ has no neighbourhood in $D \setminus M$. Then z_0 is an accumulation point of M in D. Therefore $D \setminus M$ is open. We now show that $D \setminus M$ is connected. Pick $p, q \in D \setminus M$, and choose a continuous path $\alpha\colon [0,1] \to D$ connecting p and q, i.e. $\alpha(0) = p$ and $\alpha(1) = q$. The image $\alpha([0,1])$ is a compact set in D, and hence it is bounded and can contain only finitely many elements of M, say $m + 1$ (the set M has no accumulation point in D and thus the elements of M are separated by disjoint bounded neighbourhoods). We construct a path $\widetilde{\alpha}$ joining p and q containing only m points of M. Suppose $z_0 \in \alpha([0,1]) \cap M$. Choose $\varepsilon > 0$ such that $\{z \in \mathbb{C}\colon |z - z_0| \leq \varepsilon\} \subset D$ and such that $B_\varepsilon(z_0) \cap M = \{z_0\}$. Define

$$\begin{aligned} t_0 &:= \inf\{t \in [0,1]\colon |\alpha(t) - z_0| \leq \varepsilon\}, \\ t_1 &:= \sup\{t \in [0,1]\colon |\alpha(t) - z_0| \leq \varepsilon\}. \end{aligned}$$

To obtain $\widetilde{\alpha}$, replace $\alpha|_{[t_0,t_1]}$ by an arc of $\partial B_\varepsilon(z_0)$ to surround the point z_0. Repeat this construction until you get a path joining p and q avoiding M. □

Example 6.10 The function $f(z) = \frac{1}{(z-3)\,(z-\mathrm{i})^2} \exp(z)$ is meromorphic on $\mathbb{C}$, the function $f(z) = z \exp\left(\frac{1}{z}\right)$ is not. ♣

6.2 Laurent series

The description of an holomorphic function in a Taylor series is not always suitable, when one is close to a singularity.

Example 6.11 Consider the function $f(z) = \exp(z) + \frac{1}{z^2}$ on the annulus

$$A := \{z \in \mathbb{C}\colon 0 < |z| < R\}, \qquad R > 0.$$

By Taylor's theorem, Theorem 5.1, the function f can be developed locally in a Taylor series around every point in A. But such a Taylor series will never describe f on the whole annulus - in fact by Remark 5.2 we see that the radius of convergence for the Taylor series around z is $|z|$. On the other hand, it is quite natural to write

$$f(z) = z^{-2} + \sum_{k=0}^{\infty} \frac{z^k}{k!},$$

which is valid for all $z \in A$. ♣

Definition 6.12 (Laurent series) A *Laurent series* is a series of the form

$$\sum_{k\in\mathbb{Z}} a_k(z-z_0)^k. \tag{6.1}$$

Of course, what we mean when we write a series like (6.1) is the sum of *two series*

$$\sum_{k=0}^{\infty} a_k(z-z_0)^k \qquad \text{and} \qquad \sum_{k=1}^{\infty} a_{-k}\frac{1}{(z-z_0)^k}. \tag{6.2}$$

We say that the series (6.1) converges if both of the series in (6.2) converge. Also writing the sum as in (6.2) it follows immediately, that many of the nice properties of Taylor series observed in Section 3.3 also hold for Laurent series. In fact, the first sum in (6.2) is a usual Taylor series. Hence it converges locally uniformly on a ball $\{z \in \mathbb{C}\colon |z-z_0| < R\}$ for some $R > 0$. Inside of that ball it is an holomorphic function and it can be differentiated under the sum.

But the second sum in (6.2) is nothing but the usual Taylor series $\sum_{k=1}^{\infty} a_{-k}\xi^k$ evaluated at $\xi = (z-z_0)^{-1}$. Hence it has the same nice properties as the first sum on the set $\{|\xi| < \bar{R}\}$ for some $\bar{R} > 0$. But $\xi < \bar{R}$ if and only if $|z-z_0| > \frac{1}{\bar{R}}$. Hence we can conclude:

> Laurent series converge locally uniformly on an *annulus of convergence*, i.e., a set of the form
>
> $$A = \{z \in \mathbb{C}\colon R_1 < |z| < R_2\}.$$
>
> Laurent series describe a holomorphic function on that annulus and they can be differentiated under the sum.

Of course, as in the case of Taylor series, we cannot make a general statement about the convergence at a given point on the boundary of the annulus.

The aim of this section is to show that any holomorphic function on an annulus can be developed in a Laurent series. The essential ingredient is the following generalisation of Cauchy's integral formula.

Theorem 6.13 (Cauchy for annuli) *Let $D \subset \mathbb{C}$ be open and let $f\colon D \to \mathbb{C}$ be holomorphic on D. Suppose that for some $a \in \mathbb{C}$ and radii $0 < R_1 < R_2 < \infty$, the annulus*

$$A = \{z \in \mathbb{C}\colon R_1 < |z-a| < R_2\}$$

and its boundary ∂A are contained in D, i.e., $A \cup \partial A \subset D$, where

$$\partial A = \{z \in \mathbb{C}\colon R_1 \le |z-a| \le R_2\}\,.$$

Then for any $z \in A$ we have

$$f(z) = \frac{1}{2\pi \mathrm{i}} \int_{\partial B_{R_2}(a)} \frac{f(\xi)}{(\xi - z)}\, \mathrm{d}\xi - \frac{1}{2\pi \mathrm{i}} \int_{\partial B_{R_1}(a)} \frac{f(\xi)}{(\xi - z)}\, \mathrm{d}\xi. \tag{6.3}$$

Remark 6.14 If f is holomorphic on the ball $B_{R_2}(a)$, then the second integral in (6.3) vanishes and the formula reduces to (4.3). ◇

Proof. We can assume without loss of generality that $a = 0$. Pick a $z_0 \in A$ and let $\varepsilon > 0$ be small enough to guarantee that $\overline{B}_\varepsilon(z_0) \subset A$. Then Cauchy's integral formula 4.3 implies that

$$f(z_0) = \frac{1}{2\pi \mathrm{i}} \int_{\partial B_\varepsilon(z_0)} \frac{f(\xi)}{(\xi - z_0)}\, \mathrm{d}\xi.$$

Hence, we are done as soon as we have established that

$$\int_{\partial B_{R_2}(a)} \frac{f(\xi)}{(\xi - z_0)}\, \mathrm{d}\xi - \int_{\partial B_{R_1}(a)} \frac{f(\xi)}{(\xi - z_0)}\, \mathrm{d}\xi = \int_{\partial B_\varepsilon(z_0)} \frac{f(\xi)}{(\xi - z_0)}\, \mathrm{d}\xi.$$

The proof of this fact is very similar to the construction that was used in the proof of Cauchy's integral formula, Theorem 4.25. Indeed, the inner circle is again connected to the outer ring, this time in "thinner slices" (rather than halves) which are all contained in star shaped domains on which the function $\xi \mapsto \frac{f(\xi)}{(\xi - z_0)}$ is holomorphic. The details are left to the reader: This follows from Goursat's Theorem for the images of squares. Actually, let $\varphi\colon [0, 2\pi] \times [R_1, R_2] \to A$ be a continuous mapping with $\varphi(\theta, R_1) = R_1 \frac{z_0}{|z_0|} \mathrm{e}^{\mathrm{i}\theta}$ and $\varphi(\theta, R_2) = R_2 \frac{z_0}{|z_0|} \mathrm{e}^{\mathrm{i}\theta}$ that interpolates between these two curves but "cuts out" the ball $B_\varepsilon(z_0)$ around z_0. Then the integral over the image of the boundary curve of $[0, 2\pi] \times [R_1, R_2]$ is zero. But this boundary curve consists exactly of the curves $B_{R_2}(0)$, and the two curves $\partial B_{R_1}(0)$ and $\partial B_\varepsilon(z_0)$ with opposite orientation. In fact, there are some "connecting bits" that cancel because they appear twice with opposite orientation. □

With this preliminary result in hand, we are now ready to prove the main result of this section, namely the fact that every holomorphic function defined on an annulus can be developed in a Laurent series. Before we proceed, note that (6.2) should really be interpreted as a decomposition of the series into a part that is holomorphic at a and another part that is holomorphic at ∞. Note that the right hand side of (6.3) also has that structure.

Theorem 6.15 (Laurent's theorem) *Let f be holomorphic on a neighbourhood of the annulus*

$$A = \{z \in \mathbb{C}\colon R_1 < |z-a| < R_2\}\,, \quad 0 < R_1 < R_2 < \infty, a \in \mathbb{C}\,.$$

Then, for every $z \in A$, we have

$$f(z) = \sum_{k\in\mathbb{Z}} a_k (z-a)^k.$$

For every $\varrho \in [R_1, R_2]$ the coefficients a_k are given by

$$a_k = \frac{1}{2\pi\mathrm{i}} \int_{\partial B_\varrho(a)} \frac{f(\xi)}{(\xi-a)^{k+1}}\,\mathrm{d}\xi\,, \quad k \in \mathbb{Z}\,. \tag{6.4}$$

Proof. The argument is very similar to the proof of Taylor's theorem, Theorem 5.1. We can assume without loss of generality that $a = 0$.

We fix an arbitrary $z_0 \in A$. Then Cauchy's integral formula for annuli implies that

$$f(z_0) = \frac{1}{2\pi\mathrm{i}} \int_{\partial B_{R_2}(0)} \frac{f(\xi)}{(\xi - z_0)}\,\mathrm{d}\xi - \frac{1}{2\pi\mathrm{i}} \int_{\partial B_{R_1}(0)} \frac{f(\xi)}{(\xi - z_0)}\,\mathrm{d}\xi.$$

The first term on the right hand side can be treated exactly as in the proof of Taylor's theorem. We obtain that

$$\frac{1}{2\pi\mathrm{i}} \int_{\partial B_{R_2}(0)} \frac{f(\xi)}{\xi - z_0}\,\mathrm{d}\xi = \sum_{k=0}^{\infty} \left(\frac{1}{2\pi\mathrm{i}} \int_{\partial B_{R_2}(0)} \frac{f(\xi)}{\xi^{k+1}}\,\mathrm{d}\xi \right) z_0^k.$$

To treat the second term on the right hand side we write for any $\xi \in B_{R_1}(0)$ that

$$\frac{1}{\xi - z_0} = \frac{-1}{z_0} \left(\frac{1}{1 - \frac{\xi}{z_0}} \right) = \frac{-1}{z_0} \sum_{k=0}^{\infty} \left(\frac{\xi}{z_0} \right)^k,$$

and observe that as $z_0 \in A$ we have $|z_0| > R_1$ and thus $|\frac{\xi}{z_0}| < 1$. Hence we get

$$\begin{aligned} -\frac{1}{2\pi\mathrm{i}} \int_{\partial B_{R_1}(0)} \frac{f(\xi)}{(\xi - z_0)}\,\mathrm{d}\xi &= \frac{1}{2\pi\mathrm{i}} \int_{\partial B_{R_1}(0)} \sum_{k=0}^{\infty} \frac{f(\xi)}{z_0^{k+1}} \xi^k\,\mathrm{d}\xi \\ &= \sum_{k=-\infty}^{-1} \left(\frac{1}{2\pi\mathrm{i}} \int_{\partial B_{R_1}(0)} \frac{f(\xi)}{\xi^{k+1}}\,\mathrm{d}\xi \right) z_0^k. \end{aligned} \tag{6.5}$$

As before, interchanging the summation and the integration is justified, because the geometric series converges uniformly in the integration variable ξ.

Finally, it remains to remark that the value of the integral in (6.4) is independent of the radius $\varrho \in [R_1, R_2]$. □

Remark 6.16 (a) The proof shows that the series

$$\sum_{k\geq 0} a_k(z-a)^k$$

converges for all $z \in B_{R_2}(a)$, whereas the series

$$\sum_{k<0} a_k(z-a)^k$$

converges for all $z \in \mathbb{C} \setminus \bar{B}_{R_1}(a)$.

(b) The proof also shows the uniqueness of the Laurent decomposition. Actually, this can also be seen directly: Assume that $f(z) = \sum_{k\in\mathbb{Z}} a_k(z-a)^k$ converges in the annulus A. Then for a fixed value of $n \in \mathbb{Z}$ define

$$g_n(z) := (z-a)^n f(z) = \sum_{k\in\mathbb{Z}} a_k(z-a)^{k+n}.$$

When we integrate g around any circle of radius ϱ all the summands with $k+n \neq -1$ vanish (because they have an antiderivative) and we obtain

$$\int_{\partial B_\varrho(a)} g_n(\xi)\,\mathrm{d}\xi = 2\pi\mathrm{i}\, a_{-n-1},$$

so that we recover (6.4).

◇

Example 6.17 The function $f(z) = \frac{1}{1-z}$ is holomorphic on the annuli $A_1 = \{z \in \mathbb{C}\colon |z| < 1\}$ and on $A_2 = \{z \in \mathbb{C}\colon 1 < |z| < \infty\}$. On A_1 the Laurent series is actually a usual Taylor series and is given by the geometric series

$$f(z) = \sum_{k=0}^{\infty} z^k.$$

For $|z| > 1$ this expansion is not valid. We obtain the Laurent decomposition

$$f(z) = -\frac{1}{z}\Big(\frac{1}{1-\frac{1}{z}}\Big) = -\frac{1}{z}\sum_{k=0}^{\infty} z^{-k} = \sum_{k=-\infty}^{-1} (-1)\, z^k.$$

♣

As above in Corolllary 5.6, the boundedness of f implies bounds on all of the Laurent coefficients:

Corollary 6.18 *Let f be as in the statement of Theorem 6.15. Suppose that for some $\varrho \in [R_1, R_2]$ we have $|f(\xi)| \leq M$ on $\{z \in \mathbb{C}\colon |z-a| = \varrho\}$. Then, for every $k \in \mathbb{Z}$, we have*

$$|a_k| \leq \frac{M}{\varrho^k}.$$

6.3 Classification of singularities

With Laurent's theorem in hand we can now proceed to show that indeed the three situations explained above in Example 6.2 are the only possible isolated singularities. The following theorem is an immediate consequence of Corollary 6.18.

Theorem 6.19 (Riemann's removable singularity theorem) *Let $D \subset \mathbb{C}$ be an open set and assume that $f\colon D \to \mathbb{C}$ is holomorphic on D and that f has an isolated singularity in $z_0 \in \mathbb{C} \setminus D$. Furthermore, assume that $|f|$ is bounded in a neighbourhood of z_0. Then there exists an holomorphic function $\tilde{f}$ that extends f to $D \cup \{z_0\}$. In particular, the isolated singularity in z_0 is removable.*

Proof. We can expand f in a Laurent series in a neighbourhood of z_0. That is, we have for z close enough to z_0 the expansion

$$f(z) = \sum_{k\in\mathbb{Z}} a_k(z - z_0)^k .$$

By assumption, $|f|$ is bounded on a neighbourhood of z_0, e.g., by a constant M. Then we have for all $k < 0$ and for all $\varrho > 0$ (which are sufficiently small) that

$$|a_k| \leq \frac{M}{\varrho^k}.$$

Letting ϱ tend to zero we can conclude that all the coefficients of negative order vanish and hence f is actually given locally by a usual Taylor series, which can be extended to z_0 by the value a_0. □

Corollary 6.20 *Let $f\colon D \to \mathbb{C}$ be holomorphic on D with an isolated singularity in z_0. Then the following statements are equivalent:*

(i) *z_0 is a pole.*

(ii) *at least one of the coefficients of negative order in the Laurent series around z_0 is non-zero, but at most finitely many.*

(iii) $\lim_{z\to z_0} |f(z)| = \infty$.

Proof. $\underline{(i) \Rightarrow (ii)}$. If f has a pole at z_0 then clearly at least one of the Laurent coefficients of negative order must be non-zero, because else the singularity would be removable. On the other hand by definition there is an $n < \infty$ such that $g(z) = (z - z_0)^n f(z)$ is bounded near z_0, and hence it can be extended to an holomorphic function to z_0. Let us write

$$g(z) = \sum_{k=0}^{\infty} b_k(z - z_0)^k,$$

for the Taylor series of g around z_0. But then we have $f(z) = \sum_{k=0}^{\infty} b_k(z - z_0)^{k-n}$ for any $z \neq z_0$.

<u>*(ii)* $\Rightarrow$ *(iii)*.</u> Assume that $f(z) = \sum_{k=-n}^{\infty} a_k(z - z_0)^k$ in a neighbourhood of z_0. Then $f(z) = \frac{g(z)}{(z-z_0)^n}$ for an holomorphic function g. This function is unbounded in a neighbourhood of z_0 and hence the claim follows.

<u>*(iii)* $\Rightarrow$ *(i)*.</u> Follows from the next theorem. $\square$

Theorem 6.21 (Casorati-Weierstrass theorem) *Assume that an holomorphic function f has an essential singularity in z_0. Then for any $\varepsilon > 0$ the image of the set $B_\varepsilon(z_0) \setminus \{z_0\}$ under f is dense in $\mathbb{C}$.*

Proof. Assume the opposite, i.e., suppose that there exists an $\varepsilon > 0$ and a point $w \in \mathbb{C}$ and some $\delta > 0$ such that

$$|f(z) - w| \geq \delta \qquad \text{for all} \quad z \in B_\varepsilon(z_0) \setminus \{z_0\}.$$

Then consider the function

$$g(z) = \frac{1}{f(z) - w}.$$

By assumption g is bounded in a neighbourhood of z_0 and hence it can be extended to an holomorphic function onto all of $B_\varepsilon(z_0)$. But this is not possible, because then

$$f(z) = \frac{1}{g(z)} + w,$$

cannot have an essential singularity at z_0. $\square$

Example 6.22 (Injective entire functions) The Casorati-Weierstrass theorem implies that the only *injective* entire functions f are the linear functions

$$f(z) = \alpha z + \beta$$

for $\alpha \neq 0$ and $\beta \in \mathbb{C}$. To see this, let f be an injective entire function and denote by

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

its Taylor decomposition around 0. For $z \neq 0$ define

$$g(z) = f(z^{-1}) = \sum_{k=0}^{\infty} a_k z^{-k}.$$

As the composition of two injective functions, g must also be injective on $\mathbb{C} \setminus \{0\}$. Hence, by the Casorati-Weierstrass theorem and the open mapping theorem, g cannot have an essential singularity in 0, which implies that only finitely many a_k are non-zero. Therefore, f is a polynomial. But by the *Fundamental theorem of Algebra*, Corollary 5.11, the only injective polynomials are linear. $\diamond$

Remark 6.23 Actually, there is even a much stronger theorem about the behaviour of holomorphic functions near isolated singularities: *Picard's theorem* states that an holomorphic function f attains any value in $\mathbb{C}$ with at most *one* exception in any neighbourhood of an essential singularity. For example, it is easy to see that the function $f(z) = \exp(z^{-1})$ attains any value in $\mathbb{C} \setminus \{0\}$ on $B_\varepsilon(0) \setminus \{0\}$. The proof of this result requires more effort and will be omitted.

7 Winding numbers and the Residue theorem

7.1 The winding number

We want to address the following question. Suppose $f\colon D \to \mathbb{C}$ is holomorphic on D ($D \subset \mathbb{C}$ open and connected). Let $\gamma_1, \ldots, \gamma_n$, be closed piece-wise $\mathcal{C}^1$ curves (paths) in D and denote

$$\gamma := \alpha_1\gamma_1 \oplus \cdots \oplus \alpha_n\gamma_n\,, \qquad z_i \in \mathbb{Z}, i = 1, \ldots, n,$$

to be a *cycle* which is a formal linear combination (joint of all curves taking their direction into account) of closed (piece-wise) $\mathcal{C}^1$ curves (paths). We then write

$$\int_\gamma f(z)\,\mathrm{d}z := \alpha_1 \int_{\gamma_1} f(z)\,\mathrm{d}z + \cdots + \alpha_n \int_{\gamma_n} f(z)\,\mathrm{d}z.$$

Question: Let $D \subset \mathbb{C}$ be open and connected. For which cycles γ in D we have

$$\int_\gamma f(z)\,\mathrm{d}z = 0 \ \text{ for all } \ f \in \mathcal{H}(D)?$$

Let $z_0, z_1 \in \mathbb{C} \setminus \{0\}$ with $z_0/|z_0| \neq -z_1/|z_1|$. Then there exists a unique $\theta \in (-\pi, \pi)$ such that

$$\frac{z_0}{|z_0|}\mathrm{e}^{\mathrm{i}\theta} = \frac{z_0}{|z_1|}.$$

see Figure 15.

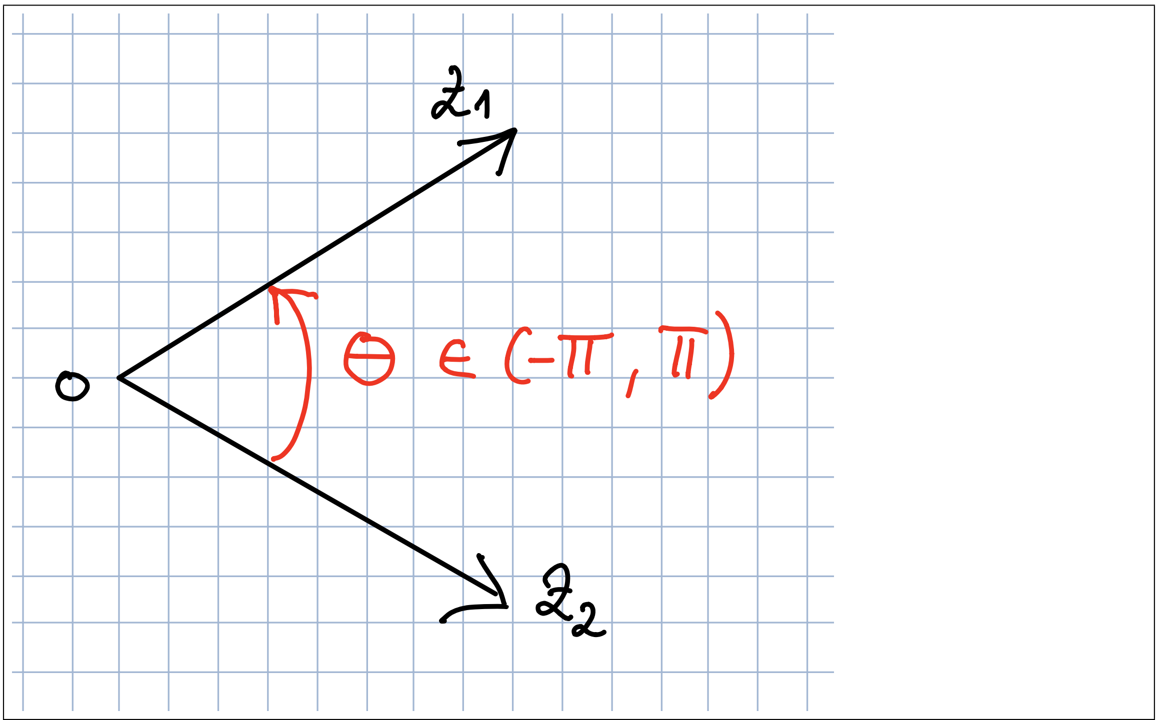

Figure 15:

We write $\measuredangle_{z_0}^{z_1} = \theta$.

Definition 7.1 (a) The (piece-wise) $\mathcal{C}^1$ curve (path) $\gamma\colon [t_0, t_1] \to \mathbb{C}$ is a *half-plane curve* half-plane curve if $\gamma([t_0, t_1])$ is wholly contained in a half-plane whose boundary line goes through the origin 0, see Figure 16.

Define

$$\measuredangle\gamma := \measuredangle_{\gamma(t_0)}^{\gamma t_1}.$$

(b) Let $\gamma\colon [t_0, t_1] \to \mathbb{C} \setminus \{0\}$ be a (piece-wise) $\mathcal{C}^1$ curve (path). Suppose that $t_0 = \tau_0 \leq \tau_1 \leq \cdots \leq \tau_n = t_1, n \in \mathbb{N}$, is a partition of the interval $[t_0, t_1]$ such that

$$\gamma|_{[\tau_{i-1}, \tau_i]} \text{ half-plane curve for all } i = 1, \ldots, n.$$

Then define

$$\measuredangle\gamma := \sum_{i=1}^{n} \measuredangle\gamma|_{[\tau_{i-1}, \tau_i]}. \tag{7.1}$$

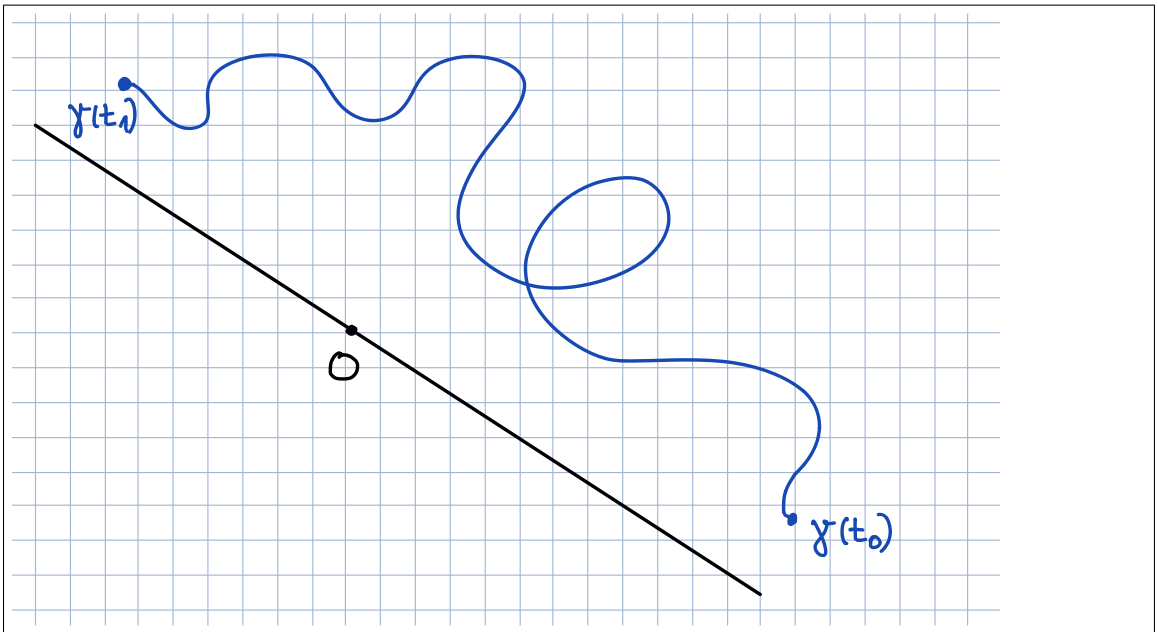

Figure 16:

Note that Definition 7.1 is independent of the choice of the partition of the interval $[t_0, t_1]$ (Exercise!).

Lemma 7.2 *Suppose $\gamma\colon [t_0, t_1] \to \mathbb{C} \setminus \{0\}$ is a closed (piece-wise) $\mathcal{C}^1$ curve (path). Then there exists* **Ind**$(\gamma, 0) \in \mathbb{Z}$ *such that*

$$\measuredangle\gamma =: 2\pi\, \mathbf{Ind}(\gamma, 0).$$

Proof. Define

$$\theta_i := \measuredangle_{\gamma(\tau_{i-1})}^{\gamma(\tau_i)}, \qquad i = 1, \ldots, n,$$

for some partition $\tau_0 \leq \tau_1 \leq \cdots \leq \tau_n = t_1$ of the interval $[t_0, t_1]$ such that each

$$\gamma|_{[\tau_{i-1},\tau_i]}$$

is a half-plane curve. Then

$$\frac{\gamma(t_0)}{|\gamma(t_0)|}\mathrm{e}^{\mathrm{i}(\theta_1+\cdots+\theta_n)} = \frac{\gamma(t_1)}{|\gamma(t_1)|}.$$

Now, as γ is closed, we have that $\theta := \sum_{i=1}^n \theta_i = 2\pi k$ for some $k \in \mathbb{Z}$, and we set $\mathbf{Ind}(\gamma, 0) = k$. □

Remark 7.3 For any $a \in \mathbb{C}$, let $\widetilde{\gamma}$ be the curve (path) $\widetilde{\gamma} = \gamma - a, \widetilde{\gamma}(t) = \gamma(t) - a$ for all $t \in [t_0, t_1]$, where $\gamma\colon [t_0, t_1] \to \mathbb{C} \setminus \{a\}$ is a closed (piece-wise) $\mathcal{C}^1$ curve (path). Then

$$\mathbf{Ind}(\gamma, a) := \mathbf{Ind}(\widetilde{\gamma}, 0). \tag{7.2}$$

◇

Definition 7.4 Let $a \in \mathbb{C}$ and let $\gamma\colon [t_0, t_1] \to \mathbb{C}$ be a closed, piecewise $\mathcal{C}^1$ curve with $a \notin \gamma([t_0, t_1])$. Then the *index* of γ with respect to a is defined as the number $\mathbf{Ind}(\gamma, a)$ given in Lemma 7.2. The index $\mathrm{ind}(\gamma, a)$ is also called *winding number* of γ about a.

Proposition 7.5 *Let $a \in \mathbb{C}$ and let $\gamma\colon [t_0, t_1] \to \mathbb{C}$ be a closed, piecewise $\mathcal{C}^1$ curve with $a \notin \gamma([t_0, t_1])$. Then*

$$\mathbf{Ind}(\gamma, a) := \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{1}{z-a}\,\mathrm{d}z. \tag{7.3}$$

Proof of Proposition 7.5. We give a direct proof. A second alternative way to prove the statement is using the complex logarithm. With loos of generality we assume that $a = 0$. Consider the partition of $[0, 1]$ as in Definition 7.1 above. Namely, $a \notin \gamma([0, 1])$,

$$0 = t_0 = \tau + 0 \leq \tau_1 \leq \cdots \leq \tau_n = t_1 = 1\,.$$

Denote α_i the straight line connecting $\gamma(\tau_i)$ and $\gamma(\tau_i)/|\gamma(\tau_i)|$, see Figure 17. Denote β_i the shorter arc (of the unit circle around the origin 0) from $\gamma(\tau_{i-1})/|\gamma(\tau_{i-1})|$ to $\gamma(\tau_i)/|\gamma(\tau_i)|$. Then

$$\int_{\gamma|_{[\tau_{i-1},\tau_i]}} \frac{\mathrm{d}z}{z} = \int_{\alpha_{i-1}} \frac{\mathrm{d}z}{z} + \int_{\beta_i} \frac{\mathrm{d}z}{z} - \int_{\alpha_i} \frac{\mathrm{d}z}{z}$$

for $i = 1, \ldots, n$. Therefore,

$$\int_\gamma \frac{\mathrm{d}z}{z} = \sum_{i=1}^n \int_{\gamma|_{[\tau_{i-1},\tau_i]}} \frac{\mathrm{d}z}{z}\,.$$

Note that $\alpha_0 = \alpha_n$ because $\gamma(t_0) = \gamma(\tau_0) = \gamma(t_1) = \gamma(\tau_n)$. Finally we get

$$\sum_{i=1}^n \int_{\beta_i} \frac{\mathrm{d}z}{z} = 2\pi \mathrm{i} \sum_{i=1}^n \theta_i\,.$$

□

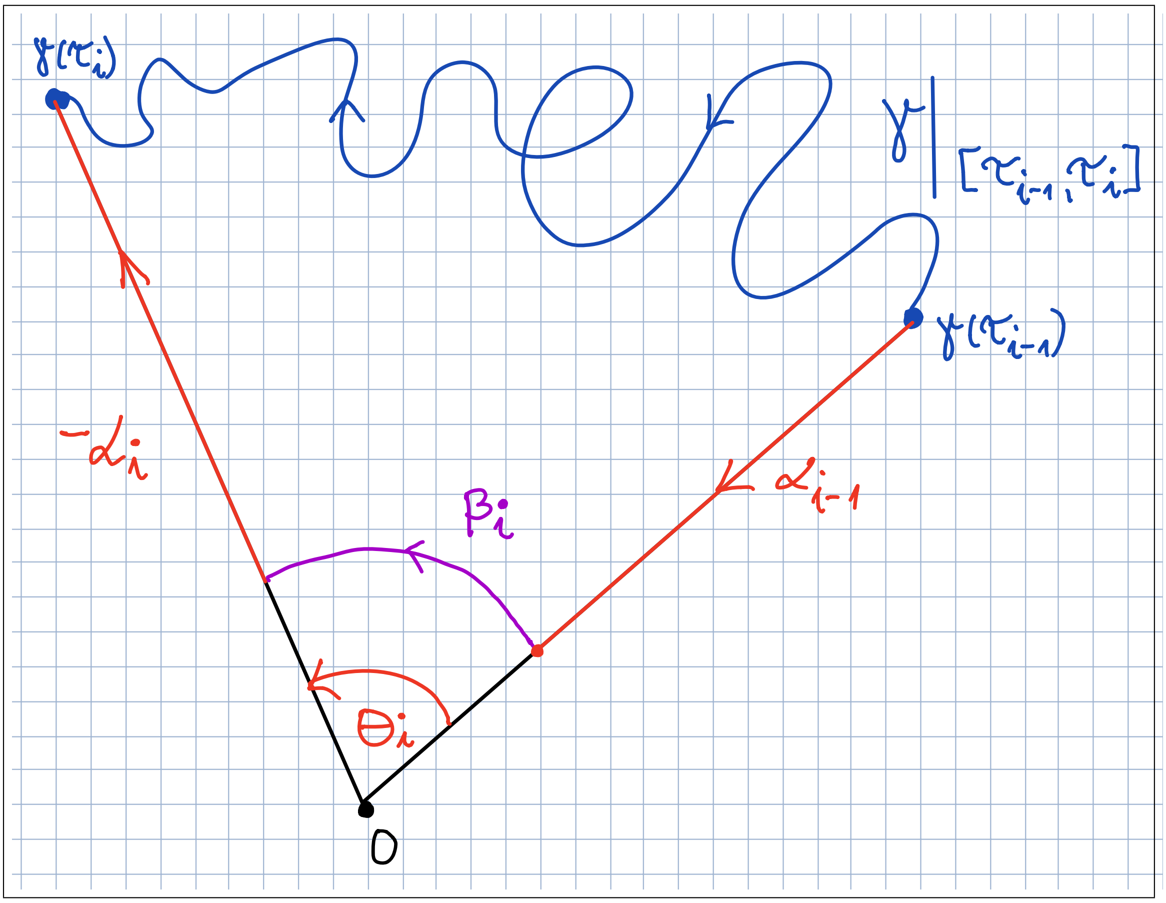

Figure 17:

Remark 7.6 (a) Proposition 7.5 holds for any cycle γ, that is,

$$\gamma = \oplus_{i=1}^{n} \alpha_i \gamma_i, \alpha_i \in \mathbb{Z},$$

is a formal linear combination, by noting that

$$\mathbf{Ind}(\gamma, a) = \sum_{i=1}^{n} \alpha_i \,\mathbf{Ind}(\gamma_i, a).$$

(b) A cycle winds around a if $\mathbf{Ind}(\gamma, a) \neq 0$.

(c) We leave the following as an Exercise for the reader. For any closed piecewise $\mathcal{C}^1$ curve γ the set $\{z \in \mathbb{C} \colon \mathbf{Ind}(\gamma, z) \neq 0\}$ is bounded.

◇

Example 7.7 Consider the curve $\gamma \colon [0, 1] \to \mathbb{C}$ given by $\gamma(t) = \mathrm{e}^{2\pi int}$. Then we have

$$\mathbf{Ind}(\gamma, 0) = \frac{1}{2\pi \mathrm{i}} \int_0^1 \frac{1}{\mathrm{e}^{2\pi \mathrm{i}nt}} (2\pi \mathrm{i}n) \mathrm{e}^{2\pi \mathrm{i}nt} \,\mathrm{d}t = n.$$

The index counts the number of times γ winds around the origin (which justifies the term *winding number*). Note that the index can be negative: If we consider instead the curve

$\tilde{\gamma}(t) = \mathrm{e}^{-2\pi i n t}$ then we have

$$\mathbf{Ind}(\tilde{\gamma}, 0) = -n.$$

♣

Definition 7.8 (a) Let $D \subset \mathbb{C}$ be open. A pice-wise $\mathcal{C}^1$ curve in D is *homologous* to 0 in D if for any $a \in \mathbb{C} \setminus D$ we have

$$\mathbf{Ind}(\gamma, a) = 0.$$

(b) A (piece-wise) $\mathcal{C}^1$ cycle is homologous to zero in D if $\mathbf{Ind}(\gamma, a)$ for every $a \in \mathbb{C} \setminus D$.

Remark 7.9 The point of this definition is that a cycle can be homologous to zero in D even if the individual curves γ_i are not. For example, in the proof of Cauchy's integral formula (Theorem 4.25) we wanted to argue that an integral over the boundary γ_1 of a large circle is the same as the integral over a smaller circle γ_2. Both of these circles have index 1 with respect to a point z_0 in the middle, and are not homologous to zero in $\mathbb{C} \setminus \{z_0\}$. But the cycle $\gamma_1 \oplus (-1)\gamma_2$ is homologous to zero in $D \setminus \{z_0\}$. ◇

Lemma 7.10 *The mapping* $a \mapsto \mathbf{Ind}(\gamma, a)$ *is locally constant in* $\mathbb{C} \setminus \gamma([t_0, t_1])$*. More precisely, if* $z_0 \in \mathbb{C} \setminus \gamma([t_0, t_1])$*, then there exists a* $\delta > 0$ *such that* $\mathbf{Ind}(\gamma, a)$ *is constant for* $a \in B_\delta(z_0)$*.*

Proof. As $\mathbb{C} \setminus \gamma([t_0, t_1])$ is open, there exists a $\delta > 0$ such that $B_{2\delta}(z_0)$ is also contained in $\mathbb{C} \setminus \gamma([t_0, t_1])$. Then on $B_\delta(z_0)$ the mapping $a \mapsto \mathbf{Ind}(\gamma, a)$ is continuous.

To see that, suppose that $a_n \in B_\delta(z_0)$ converge to $a \in B_\delta(z_0)$. We have

$$\mathbf{Ind}(\gamma, a_n) = \frac{1}{2\pi i} \int_\gamma \frac{1}{\xi - a_n} \, \mathrm{d}\xi.$$

The function $\frac{1}{\xi - a_n}$ converges uniformly in $\xi \in B^{\mathrm{c}}_{2\delta}(z_0)$ to $\frac{1}{\xi - a}$ and hence we can pass to the limit in the integrals.

Since the index only attains integer values we can conclude that it has to be constant. □

We are now ready to give the most general version of Cauchy's theorem.

Theorem 7.11 (Cauchy's theorem – homology version) *Let* $D \subset \mathbb{C}$ *be open and connected. Let* γ *be a piecewise* $\mathcal{C}^1$ *cycle that is* homologous to 0 *in* D*. Then for any holomorphic function* $f \colon D \to \mathbb{C}$ *we have*

$$\int_\gamma f(z) \, \mathrm{d}z = 0.$$

Proof. We start by reducing the theorem to the case of a single closed curve $\gamma \in D$. Indeed, assume the cycle γ is given by

$$\gamma = \alpha_1 \gamma_1 \oplus \ldots \oplus \alpha_N \gamma_N. \tag{7.4}$$

Furthermore, assume that each of the curves γ_i is parametrised over $[0,1]$. For $i = 1,\dots,n-1$ let β_i be a $\mathcal{C}^1$ curve in U that connects $\gamma_i(0)$ to $\gamma_{i+1}(0)$. Finally, let $\hat{\gamma}$ be the closed curve that is obtained by following around γ_1 α_1 times, then following β_1 to $\gamma_2(0)$. In the same way, one follows through all of the γ_i α_i times. Finally, the curve traces back along all of the β_i to its starting point $\gamma_1(0)$. The curve is closed and piecewise $\mathcal{C}^1$. Furthermore, we have $\int_\gamma f(z)\,\mathrm{d}z = \int_{\hat{\gamma}} f(z)\,\mathrm{d}z$ for every continuous function f because the auxiliary curves β_i are passed exactly once in each orientation, and hence they don't contribute. In particular, $\hat{\gamma}$ is homologous to zero in D.

For now on we are thus going to assume that γ is a single closed curve. We use the notation $\gamma([t_0,t_1]) = \Gamma$.

We can assume without loss of generality that D is bounded. Indeed if it is not replace D by $D \cap B_R(0)$ for some R large enough to ensure that Γ is contained in $B_R(0)$. By compactness of Γ the distance of Γ to $\mathbb{C} \setminus D$ is strictly positive - let us denote it by 2δ.

We put a grid of width δ on $\mathbb{C}$, i.e. we consider the open squares whose corners are given by the four points in $((k+\{0,1\})\delta, (m+\{0,1\})\delta)$. We denote by $(Q_i)_{i=1}^K$ the finitely many open squares that are fully contained in D. By the assumption on the distance to the boundary the set Γ is fully contained in $\cup_{i=1}^K \bar{Q}_i$.

We observe that any integral over $\partial \cup_{i=1}^K \bar{Q}_i$ can be written as sum over integrals over the ∂Q_i because all the *interior edges* are crossed twice with opposite direction. Also the winding number of γ about any point in $a \in \mathbb{C} \setminus \cup_{i=1}^K \bar{Q}_i$ and even any point in the boundary is zero as follows easily from Lemma 7.10.

Now let z_0 be an arbitrary point in some square Q_{i_0}. Then we have that

$$f(z_0) = \frac{1}{2\pi\mathrm{i}} \int_{\partial Q_{i_0}} \frac{f(\xi)}{\xi - z_0}\,\mathrm{d}\xi.$$

(In fact, we have not stated Cauchy's integral formula for squares above, but the proof is identical to the argument for Theorem 4.25). For any other square Q_i for $i \neq i_0$ we have

$$\frac{1}{2\pi\mathrm{i}} \int_{\partial Q_i} \frac{f(\xi)}{\xi - z_0}\,\mathrm{d}\xi = 0.$$

Hence summing over all boxes we get

$$f(z_0) = \frac{1}{2\pi\mathrm{i}} \sum_{i=1}^K \int_{\partial Q_i} \frac{f(\xi)}{\xi - z_0}\,\mathrm{d}\xi = \frac{1}{2\pi\mathrm{i}} \int_{\partial \cup_{i=1}^K \bar{Q}_i} \frac{f(\xi)}{\xi - z_0}\,\mathrm{d}\xi.$$

By continuity it is easy to see that this formula holds for any point in the interior of $\cup_i \bar{Q}_i$ (i.e. also on the interior boundaries of squares).

Now we integrate this identity over γ and obtain

$$\int_\gamma f(z)\,\mathrm{d}z = \int_\gamma \frac{1}{2\pi\mathrm{i}} \left(\int_{\partial \cup_{i=1}^K \bar{Q}_i} \frac{f(\xi)}{\xi - z}\,\mathrm{d}\xi \right) \mathrm{d}z = \int_{\partial \cup_{i=1}^K \bar{Q}_i} f(\xi) \left(\frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{1}{\xi - z}\,\mathrm{d}z \right) \mathrm{d}\xi.$$

Here the interchange of integrals is justified by Fubini's theorem, because all the integrands involved are bounded. Now we see that for any ξ the inner integral is minus the index of γ with respect to the point ξ on the boundary of $\cup_i \bar{Q}_i$. We had seen above that this is 0. $\square$

Corollary 7.12 (Cauchy's integral formula –general version) *Let $D \subset \mathbb{C}$ be open and connected and let γ be a closed piecewise $\mathcal{C}^1$ curve in D that is homologous to 0 in D. Then for any holomorphic function $f\colon D \to \mathbb{C}$ and for any $z \in D$ we have*

$$\frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{f(\xi)}{\xi - z}\,\mathrm{d}\xi = f(z)\,\mathbf{Ind}(\gamma, z).$$

Proof. The function

$$g(\xi) = \frac{f(\xi) - f(z)}{\xi - z}$$

is holomorphic on $D \setminus \{z\}$ with a removable singularity at z. Hence it can be extended to an holomorphic function onto z. Then Theorem 7.11 implies that

$$\int_\gamma g(\xi)\,\mathrm{d}\xi = 0.$$

Then we can write that

$$\frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{f(\xi)}{\xi - z}\,\mathrm{d}\xi = \frac{1}{2\pi\mathrm{i}} \int_\gamma g(\xi)\,\mathrm{d}\xi + \frac{f(z)}{2\pi\mathrm{i}} \int_\gamma \frac{1}{\xi - z}\,\mathrm{d}\xi = f(z)\,\mathbf{Ind}(\gamma, z)$$

as claimed. □

7.2 The Residue theorem

The residue theorem is a consequence of the general Cauchy theorem and it concerns integrals over functions that have singularities. Its derivation is not difficult at this stage. Still it is remarkably useful as we will see below.

We start with a definition.

Definition 7.13 Let f be a holomorphic with an isolated singularity at z_0. The *residue* of f at z_0 is defined as

$$\mathbf{res}(f, z_0) := \frac{1}{2\pi\mathrm{i}} \int_{\partial B_\varepsilon(z_0)} f(\xi)\,\mathrm{d}\xi, \tag{7.5}$$

for $\varepsilon > 0$ small enough (such that f is holomorphic on a ball $B_\delta(z_0) \setminus \{z_0\}$ for some $\delta > \varepsilon$).

Remark 7.14 (a) Note that the integral in (7.5) is independent of the choice of ε as long as f is holomorphic on $B_\varepsilon(z) \setminus \{z\}$.

(b) The expression (6.4) in Laurent's theorem shows that res(f, z_0) is the coefficient of order -1 in the Laurent expansion of f around z_0. In practice, this gives a much more direct way to calculating residues than the evaluation of (7.5). Actually, suppose that f has a pole of order n in z_0 and that $f(z) = \sum_{k=-n}^{\infty} a_k(z - z_0)^k$ is the Laurent series of f around z_0. Then the function $g(z) = (z - z_0)^n f(z) = \sum_{k=0}^{\infty} a_{k-n}(z - z_0)^k$ is holomorphic in z_0. Differentiating $(n-1)$ times we obtain

$$\mathbf{res}(f, z_0) = a_{-1} = \frac{g^{(n-1)}(z_0)}{(n-1)!} = \lim_{z \to z_0} \frac{1}{(n-1)!} \frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}}\Big((z - z_0)^n f(z)\Big).$$

In the case where $f(z) = \frac{h_1(z)}{h_2(z)}$ and h_2 has a simple zero in z_0, this expression becomes

$$\mathbf{res}(f, z_0) = \lim_{z \to z_0} \frac{h_1(z)}{(z - z_0)^{-1} h_2(z)} = \frac{h_1(z_0)}{h_2'(z_0)}.$$

◇

Example 7.15 (a) The function

$$f(z) = \frac{1}{(z^2 + 1)(z - 4)^3}$$

has a simple pole at $\pm\mathrm{i}$ and a pole of order 3 in 4. In a neighbourhood of i we write

$$f(z) = \underbrace{\frac{1}{z - \mathrm{i}}}_{\text{simple pole}} \quad \underbrace{\frac{1}{(z + \mathrm{i})(z - 4)^3}}_{\text{holomorphic near i}},$$

which yields $\mathbf{res}(f, \mathrm{i}) = \frac{1}{2\mathrm{i}(\mathrm{i}-4)^3}$. In the same way, one obtains that $\mathrm{res}(f, -i) = \frac{1}{-2\mathrm{i}(-\mathrm{i}-4)^3}$. Near 4 we can write

$$g(z) := (z - 4)^3 f(z) = \frac{1}{z^2 + 1}.$$

Hence

$$\mathbf{res}(f, 4) = \frac{1}{2} g''(4) = \frac{1}{2} \frac{6z^2 - 2}{(z^2 + 1)^3}\bigg|_{z=4} = \frac{47}{4913}.$$

(b) The function $f(z) = z^n \exp\left(\frac{1}{z}\right)$ has an essential singularity at 0 and hence the strategy using derivatives outlined in Remark 7.14 does not apply directly. Still it is easy to see that

$$z^n \exp\left(\frac{1}{z}\right) = \sum_{k=0}^{\infty} \frac{1}{k!} z^{n-k}.$$

Hence we can read off that $\mathbf{res}(f, 0) = \frac{1}{(n+1)!}$.

♣

The residue is important because of the following theorem.

Theorem 7.16 (Residue theorem) *Let D be an open and connected set. Assume that f is holomorphic on D except for a discrete set S of isolated singularities. Let γ be a closed piecewise $\mathcal{C}^1$ curve in D that is homologous to 0 in D and that does not go through any of the singularities in S.*

Then γ winds around at most a finite number of singularities in S and we have

$$\int_\gamma f(z)\,\mathrm{d}z = 2\pi\mathrm{i} \sum_{a \in S} \mathbf{Ind}(\gamma, a)\, \mathbf{res}(f, a). \tag{7.6}$$

Proof. We start by showing that γ winds around at most finitely many of the isolated singularities. In fact, by Remark 7.6 (c), we see that the set

$$A := \{a \in S \colon \operatorname{ind}(\gamma, a) \neq 0\}$$

is bounded. Hence, if we assume that there exists a sequence (a_n) of pairwise distinct points in A, there must be a subsequence of the a_n that converges to a point a. This point a is either in the image of γ, or $\mathbf{Ind}(\gamma, a) \neq 0$ because the index is locally constant. In either case $a \in D$. But this is a contradiction, because by assumption either f is holomorphic in a and hence in a whole neighbourhood, or there is a singularity at a which is isolated.

To show the residue formula (7.6), let $a_1, \ldots, a_N$ be the isolated singularities that γ winds about. Denote by $n_i = \mathbf{Ind}(\gamma, a_i)$. Furthermore, let $\varepsilon > 0$ be small enough to ensure that the balls of radius ε around the a_i do not touch the image of γ. Then for $i = 1, \ldots, N$ let γ_i, be the curve

$$\gamma_i(t) = a_i + \varepsilon \mathrm{e}^{\mathrm{i}2\pi t} \qquad \text{for } t \in [0, 1],$$

We consider the cycle

$$\lambda = \gamma \oplus (-n_1)\gamma_1 \oplus (-n_2)\gamma_2 \oplus \ldots \oplus (-n_N)\gamma_N.$$

By construction, the cycle λ does not wind about any point in S. Hence by the general Cauchy theorem, Theorem 7.11 we have that $\int_\lambda f(z)\,\mathrm{d}z = 0$.

Hence we can conclude that

$$\int_\gamma f(z)\,\mathrm{d}z = \sum_{i=1}^{N} n_i \int_{\gamma_i} f(z)\,\mathrm{d}z.$$

By the definition of the γ_i and the definition of the residue in (7.5), we obtain the desired expression (7.6). □

Example 7.17 We give the calculation of real integrals as the first application of the residue theorem. We treat the simplest case only and leave more tricky examples for the exercises. Let p be a real polynomial of degree at least 2 without any zeros on $\mathbb{R}$ (in particular p has even degree). The Residue Theorem gives a way to explicitly evaluate the integral

$$\int_{-\infty}^{\infty} \frac{1}{p(x)}\,\mathrm{d}x.$$

Note that the conditions on p imply that $\frac{1}{p}$ is indeed integrable. Furthermore, we have

$$\int_{-\infty}^{\infty} \frac{1}{p(x)}\,\mathrm{d}x = \lim_{R\to\infty} \int_{-R}^{R} \frac{1}{p(x)}\,\mathrm{d}x.$$

In order to apply the Residue Theorem we introduce the auxiliary curves (see Figure 18)

$$\gamma_1^R(t) = tR \qquad t \in [-1, 1] \qquad \gamma_2^R(t) = R\mathrm{e}^{\mathrm{i}\pi t} \quad t \in [0, 1]$$

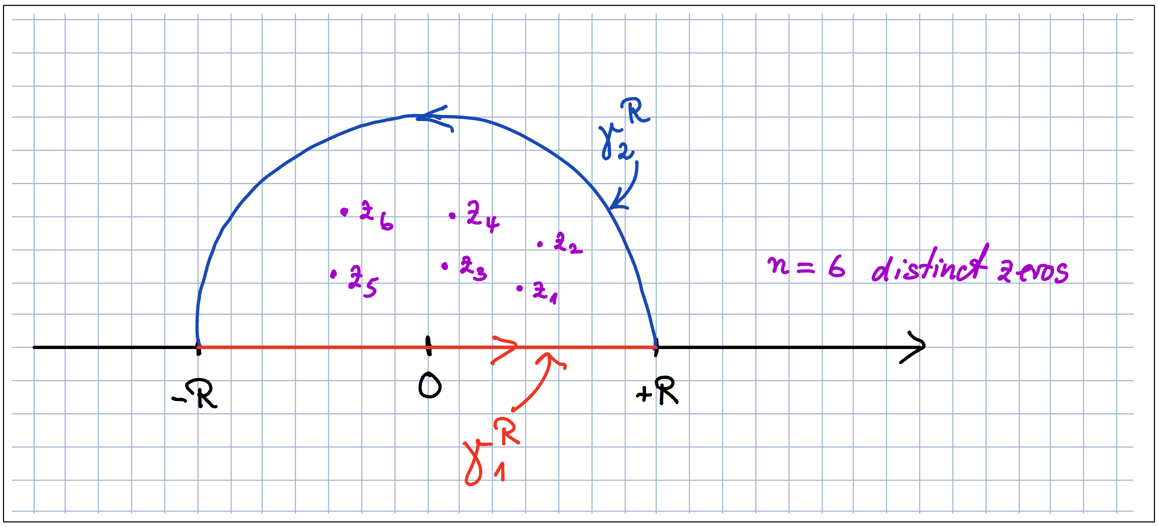

Figure 18:

and we denote by γ^R the concatenation of these two curves. Denote by $z_1, \ldots, z_n$ the distinct zeros of p in the upper half plane. Then for R large enough the residue theorem implies that

$$\int_{\gamma^R} \frac{1}{p(z)}\,\mathrm{d}z = 2\pi\mathrm{i}\sum_{j=1}^{n} \mathbf{res}\Big(\frac{1}{p}, z_j\Big).$$

On the other hand, as p is of degree at least 2 there exists an $R_0 > 0$ and a C such that if $|z| > R_0$, then $|p(z)| \geq C|z|^2$. Hence we obtain, for $R > R_0$,

$$\int_{\gamma_2^R} \frac{1}{p(z)}\,\mathrm{d}z \leq R\pi\frac{1}{CR^2} \to 0 \qquad \text{as } R \to \infty.$$

So finally we can conclude that

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1}{p(x)}\,\mathrm{d}x &= \lim_{R\to\infty}\int_{-R}^{R} \frac{1}{p(x)}\,\mathrm{d}x = \lim_{R\to\infty}\Big(\int_{\gamma^R} \frac{1}{p(z)}\,\mathrm{d}z - \int_{\gamma_2^R} \frac{1}{p(z)}\,\mathrm{d}z\Big)\\ &= 2\pi\mathrm{i}\sum_{j=1}^{n} \mathbf{res}\Big(\frac{1}{p}, z_j\Big).\end{aligned}$$

♣

Example 7.18 Show that

$$\int_0^{\infty} \frac{\sin(x)}{x}\,\mathrm{d}x = \frac{\pi}{2}.$$

$$f(z) := \frac{\mathrm{e}^{\mathrm{i}z}}{z}$$

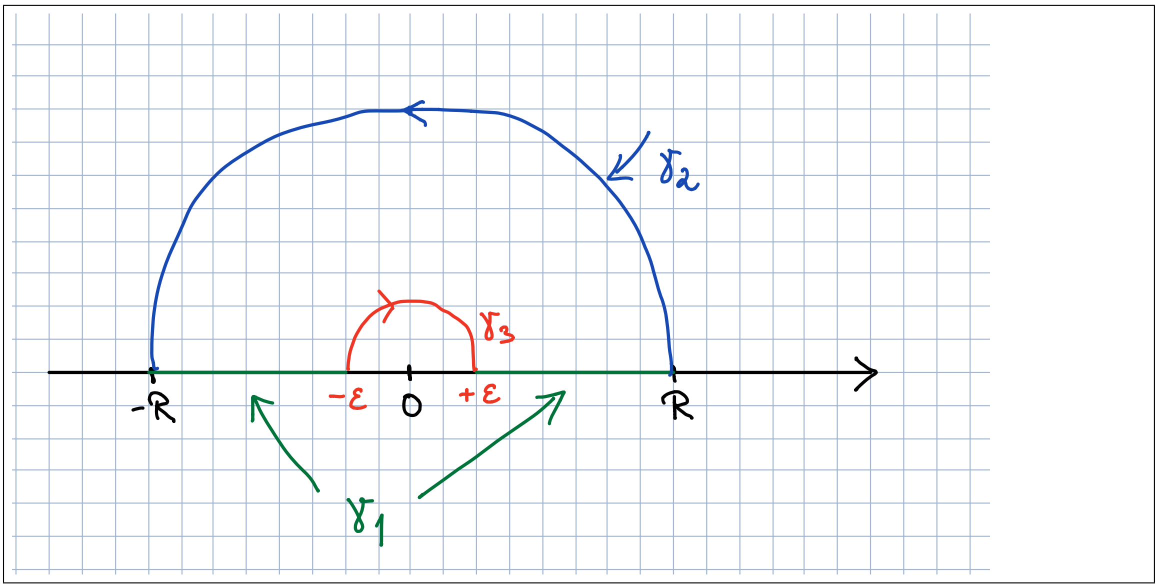

Figure 19:

has an isolated singularity at $z = 0$. Define the following curves (see Figure 19):

$$\begin{aligned}\gamma_2(t) &= R\mathrm{e}^{\mathrm{i}t\pi}, \quad t \in [0,1],\\ \gamma_1(t) &= Rt, \quad t \in [-1,-\varepsilon] \cup [\varepsilon, 1],\\ \gamma_\varepsilon(t) &= \varepsilon \mathrm{e}^{-\mathrm{i}\pi t}, \quad t \in [0,1],\\ \gamma &= \text{ concatenation of } \gamma_2, \gamma_\varepsilon, \text{ and } \gamma_1.\end{aligned}$$

As $0 \notin \mathrm{I}(\gamma)$, we have

$$\int_\gamma f(z)\,\mathrm{d}z = 0.$$

Using the Lemma 7.19 below, we conclude (details are left as an exercise). ♣

Lemma 7.19 *Suppose that $f\colon \mathbb{C} \to \mathbb{C}$ is holomorphic and*

$$|f(z)| \le \frac{M}{R^k}, \qquad \textit{for } |z| = R,$$

for some $R > 0, M > 0, k \in \mathbb{N}$. Then

$$\lim_{R\to\infty} \int_{\gamma_R} \mathrm{e}^{\mathrm{i}mz} f(z)\,\mathrm{d}z = 0, \quad m > 0,$$

where $\gamma_R\colon [0,\pi] \to \mathbb{C}, t \mapsto \gamma_R(t) = R\mathrm{e}^{\mathrm{i}t}$.

Proof. We are going to use $\sin(\theta) \ge 2\theta/\pi$ for $\theta \in [0, \pi/2]$ in the following:

$$\int_{\gamma_R} \mathrm{e}^{\mathrm{i}mz} f(z)\,\mathrm{d}z = \int_0^\pi \mathrm{e}^{\mathrm{i}mR\mathrm{e}^{\mathrm{i}\theta}} f(R\mathrm{e}^{\mathrm{i}\theta})\mathrm{i}R\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta$$

and thus

$$\begin{aligned}\Big|\int_{\gamma_R} \mathrm{e}^{\mathrm{i}mz} f(z)\,\mathrm{d}z\Big| &\le \int_0^\pi \Big|\mathrm{e}^{\mathrm{i}mR\mathrm{e}^{\mathrm{i}\theta}}\Big|\,\Big|f(R\mathrm{e}^{\mathrm{i}\theta})\Big| R\,\mathrm{d}\theta \\ &= \int_0^\pi \mathrm{e}^{-mR\sin(\theta)}\,|f(R\mathrm{e}^{\mathrm{i}\theta})|R\,\mathrm{d}\theta \\ &\le \frac{M}{R^{k-1}}\int_0^\pi \mathrm{e}^{-mR\sin(\theta)}\,\mathrm{d}\theta = \frac{2M}{R^{k-1}}\int_0^{\pi/2} \mathrm{e}^{-mR\sin(\theta)}\,\mathrm{d}\theta \\ &\le \frac{2M}{R^{k-1}}\int_0^{\pi/2} \mathrm{e}^{-2mR\theta/\pi}\,\mathrm{d}\theta = \frac{M}{mR^k}(1-\mathrm{e}^{-mR}).\end{aligned}$$

□

Notation 7.20 Properties of a function at ∞ are seen as properties of $f(\frac{1}{z})$ at 0.

$$\infty \text{ isolated singularity of } f \leftrightarrow 0 \text{ isolated singularity of } f(\frac{1}{z})$$

$$\infty \text{ pole of } f \leftrightarrow o \text{ pole of } f(\frac{1}{z})$$

$$\infty \text{ zero of order } k \leftrightarrow 0 \text{ zero of order } k \text{ for } f(\frac{1}{z})$$

Example 7.21 Let $0 < \alpha < 1$ and $R(z)$ a rational function with the following properties:
(i) $|R(z)| \le \frac{C}{|z|^2}$ for $|z| \to \infty$,
(ii) R holomorphic at 0, or R has a simple pole at $z = 0$,
(iii) R has no poles on $\mathbb{R}_+$.
Then

$$\int_0^\infty x^\alpha R(x)\,\mathrm{d}x = \frac{2\pi\mathrm{i}}{1-\mathrm{e}^{2\pi\mathrm{i}\alpha}} \sum_{z_0\in S\cap\mathbb{C}\setminus\mathbb{R}_+} \mathbf{res}(z^\alpha R(z), z_0).$$

where S is the set of poles.

♣

Example 7.22 (Exercise) Consider the real integral

$$\int_0^{2\pi} R(\cos(\theta), \sin(\theta))\,\mathrm{d}\theta$$

with R being a rational function. Then

$$\int_0^{2\pi} R(\cos(\theta), \sin(\theta))\,\mathrm{d}\theta = 2\pi\mathrm{i} \sum_{|z|<1} \mathbf{res}(Q, z)$$

when the function

$$Q(z) := \frac{1}{z} R\Big(\frac{1}{2}(z+1/z), \frac{1}{2\mathrm{i}}(z-1/z)\Big)$$

has no poles on $\partial\mathbf{\Delta}$. Hint: $z + \bar{z} = 2\cos(\theta)$ and $\bar{z} = \frac{1}{z}$ on $\partial\mathbf{\Delta}$. ♣

A more theoretical application of the residue theorem concerns the location of zeros of holomorphic functions.

Definition 7.23 Let f be a holomorphic function which is not 0 everywhere, defined on an open and connected set D. Then the *logarithmic derivative* of f is the function $\frac{f'(z)}{f(z)}$.

Remark 7.24 Locally around any point with $f(z) \neq 0$ it is always possible to define an holomorphic branch of $\log(f)$. Then it is indeed true that $\log(f)' = \frac{f'}{f}$. The point is though that the logarithmic derivative always makes sense *globally* (except for possibly some isolated singularities), the logarithm can in general only be defined locally.

Let us consider the behaviour of the logarithmic derivative near a pole or a zero of f at z_0. In either case we can write for z in a neighbourhood of z_0

$$f(z) = (z - z_0)^n g(z),$$

for some holomorphic function g which does not vanish at z_0. The exponent $n \neq 0$ is the order of f in z_0. It is positive in the case of a zero and negative in the case of a pole. For z close to z_0 the logarithmic derivative then yields

$$\frac{f'(z)}{f(z)} = \frac{1}{f(z)}\Big(n(z - z_0)^{n-1} g(z) + (z - z_0)^n g'(z)\Big) = \frac{n}{z - z_0} + \frac{g'(z)}{g(z)}.$$

As $g(z_0) \neq 0$ we can conclude that $\frac{f'}{f}$ has a simple pole with residue n at z_0. The residue theorem then immediately implies the following theorem.

Theorem 7.25 (Argument principle) *Let D be an open and connected set and let f be a meromorphic function on D. Let $A \subset D$ be open. We assume that the boundary of A is a closed piecewise $\mathcal{C}^1$ curve γ which is fully contained in D and that none of the zeros or poles of f lie on γ.*

Denote by $Z_A(f)$ the number of zeros of f in A counting multiplicity and by $P_A(f)$ the number of poles counting multiplicity. Then we have

$$\frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{f'(z)}{f(z)} \, \mathrm{d}z = Z_A(f) - P_A(f).$$

Proof. This follows immediately from the discussion above. □

The argument principle has a nice geometric interpretation. Indeed, as can be checked easily using the chain rule, we have for any piecewise $\mathcal{C}^1$ curve γ that

$$\int_\gamma \frac{f'(z)}{f(z)} \, \mathrm{d}z = \int_{f\circ\gamma} \frac{1}{z} \, \mathrm{d}z.$$

Therefore, the argument principle states that $Z_A(f) - P_A(f)$ is the winding number of $f \circ \gamma$ about 0.

Example 7.26 Consider $f(z) = z^n$ on the unit disc $\mathbf{\Delta} = \{z \in \mathbb{C}\colon |z| < 1\}$. The function f has no poles and a zero of order n in 0, so that $Z_{\mathbf{\Delta}}(f) - P_{\mathbf{\Delta}}(f) = n$. The boundary of $\mathbf{\Delta}$ is given by the curve $\gamma(t) = \mathrm{e}^{\mathrm{i}2\pi t}$, $t \in [0,1]$ and $f \circ \gamma(t) = \mathrm{e}^{\mathrm{i}2\pi n t}$. This curve winds exactly n times around 0.

This interpretation has a remarkable consequence.

Theorem 7.27 (Rouché's theorem) *Let $A \subset \mathbb{C}$ be open and connected and let $A \subset D$ be open with boundary curve γ which is closed, piecewise $\mathcal{C}^1$, and fully contained in D. Let f and g be two holomorphic functions on A. Then if for all z in the image of γ we have $|g(z)| < |f(z)|$, then f and $f+g$ have the same number of zeros in A.*

Remark 7.28 Note that the assumption $|g(z)| < |f(z)|$ implies that f and $f+g$ both have no zeros on the boundary curve γ.

Proof. Define the meromorphic function $F(z) = \frac{f(z)+g(z)}{f(z)} = 1 + \frac{g(z)}{f(z)}$. It is sufficient to show that $Z_A(F) - P_A(F) = 0$.

By the argument principle, $Z_A(F) - P_A(F)$ is the winding number of $F \circ \gamma$ about 0. But for all z in the image of γ we have by assumption that $|F(z) - 1| < 1$. Hence the image of γ under F is contained in a ball of radius one around 1 and in particular the winding number about 0 must be 0. □

Example 7.29 Rouché's theorem gives an easy way to calculate the number of zeros of a given function in a given set without solving any complicated equations. For example, assume we want to know how many zeros the function $h(z) = 10z^4 + \exp(z)^2 + (z^2+1)$ has in the unit disc. Solving the equation $h(z) = 0$ seems quite cumbersome. But if we set $f(z) = 10z^4$ and $g(z) = \exp(z)^2 + (z^2+1)$, then we have for $|z| = 1$ that

$$10 = |f(z)| > \mathrm{e}^2 + 2 \geq |g(z)|.$$

As f has 4 zeros in the unit disc (one zero of multiplicity 4), so does $h = f + g$. Of course, this does not imply that h has a single zero of multiplicity four. The four zeros will in general be at distinct points in the unit disc. ♣

Example 7.30 If we already know (for example by Rouché's theorem) that a holomorphic function f has only a single zero in an open and connected set A (with boundary curve γ as above), then the location of this zero can also be calculated using an integral. If z_0 denotes the location of the zero, then we have

$$z_0 = \frac{1}{2\pi\mathrm{i}} \int_\gamma z \frac{f'(z)}{f(z)} \,\mathrm{d}z.$$

To see this, note that $\frac{zf'(z)}{f(z)}$ is holomorphic outside of z_0 and at z_0 it as a simple pole of residue z_0. Hence, the result follows from the residue theorem.

♣

8 Sequences of holomorphic functions

8.1 Locally uniform convergence of holomorphic functions

In this section we treat convergence properties of holomorphic functions. Recall that in real analysis one usually has to be very careful about which kind of convergence implies which kind of behaviour of the limit. For example, let f^n be a sequence of smooth functions (say real valued) on $\mathbb{R}^n$, converging uniformly to a function f. Then f will be continuous but in general not more regular than that. In a way the most extreme statement in this direction is the Stone-Weierstrass theorem which states that any continuous real valued function on a compact subset of $\mathbb{R}^n$ can be approximated uniformly by polynomials.

This situation is very different in the case of holomorphic functions. Before we state the next theorem, recall that a sequence of functions $f_n\colon D \to \mathbb{C}$, defined on an open set D converges locally uniformly to f if for every compact set $K \subseteq D$ the sequence of restricted functions $f_n|_K$ converges uniformly to $f|_K$.

Theorem 8.1 (Weierstrass convergence theorem) *Let $D \subset \mathbb{C}$ be an open and connected set, and let $f_n \in \mathcal{H}(D)$ be holomorphic functions on D. If f_n converges locally uniformly to a function f, then f is holomorphic on D.*

Proof. Fix $z_0 \in D$ and a $\delta > 0$ such that $\bar{B}_\delta(z_0) \subset D$. As complex-differentiability is a local property it is sufficient to prove that f is holomorphic on $B_\delta(z_0)$.

Recall that according to Morera's theorem, Theorem 5.12, it is sufficient to show that for any closed, piecewise $\mathcal{C}^1$-curve γ in $B_\delta(z_0)$ we have

$$\int_\gamma f(z)\,\mathrm{d}z = 0.$$

Indeed, Morera's theorem is stated for boundaries of triangles, but of course if this is true for any closed curve, then in particular for boundaries of triangles.

We have by the assumptions that for any n

$$\int_\gamma f(z)\,\mathrm{d}z = \underbrace{\int_\gamma f_n(z)\,\mathrm{d}z}_{=0 \text{ as } f_n \in \mathcal{H}(D)} + \underbrace{\int_\gamma (f(z) - f_n(z))\,\mathrm{d}z}_{\substack{\to 0 \text{ as } n\to\infty \\ \text{by uniform convergence on } \bar{B}_\delta}}.$$

Hence, the claim is proved. $\square$

We get the following even stronger statement.

Theorem 8.2 *Let $D \subset \mathbb{C}$ be an open and connected set, and let f_n be holomorphic functions on D. If f_n converges locally uniformly to a function f, then for every $k \geq 1$ the k-th derivatives $f_n^{(k)}$ converge locally uniformly to $f^{(k)}$.*

Proof. Let $z_0 \in D$ and $\delta > 0$ be such that $\overline{B}_{2\delta}(z_0) \subseteq D$. It is sufficient to prove the uniform convergence on the smaller ball $\overline{B}_\delta(z_0)$ (Exercise!).

For any $z \in \overline{B}_\delta(z_0)$ we have that

$$f_n^{(k)}(z) - f^{(k)}(z) = \frac{k!}{2\pi \mathrm{i}} \int_{\partial B_{2\delta}(z_0)} \frac{f_n(\xi) - f(\xi)}{(\xi - z)^{k+1}} \, \mathrm{d}\xi.$$

Noting that for $z \in \overline{B}_\delta(z_0)$ and $\xi \in \partial B_{2\delta}(z_0)$ we have $|z - \xi| \geq \delta$, we can conclude that, uniformly for $z \in B_\delta(z_0)$, we have

$$|f_n^{(k)}(z) - f^{(k)}(z)| \leq \frac{k!}{2\pi}(4\pi\delta) \sup_{\xi \in \bar{B}_{2\delta}(z_0)} |f_n(\xi) - f(\xi)| \frac{1}{\delta^{k+1}} = \frac{2k!}{\delta^k} \sup_{\xi \in \bar{B}_{2\delta}(z_0)} |f_n(\xi) - f(\xi)|.$$

The quantity on the right hand side goes to zero as n goes to ∞ by assumption. □

Another remarkable property of sequences of holomorphic functions is that in the limit the number of zeros cannot increase.

Theorem 8.3 (Hurwitz' theorem) *Let $D \subset \mathbb{C}$ be open and connected and suppose that $f_n \colon D \to \mathbb{C}$ are holomorphic on D and converge locally uniformly to f (which is necessarily a holomorphic function on D by Theorem 8.1). Suppose that, for some $k \in \mathbb{N}_0$, none of the f_n has more than k zeros (counting multiplicity). Then either f is constant or f has at most k zeros (counting multiplicity).*

Remark 8.4 (a) In particular, if the f_n have no zeros, then either f is constant or it does not have any zeros either.

(b) The number of zeros can decrease in the limit. For example, if

$$D = \mathbf{\Delta} = \{z \in \mathbb{C} \colon |z| < 1\}$$

and $f_n(z) := (z - 1 + \frac{1}{n})$, then for any n the function f_n has a zero in $\mathbf{\Delta}$. But f_n converge locally uniformly to $f(z) = z - 1$ which does not have a zero in $\mathbf{\Delta}$ (because the zero has "wandered out of the domain"). In the same way it is easy to construct examples in which the number of zeros drops by an arbitrary natural number in the limit.

◇

Proof. Suppose that f is not constant. Then all its zeros are isolated (note that here we use the fact that D is connected). Suppose that f has zeros of multiplicity $m_1, m_2, \ldots, m_K$, at distinct points $z_1, z_2, \ldots, z_K$.

Let us also fix a (small) $\delta > 0$ such that there are no zeros in the sets $B_\delta(z_i) \setminus \{z_i\}$ for all $i = 1, \ldots, K$. Let

$$\varepsilon := \inf_{i=1,\ldots,K} \inf_{\xi \in \partial B_\delta(z_i)} |f(\xi)| > 0$$

Then for all n large enough (say larger than a n_0) we know that

$$\sup_{i=1,\ldots,K} \sup_{\xi \in \partial B_\delta(z_i)} |f_n(\xi) - f(\xi)| < \frac{\varepsilon}{2},$$

so Rouché's theorem 7.27 implies that f_n also have exactly m_i zeros in the balls $B_\delta(z_i)$.

This argument shows that the number of zeros cannot go up in the limit. □

Corollary 8.5 *The locally uniform limit of* injective *holomorphic functions f_n (defined on an open connected set $D \subset \mathbb{C}$) is either constant or injective.*

Proof. Assume that f_n converge locally uniformly to f and that f is not constant and not injective. Then there exists at least one pair $z_1 \neq z_2$ with $f(z_1) = f(z_2) =: w$. In particular, the function $f(z) - w$ has at least two zeros in D. But this implies that the function $f_n - w$ also has at least two zeros for n large enough (at least in a subsequence), so the f_n cannot be injective. □

8.2 Compactness

Recall that any bounded sequence in $\mathbb{R}^n$ has the property that one can select a converging subsequence. This property is crucial in many situations, for example, when proving that a certain function admits a minimiser. Recall that a subset of an arbitrary metric space that has the property that every sequence has a convergent subsequence is called *compact*.

For sequences of *functions* statements of this type are also very important (e.g. in the calculus of variations) but in general one has to be much more cautious as the following example illustrates.

Example 8.6 Consider the following sequence of continuous real valued functions on $[0, 1]$: Let

$$f_1(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq \frac{1}{2}, \\ 2 - 2x & \text{if } \frac{1}{2} < x \leq 1, \end{cases}$$

i.e. f_1 goes up from 0 to 1 on $[0, \frac{1}{2}]$ as a straight line and down from 1 to 0 on $[\frac{1}{2}, 1]$. Then we can define recursively the functions

$$f_n(x) = \begin{cases} f_{n-1}(2x) & \text{if } 0 \leq x \leq \frac{1}{2}, \\ f_{n-1}(2x - 1) & \text{if } \frac{1}{2} < x \leq 1. \end{cases}$$

The function f_n "goes up and down" on any dyadic interval of the form

$$[k2^{-n+1}, (k + 1)2^{-n+1}].$$

Of course, we have for any $x \in [0, 1]$ and any $n \in \mathbb{N}$ that $|f_n(x)| \leq 1$. The sequence (f_n) is bounded in uniform norm.

Still, we claim that there is no subsequence of the (f_n) that converges (locally) uniformly. To see that assume the opposite, that some subsequence $(f_{n_i})_i$ converges uniformly to a function f which is necessarily continuous. Then there exists a $\delta > 0$ such that for $x \in [\frac{1}{2} - \delta, \frac{1}{2} + \delta]$ we have $|f(x) - f(\frac{1}{2})| \leq \frac{1}{4}$. On the other hand for i large enough we have uniformly for $x \in [\frac{1}{2} - \delta, \frac{1}{2} + \delta]$ that $|f_{n_i}(x) - f(x)| \leq \frac{1}{4}$. But this is a contradiction, because for i large enough a whole interval of the type $[k2^{-n_i+1}, (k + 1)2^{-n_i+1}]$ is contained in $[\frac{1}{2} - \delta, \frac{1}{2} + \delta]$, and hence f_{n_i} attains all values in $[0, 1]$.

Remark 8.7 The important *Arzela-Ascoli theorem* states that a closed set $\mathcal{A}$ in the space of continuous functions over a compact metric space is *compact* if and only if $\mathcal{A}$ it is bounded and equicontinuous. Recall that equicontinuous means that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for all x, y with $d(x, y) < \delta$ and *for all* $f \in \mathcal{A}$ we have $|f(x) - f(y)| < \varepsilon$. The point of this assumption is that one can chose the same δ that works for all f. This is what fails in Example 8.6.

We will now see that the behaviour illustrated in Example 8.6 cannot appear for holomorphic functions.

Lemma 8.8 *Let $D \subset \mathbb{C}$ be open and connected and for any $n \geq 1$ let $f_n \colon D \to \mathbb{C}$ be holomorphic on D. Assume that:*

- *The sequence (f_n) is locally bounded i.e. for every compact set $K \subseteq D$ there is a constant $C > 0$ such that for all $n \geq 1$ and for all $z \in K$*

$$|f_n(z)| \leq C.$$

- *There exists a dense subset $\mathcal{D} \subset D$ such that for every $z \in \mathcal{D}$ the sequence $f_n(z)$ converges.*

Then the whole sequence converges locally uniformly to a holomorphic function f.

Proof. Fix $z_0 \in D$ and $\delta > 0$ small enough such that $\overline{B}_{2\delta}(z_0) \subset D$. It is sufficient to prove (Exercise!) that for every $\varepsilon > 0$ there exists an $N > 0$ such that for all $n \geq N$ and all $z \in B_\delta(z_0)$ we have

$$|f_n(z) - f_m(z)| \leq \varepsilon. \tag{8.1}$$

Let a be any point in $\mathcal{D} \cap B_\delta(z_0)$. Then, by triangle inequality, we have that

$$|f_n(z) - f_m(z)| \leq |f_n(z) - f_n(a)| + |f_n(a) - f_m(a)| + |f_m(z) - f_m(a)|.$$

Our aim is to derive a uniform control on the modulus of continuity of the f_n (compare Remark 8.7) that allows to give a uniform in n bound on the terms $|f_n(z) - f_n(a)|$ that only depends on $|z - a|$.

Let $z, z' \in B_\delta(z_0)$. Then we obtain from Cauchy's integral formula (4.3) that for any n

$$\begin{aligned} f_n(z) - f_n(z') =& \frac{1}{2\pi \mathrm{i}} \int_{\partial B_{2\delta}(z_0)} \frac{f_n(\xi)}{(\xi - z)} \,\mathrm{d}\xi - \frac{1}{2\pi \mathrm{i}} \int_{\partial B_{2\delta}(z_0)} \frac{f_n(\xi)}{(\xi - z')} \,\mathrm{d}\xi \\ =& \frac{z - z'}{2\pi \mathrm{i}} \int_{\partial B_{2\delta}(z_0)} \frac{f_n(\xi)}{(\xi - z)(\xi - z')} \,\mathrm{d}\xi. \end{aligned}$$

According to the assumptions there exists a $C > 0$ such that for all n and all $z \in \overline{B}_{2\delta}(z_0)$ we have $|f_n(z)| \leq C$. This yields that bound

$$|f_n(z) - f_n(z')| \leq |z - z'| \frac{1}{2\pi} (4\pi\delta) \frac{C}{\delta^2} = |z - z'| \frac{2C}{\delta}. \tag{8.2}$$

With the bound (8.2) in hand it is now straightforward to conclude: For a fixed $\varepsilon > 0$ there exists *finitely* many $a_1, \ldots, a_K \in \mathcal{D}$ such that the open balls of radius $\frac{\varepsilon\delta}{6C}$ cover all of $\bar{B}_\delta(z_0)$. Also there exists a $N > 0$ such that for all $n, m \geq N$ and for all $i = 1, \ldots, K$ we have $|f_n(a_i) - f_m(a_i)| \leq \frac{\varepsilon}{3}$.

Then for any fixed $z \in B_\delta(z_0)$ there exists a a_i with $|z - a_i| \leq \frac{\varepsilon\delta}{6C}$ and we get for $n, m \geq N$ that

$$|f_n(z) - f_m(z)| \leq \underbrace{|f_n(z) - f_n(a_i)|}_{\leq |z - a_i| \frac{2C}{\delta} \leq \frac{\varepsilon}{3}} + \underbrace{|f_n(a_i) - f_m(a_i)|}_{\leq \frac{\varepsilon}{3} \text{ by assumption on } N} + \underbrace{|f_m(z) - f_m(a_i)|}_{\leq \frac{\varepsilon}{3}} \leq \varepsilon.$$

$\square$

An immediate consequence of this lemma is the following theorem.

Theorem 8.9 (Montel's theorem) *Every locally bounded sequence of holomorphic functions on an open, connected set has a locally uniformly convergent subsequence.*

Proof. Let (a_i) be a dense subsequence of D. Then the sequence $f_n(a_1)$ of complex numbers is bounded by assumption, and hence there exists a convergent (in $\mathbb{C}$) subsequence, denoted by $f_n^1(a_1)$. Plugging a_2 into the functions f_n^1 we get a new bounded sequence of complex numbers that will again admit a convergent subsequence, denoted by $f_n^2(a_2)$. By iterating this procedure we obtain sequences f_n^m of holomorphic functions such that for every m and any $i \leq m$ the sequences $(f_n^m(a_i))_n$ converge. Then the diagonal sequence f_n^n converges in all points a_i. By Lemma 8.8 the diagonal sequence f_n^n then also converges locally uniformly. □

Remark 8.10 (a) Note that it is not possible to remove the "local" from the statement of Montel's theorem. Take for example the sequence $f_n(z) = z^n$ defined on $\mathbf{\Delta} = \{z \in \mathbb{C} \colon |z| < 1\}$. This sequence is clearly globally bounded, and it converges locally uniformly to 0. But it does not converge uniformly.

(b) It is interesting to briefly discuss the connection to the Stone-Weierstrass theorem, already mentioned above, at the beginning of this chapter. Let f be a continuous function on $[0, 1]$. Imagine a rather rough function (say the values of a stock price over a day), i.e. let us say that f is continuous but not differentiable. Of course, f cannot be extended to a holomorphic function onto any open neighbourhood of the interval $[0, 1]$ in the complex plane, because then it would have to be smooth.

Nonetheless, f can be approximated uniformly on $[0, 1]$ by polynomials. Of course, these polynomials can be viewed as holomorphic functions in a small neighbourhood D of $[0, 1]$ in the complex plane. But it is not possible for the polynomials to converge locally uniformly on D, because otherwise the limit would have to be holomorphic and to coincide with f on $[0, 1]$ which is impossible. Actually, by the same argument the polynomials cannot even have a locally uniformly convergent subsequence. By Montel's theorem, we can conclude that the polynomials must be unbounded in any small neighbourhood of $[0, 1]$.

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9 Some special functions

9.1 The Gamma function

Definition 9.1 For $\Re(z) > 0$ we can define the gamma function as

$$\Gamma(z) = \int_0^\infty t^{z-1} \mathrm{e}^{-t} \, \mathrm{d}t. \tag{9.1}$$

Recall that t^z is defined as $t^z := \exp(\log(t) z)$. The variable t is real valued and log should be interpreted as the *usual* real valued logarithm on $\mathbb{R}_+$.

Theorem 9.2 *The* Γ- *integral* (9.1) *defines an holomorphic function on* $\mathsf{H}_R := \{z \in \mathbb{C}\colon \Re(z) > 0\}$. *The* k*-th derivative is given by*

$$\Gamma^{(k)}(z) = \int_0^\infty t^{z-1}\,(\log(t))^k\,\mathrm{e}^{-t}\,\mathrm{d}t.$$

We note that for fixed $t > 0$ the function $z \mapsto t^{z-1}\mathrm{e}^{-t}$ is differentiable on all of $\mathbb{C}$ and the k-th derivative is given by $t^{z-1}(\log(t))^k\mathrm{e}^{-t}$. Hence the proof consists of justifying the differentiation under the integral. The following lemma from real analysis is a key ingredient to this end.

Lemma 9.3 *Let* $f(t,z)$ *be a real valued function defined on* $[t_0,t_1]\times(a,b)$ *for some* $t_0 < t_1$ *and* $a < b$. *Assume that* f *is continuous. Assume furthermore, that* $\partial_z f$ *exists and is continuous on all of* $[t_0,t_1]\times(a,b)$. *Then the function*

$$g(z) = \int_{t_0}^{t_1} f(t,z)\,\mathrm{d}t$$

is continuously differentiable on (a,b) *and the derivative is given by*

$$\partial_z g(z) = \int_{t_0}^{t_1} \partial_z f(t,z)\,\mathrm{d}t.$$

Proof of Lemma 9.3 . Let $z_0 \in (a,b)$ and for simplicity assume $z > z_0$ (the argument for $z < z_0$ is the same). According to the *mean value theorem* for every $t \in [t_0,t_1]$ there exists a $\xi(t) \in [z_0,z]$ such that for $f(t,z) - f(t,z_0) = \partial_z f(t,\xi(t))(z-z_0)$. Hence we get

$$\begin{aligned}\left|\frac{g(z)-g(z_0)}{z-z_0} - \partial_z g(z)\right| &= \left|\int_{t_0}^{t_1} \frac{f(t,z)-f(t,z_0)-\partial_z f(t,z)(z-z_0)}{z-z_0}\,\mathrm{d}t\right| \\ &\le \int_{t_0}^{t_1} \sup_{\xi\in[z_0,z]} |\partial_z f(t,\xi) - \partial_z f(t,z)|\,\mathrm{d}t. \end{aligned} \tag{9.2}$$

Now, by assumption the function $\partial_z f(t,z)$ is continuous and hence it is uniformly continuous on the compact set $[t_0,t_1]\times[z_0-\delta_0, z_0+\delta_0]$ for any $\delta_0 > 0$ (which is small enough to ensure that $[z_0-\delta_0,z_0+\delta_0] \subseteq (a,b)$). Hence, for any $\varepsilon > 0$ we can find a $\delta > 0$ such that for $|z - z_0| < \delta$ we have *uniformly in* t that

$$\sup_{\xi\in[z_0,z]} |\partial_z f(t,\xi) - \partial_z f(t,z)| < \varepsilon.$$

For such z the right hand side of (9.2) is bounded by $\varepsilon(t_1 - t_0)$ which proves the differentiability of g. The continuity of $\partial_z g$ follows in the same way. $\square$

The following conclusion for complex valued functions follows easily.

Lemma 9.4 *Let* $f(t,z)$ *be a complex valued function defined on* $[t_0,t_1]\times D$ *for some* $t_0 < t_1$ *and for an open set* $D \subset \mathbb{C}$. *Assume furthermore that for fixed* t *the function*

$z \mapsto f(t,z)$ is complex differentiable in D and that f as well as the complex derivative $\partial_z f(t,z)$ are continuous on all of $[t_0,t_1]\times D$. Then the function

$$g(z) = \int_{t_0}^{t_1} f(t,z)\,\mathrm{d}t$$

is holomorphic on D and the derivative is given by

$$\partial_z g(z) = \int_{t_0}^{t_1} \partial_z f(t,z)\,\mathrm{d}t.$$

Proof of Lemma 9.4. Write $g(z) = u(z)+iv(z)$. Then according to the previous Lemma (9.3) both u and v are continuously differentiable in the sense of real analysis (simply consider the functions for fixed real and imaginary part separately). The Cauchy-Riemann equations hold for fixed t and are preserved under integration. □

With Lemma 9.4 in hand we are now ready to finish the proof of Theorem 9.2.

Proof of Theorem 9.2. For any $N > 0$ set

$$\Gamma_N(z) = \int_{\frac{1}{N}}^{N} t^{z-1}\mathrm{e}^{-t}\,\mathrm{d}t.$$

Lemma 9.4 applied to $(t,z)\mapsto t^{z-1}\mathrm{e}^{-t}$ implies that for any N the function Γ_N is entire and the k-th derivative is given by

$$\Gamma_N^{(k)}(z) = \int_{\frac{1}{N}}^{N} t^{z-1}\,(\log(t))^k\,\mathrm{e}^{-t}\,\mathrm{d}t.$$

In order to apply the *Weierstrass convergence theorem*, Theorem 8.1, we need to show that the Γ_N converge locally uniformly on the half-space $\mathsf{H}_R := \{z\in\mathbb{C}\colon \Re(z)>0\}$. To this end we fix $0<R_1<R_2$. We claim that Γ_N converges *uniformly* on

$$\{z\in\mathbb{C}\colon R_1<\Re(z)<R_2\}.$$

To see this, we write for $R_1<\Re(z)<R_2$,

$$\begin{aligned}|\Gamma_N(z)-\Gamma(z)| &\le \int_0^{\frac{1}{N}} |t^{z-1}\mathrm{e}^{-t}|\,\mathrm{d}t + \int_N^\infty |t^{z-1}\mathrm{e}^{-t}|\,\mathrm{d}t\\ &\le \underbrace{\int_0^{\frac{1}{N}} |t^{R_1-1}|\,\mathrm{d}t}_{\le \frac{1}{R_1}N^{-R_1}} + \int_N^\infty |t^{R_2-1}\mathrm{e}^{-t}|\,\mathrm{d}t.\end{aligned}$$

The second term on the right hand side does not depend on the specific choice of z and it goes to zero as $N\to\infty$. This finishes the proof of Theorem 9.2. □

An integration by parts (which can be justified easily by considering real and imaginary part separately) shows that the Gamma function satisfies the functional equation:

$$\boxed{\begin{aligned}\Gamma(z+1) &= z\Gamma(z),\\ \Gamma(1) &= 1.\end{aligned}} \tag{9.3}$$

which yields that for every $n \in \mathbb{N}$ we have $\Gamma(n+1) = n!$. We can use the functional equation (9.3) to extend the Gamma function to the whole half plane. Indeed for any $\Re(z) > 0$ and for any $n \in \mathbb{N}$ we have

$$\Gamma(z) = \frac{\Gamma(z+n)}{z(z+1)(z+2)\dots(z+n-1)}. \tag{9.4}$$

The right hand side of this equation extends to an meromorphic function on $\{z \in \mathbb{C} \colon \Re(z) > -n\}$ with poles at $0, -1, -2, \dots, -(n-1)$, of order 1.

Theorem 9.5 *The Gamma function extends to a meromorphic function on all of* $\mathbb{C}$. *It has poles at* $0, -1, -2, -3, \dots$ *of order* 1 *and residue*

$$\textbf{Res}\,(\Gamma, -n) = \frac{(-1)^n}{n!}.$$

Proof. Let $n \in \mathbb{N}_0$. Then using (9.3) and (9.4), we extend Γ via the right hand side of (9.4) as

$$\Gamma(z) = \frac{\Gamma(z+n+1)}{z(z+1)\cdots(z+n)} \qquad \Re(z) > -n.$$

Thus Γ extends to a meromorphic function on $\mathbb{C}$. The function has a pole at $z = -n, n \in \mathbb{N}_0$, of order 1 because

$$\lim_{z \to -n} (z - (-n)) \frac{\Gamma(z+n+1)}{z(z+1)\cdots(z+n)} = \frac{(-1)^n}{n!}.$$

□

In the next section we give an alternative representation of the Γ-function as an infinite product.

9.2 Infinite products

In this section we discuss general properties of infinite products of the type

$$\prod_{n=1}^{\infty} b_n\,, \qquad b_n \in \mathbb{C}.$$

Before we develop that further , we discuss the following example which is important for your upcoming homework sheet 4 (Exercise 16).

Example 9.6 We shall show the claim

$$\sin(z) = z \prod_{k=1}^{\infty} \Big(1 - \frac{z^2}{\pi^2 k^2}\Big)$$

The following list are useful hints which can be worked out as an exercise and to solve the homework.

(i)

$$\cot(z) = \log(\sin(z))'$$

(ii)

$$f(z) := \cot(\pi z) = \frac{\cos(\pi z)}{\sin(\pi z)}$$

is a meromorphic function with $\mathcal{P}_f = \mathcal{Z}_{\sin(\pi z)}$. For all $n \in \mathbb{Z}$, the function πf has a pole at $z = n$:

$$\lim_{z \to n} (z-n)\pi \frac{\cos(\pi z)}{\sin(\pi z)} = \pi \frac{\cos(\pi n)}{\pi \cos(\pi n)} = \mathbf{Res}\,(\pi f, n) = 1.$$

Thus

$$\cot(\pi z) = \frac{1}{z-n} + \text{power series in } (z-n).$$

(iii)

$$g(z) := \cot(z) - \frac{1}{z} = \frac{z\cos(z) - \sin(z)}{z\sin(z)}.$$

The function g has poles at $z = \pi n$, $n \in \mathbb{Z} \setminus \{0\}$, and

$$\mathbf{Res}\,(g, n\pi) = \lim z \to n\pi (z - n\pi) \frac{z\cos(z) - \sin(z)}{z\sin(z)} = \frac{n\pi\cos(n\pi) - \sin(n\pi)}{\sin(n\pi) + n\pi\cos(n\pi)} = 1\,.$$

Furthermore, (L'Hospital rule)

$$\lim_{z \to 0} g(z) = 0,$$

and thus g has a removable singularity at $z = 0$, and we put $g(0) := 0$. The Residue theorem implies that

$$\cot(z) - \frac{1}{z} = \sum_{n \in \mathbb{Z}\setminus\{0\}} \Big(\frac{1}{z - n\pi} + \frac{1}{n\pi}\Big).$$

(iv)

$$\frac{1}{z} + \lim_{N\to\infty} \Big(\sum_{k=-N}^{-1} \Big(\frac{1}{z - k\pi} + \frac{1}{k\pi}\Big) + \sum_{k=1}^{N} \Big(\frac{1}{z - k\pi} + \frac{1}{k\pi}\Big)\Big)$$
$$= \frac{1}{z} + \lim_{N\to\infty} \Big\{ \frac{2z}{z^2 - \pi^2} + \cdots + \frac{2z}{z^2 - N^2\pi^2} \Big\}$$

Thus

$$\cot(z) = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{2z}{z^2 - \pi^2 k^2}.$$

♣

Definition 9.7 (a) An *infinite product* $\prod_{n=1}^{\infty} w_n$ of complex numbers $w_n \in \mathbb{C}$ converges if only finitely many of the $w_n = 0$ and if the sequence of partial products converges with a non-vanishing limit, that is,

$$\prod_{n=1}^{\infty} w_n := \lim_{N\to\infty} \prod_{n=1}^{N} w_n \neq 0\,.$$

(b) The product $\prod_{n=1}^{\infty} w_n$ converges absolutely if there exists $n_0 \in \mathbb{N}$ such that

$$\sum_{k=n_0}^{\infty} \log(w_k)$$

converges.

Remark 9.8 (a) If only finitely many of the w_n vanish we are relabelling the sequence as follows: suppose $w_{n_0} = 0$ is the last member of the sequence $(w_n)_{n\in\mathbb{N}}$, that is, $w_n \neq 0$ for all $n > n_0$. Then we just relabel $(n_0+1 \mapsto 1, n_0+2 \mapsto 2, \ldots)$ and denote that sequence $(w_n)_{n\in\mathbb{N}}$.

(b) Suppose $(w_n)_{n\in\mathbb{N}}$ converges, then

$$w_n = \frac{\prod_{k=1}^{n} w_k}{\prod_{k=1}^{n-1} w_k} \to 1 \text{ as } n \to \infty,$$

as numerator and denominator have the same limit $\neq 0$.

(c) Recall $\mathbb{C}_\Pi = \mathbb{C} \setminus \{z \in \mathbb{C} \colon \Im(z) = 0, \Re(z) \leq 0\}$, and

$$\log\left(r\mathrm{e}^{\mathrm{i}\theta}\right) = \log r + \mathrm{i}\theta \qquad \theta \in (-\pi, \pi).$$

◇

Proposition 9.9 *(a) The infinite product* $\prod_{n=1}^{\infty} w_n$ *with* $w_n \in \mathbb{C}_\Pi, n \in \mathbb{N}$, *converges* $\Leftrightarrow$ $\sum_{n=1}^{\infty} \log w_n$ *converges.*

(b) $\prod_{k=1}^{\infty}(1 + a_k), a_k \in \mathbb{C}$, *converges absolutely* $\Leftrightarrow \sum_{k=1}^{\infty}|a_k|$ *converges.*

Proof. (a) Suppose the series converges. Then

$$\mathrm{e}^{\sum_{k=1}^{m} \log w_k} = \prod_{k=1}^{m} w_k$$

converges to $e^{\sum_{k=1}^{\infty} \log w_k} \neq 0$ for $m \to \infty$, and thus the product converges. Conversely, suppose that the product converges, that is,

$$P_\infty := \lim_{N\to\infty} P_N \neq 0\,, \qquad P_N := \prod_{n=1}^{N} w_n.$$

This time, there is no simple application of the logarithm as the partial products could be not all in $\mathbb{C}_\Pi$. Choose $n_0 \in \mathbb{N}$ such that

$$\Big|\frac{P_n}{P_m} - 1\Big| < \frac{1}{2}\,, \qquad \text{for all } m, n \geq n_0,$$

and thus

$$|P_n - P_m| < \frac{1}{4}|P_\infty| < \frac{1}{2}|P_m|\,, \qquad \text{for all } m, n \geq n_0.$$

Therefore

$$P_m = \prod_{k=1}^{m} w_k \in K \qquad K := \{z \in \mathbb{C} \colon |z-1| < \frac{1}{2}\}\,,$$

and thus ensuring the convergence of $\log\Big(\prod_{k=1}^m w_k\Big) = \sum_{k\leq m} \log(w_k)$ in the limit $m \to \infty$.

(b) Exercise! □

Remark 9.10 We note that an absolutely convergent product vanishes if and only if at least one of its factors vanishes. The assumption of *absolute convergence* is essential for the last statement to be true. Consider for example the limit

$$\lim_{N\to\infty} \prod_{n=1}^{N} \frac{1}{n} = \lim_{N\to\infty} \frac{1}{N!} = 0.$$

This product is of course not absolutely convergent. ◇

After these preliminary considerations about infinite products, we now pass to the product representation of the Γ function. We have seen above that Γ has simple poles at all the non-positive integers. Hence it seems like a reasonable guess that there may be a connection between $\frac{1}{\Gamma}$ and the product $(1+z)(1+\frac{z}{2})(1+\frac{z}{3})\cdots$. Unfortunately, this product fails to converge. But it does converge if one modifies it slightly.

Lemma 9.11 *The infinite product*

$$H(z) = \prod_{n=1}^{\infty} \Big(1 + \frac{z}{n}\Big) e^{-\frac{z}{n}}$$

converges absolutely and locally uniformly on all of $\mathbb{C}$. In particular, H is an entire function.

Proof. For any z we can write $\mathrm{e}^{-\frac{z}{n}} = 1 - \frac{z}{n} + E(z)$, with an error term that is bounded locally uniformly in z by $|E(z)| \leq \frac{Cz^2}{n^2}$. Hence, we can write

$$\Big|\Big(1+\frac{z}{n}\Big)\mathrm{e}^{-\frac{z}{n}} - 1\Big| = \frac{|z|^2}{n^2} + \Big(1+\frac{|z|}{n}\Big)E(z).$$

This quantity is summable locally uniformly in z. □

Corollary 9.12 *The functions*

$$G_N(z) = z\mathrm{e}^{-z\log N}\prod_{n=1}^{N}\Big(1+\frac{z}{n}\Big) = z\mathrm{e}^{-z(\log N - \sum_{n=1}^{N}\frac{1}{n})}\prod_{n=1}^{N}\Big(1+\frac{z}{n}\Big)\mathrm{e}^{-\frac{z}{n}}$$

converge locally uniformly as $N \to \infty$ to an entire function G.

Proof. It only remains to observe that

$$\lim_{N\to\infty}\sum_{n=1}^{N}\frac{1}{n} - \log N = \gamma,$$

where γ is the Euler-Mascheroni constant, whose approximate value is

$$\gamma \approx 0,5772156649015328606065 12\ldots .$$

□

We finally get the desired product form for the Γ function.

Theorem 9.13 *For any $z \in \mathbb{C}\setminus\{0,-1,-2,-3,\ldots\}$ we have*

$$\frac{1}{\Gamma(z)} = G(z) = \lim_{N\to\infty}\frac{N^{-z}}{N!}z\,(z+1)\,(z+2)\ldots(z+N)\,. \tag{9.5}$$

In particular we get the following.

Corollary 9.14 *The Γ function has no zeros.*

We are not going to provide a full proof of Theorem 9.13 but we are only going to give a sketch. For the details we refer the reader to [FBF05].

Proof of Theorem 9.13 (Sketch). By induction over N it is easy to check that

- $G_N(1) = (1+\frac{1}{N})$,
- $zG_N(z+1) = \frac{z+N+1}{N}G_N(z)$.

Hence, by passing to the limit $N \to \infty$ we get $\frac{1}{G(1)} = 1$ and $\frac{1}{G(z+1)} = \frac{z}{G(z)}$. Furthermore, one can check that $|\frac{1}{G}|$ is bounded in the strip $\{z \in \mathbb{C}\colon 1 \leq \Re(z) \leq 2\}$. Due to a theorem of Wielandt (which was presented in an exercise) this already characterises the Γ function. □

There are many more nice identities for the Γ function. We are going to give one more of these with a sketch of proof. Again, we refer the reader to [FBF05] for the details.

Theorem 9.15 *For every $z \in \mathbb{C} \setminus \mathbb{Z}$ we have*

$$\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}. \tag{9.6}$$

Proof (Sketch). As the Γ function has poles of order one at all the non-positive integers, the product on the left hand side of (9.6) has simple poles at all integers. Furthermore, the functional equation of the Γ function implies that this product is odd. To see this write

$$\Gamma(-z)\,\Gamma(1+z) = \left(\frac{\Gamma(1-z)}{-z}\right) z\Gamma(z).$$

A similar calculation shows that for $n \in \mathbb{Z}$ we have $\Gamma(z+2n) = \Gamma(1-z-2n)$. An optimistic reader might start to believe at this point that (9.6) could be true.

In order to show this, consider the function

$$h(z) = \Gamma(z)\,\Gamma(1-z) - \frac{\pi}{\sin(\pi z)}.$$

All the singularities at integers are removable for h and therefore h can be extended to an entire function. Furthermore, h inherits the $2\mathbb{Z}$ periodicity from $\Gamma(z)\,\Gamma(1-z)$ and $\frac{\pi}{\sin(\pi z)}$. A calculation shows that $|h|$ is bounded in $\{z \in \mathbb{C}\colon 1 < \Re(z) < 3\}$. As h is periodic this implies that h is bounded overall, and hence it is constant by Liouville's theorem. As h is also odd it has to be zero. $\square$

9.3 The ζ function

For any $z \in \mathbb{C}$ with $\Re(z) > 1$ we can define the *Riemann zeta function* as

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}. \tag{9.7}$$

As above the power n^z can simply be defined as $n^z := \exp(\log(n)z)$ and poses no problem at all. This also shows that any partial sum of ζ is a holomorphic function and as the sum converges locally uniformly, the zeta function is holomorphic on $\{z \in \mathbb{C}\colon \Re(z) > 1\}$. The zeta function is one of the central objects in analytic number theory, because of its connection with prime numbers that will be illustrated in the following theorem.

Theorem 9.16 *For any z with $\Re(z) > 1$ we have*

$$\frac{1}{\zeta(z)} = \prod_{p \in \mathcal{P}} \left(1 - \frac{1}{p^z}\right).$$

Here $\mathcal{P} = \{2, 3, 5, 7, 11, \ldots\}$ denotes the set of prime numbers.

Proof. We have the obvious bound

$$\sum_{p\in\mathcal{P}} \frac{1}{|p^z|} \le \sum_{n\in\mathbb{N}} \frac{1}{n^{\Re(z)}} < \infty$$

and hence the infinite product converges absolutely.

We recalling the definition of the ζ function in (9.7) we get

$$\begin{aligned}\Big(1-\frac{1}{2^z}\Big)\zeta(z) =&\Big(1+\frac{1}{2^z}+\frac{1}{3^z}+\frac{1}{4^z}+\frac{1}{5^z}+\frac{1}{6^z}+\dots\Big)\\ &\qquad\qquad -\Big(\frac{1}{2^z}+\frac{1}{4^z}+\frac{1}{6^z}+\frac{1}{8^z}+\frac{1}{10^z}+\dots\Big)\\ =&1+\sum_{\substack{n=3\\ n\,odd}}^{\infty}\frac{1}{n^z}.\end{aligned}$$

Then adding the next factor we get

$$\begin{aligned}\Big(1-\frac{1}{3^z}\Big)\Big(1-\frac{1}{2^z}\Big)\zeta(z) =& \Big(1+\frac{1}{3^z}+\frac{1}{5^z}+\frac{1}{7^z}+\frac{1}{9^z}+\dots\Big)\\ &\qquad -\Big(\frac{1}{3^z}+\frac{1}{9^z}+\frac{1}{15^z}+\frac{1}{21^z}+\dots\Big)\\ =&1+\sum\frac{1}{n^z},\end{aligned}$$

where the sum is over all odd $n>1$ that are not divisible by 3. If we iterate this procedure we obtain for the N-th prime number

$$\left|\Big(1-\frac{1}{p_N^z}\Big)\cdots\Big(1-\frac{1}{3^z}\Big)\Big(1-\frac{1}{2^z}\Big)\zeta(z)-1\right| \le \sum\left|\frac{1}{n^z}\right|,$$

where the sum on the right hand side goes over all natural numbers $n>p_N$ that are not divisible by any of the prime numbers $2,3,\dots,p_N$. This sum converges rapidly to 0 as $N\to\infty$ which yields the result. □

As in the case of the Γ function we get the following conclusion from the product representation.

Corollary 9.17 *The ζ-function does not have any zeros in $\{z\in\mathbb{C}\colon \Re(z)>1\}$.*

Proof. None of the factors in the convergent product is zero. □

As a (modest) number theoretic application of this we obtain the following result.

Theorem 9.18 *We have*

$$\sum_{p\in\mathcal{P}}\frac{1}{p}=\infty.$$

Remark 9.19 It is of course easy and well known since ancient times that there are infinitely many primes. This stronger statement is not as easy to prove by purely number theoretical methods. The connection with the ζ function allows to obtain this result without much effort.

Proof of Theorem 9.18. Assume the contrary, namely that $\sum_{p\in\mathcal{P}} \frac{1}{p} < \infty$. Hence the product

$$\prod_{p\in\mathcal{P}} \left(1-\frac{1}{p}\right) = \exp\left(\sum_{p\in\mathcal{P}} \log\left(1-\frac{1}{p}\right)\right)$$

converges absolutely to a value $x > 0$. By monotonicity it is easy to see that for any $z > 1$ we have

$$x < \prod_{p\in\mathcal{P}} \left(1-\frac{1}{p^z}\right) = \frac{1}{\zeta(z)}.$$

But the expression on the right hand side converges to 0 as $z \downarrow 1$ which yields a contradiction. □

We will now discuss that the ζ function can be extended to a meromorphic function on all of $\mathbb{C}$. We start by the following lemma that establishes a connection between the ζ and the Γ function.

Lemma 9.20 *For all z with $\Re(z) > 1$ we have*

$$\zeta(z) = \frac{1}{\Gamma(z)} \int_0^\infty \frac{t^{z-1}\mathrm{e}^{-t}}{1-\mathrm{e}^{-t}}\,\mathrm{d}t. \tag{9.8}$$

Proof. For $\Re(z) > 1$ the integral converges and describes a holomorphic function. The proof of this fact follows in the same way as the calculation for the Γ integral in Theorem 9.2 above. Note that the denominator of the integrand diverges like $\frac{1}{t}$ near 0 so that the stronger condition $\Re(z) > 1$ is necessary to ensure convergence.

For any $n \in \mathbb{N}$ and for $\Re(z) > 0$ we observe that

$$\int_0^\infty t^{z-1}\mathrm{e}^{-nt}\,\mathrm{d}t = n^{-z}\int_0^\infty (nt)^{z-1}\mathrm{e}^{-nt}\,n\,\mathrm{d}t = n^{-z}\int_0^\infty t^{z-1}\mathrm{e}^{-t}\,\mathrm{d}t = n^{-z}\,\Gamma(z).$$

Summing over n we obtain for $\Re(z) > 1$

$$\zeta(z) = \sum_{n=1}^\infty \frac{1}{n^z} = \frac{1}{\Gamma(z)}\sum_{n=1}^\infty \int_0^\infty t^{z-1}\mathrm{e}^{-nt}\,\mathrm{d}t.$$

It remains to interchange the infinite sum with the integral. To this end we write for finite N

$$\begin{aligned}\sum_{n=1}^N \int_0^\infty t^{z-1}\mathrm{e}^{-nt}\,\mathrm{d}t &= \int_0^\infty t^{z-1}\frac{\mathrm{e}^{-t}}{1-\mathrm{e}^{-t}}(1-\mathrm{e}^{-Nt})\,\mathrm{d}t \\ &= \int_0^\infty t^{z-1}\frac{\mathrm{e}^{-t}}{1-\mathrm{e}^{-t}}\,\mathrm{d}t - \int_0^\infty t^{z-1}\frac{\mathrm{e}^{-t}}{1-\mathrm{e}^{-t}}\mathrm{e}^{-Nt}\,\mathrm{d}t.\end{aligned}$$

The first integral already has the desired form. The second integral goes to zero as $N \to \infty$ (either by the dominated convergence theorem or by performing a simple estimate by hand). □

At first glance the expression (9.8) does not seem to help us to extend the ζ function to a larger domain. Indeed, the integral converges exactly for those values where the sum (9.7) converges as well. The key observation is that for z with $\Re(z) < 1$ the integral in (9.8) diverges because of a blowup of the integrand near zero. There is no integrability problem at ∞. The following definition elegantly circumvents this problem by replacing the integral to zero by an integral over a contour that *surrounds* zero with a small but strictly positive radius.

This contour is defined as follows: Fix $0 < \delta < \varepsilon < 2\pi$. Then the so called *Hankel contour* $C_{\varepsilon,\delta}$ is the following curve

$$C_{\varepsilon,\delta}(t) = \begin{cases} -t + \mathrm{i}\delta & \text{if } t \in (-\infty, -\tilde{\varepsilon}], \\ \varepsilon \mathrm{e}^{\mathrm{i}\pi\alpha(t)} & \text{if } t \in (-\tilde{\varepsilon}, \tilde{\varepsilon}), \\ t - \mathrm{i}\delta & \text{if } t \in [\tilde{\varepsilon}, \infty). \end{cases}$$

Here, $\tilde{\varepsilon}$ is the real part of the point z at the intersection of the circle $\{z \in \mathbb{C}\colon |z| = \varepsilon\}$ with the half-line $\{z \in \mathbb{C}\colon \Im(z) = \delta,\ \Re(z) > 0\}$ and $\alpha(t)$ is the affine reparametrisation of the interval $(-\tilde{\varepsilon}, \tilde{\varepsilon})$ that ensures that the curve surrounds the origin once in positive orientation and is continuous at $\pm\tilde{\varepsilon}$. The precise form of $\tilde{\varepsilon}$ and α does not matter for us.

For any fixed $z \in \mathbb{C}$ we define the function

$$u(w) = \frac{(-w)^{z-1}\mathrm{e}^{-w}}{1 - \mathrm{e}^{-w}}.$$

Note the resemblance of this function with the integrand in (9.8). Here the function $(-w)^{z-1}$ is defined as $\exp(\log(-w)(z-1))$ where log denotes the principal branch of the complex logarithm. Therefore, u is defined on $\mathbb{C} \setminus [0, \infty)$ with a discontinuity near the positive real axis. The function u has simple poles at all $2n\pi\mathrm{i}$ for $n \in \mathbb{Z} \setminus \{0\}$.

Finally, we define the *Hankel function*

$$H_{\varepsilon,\delta}(z) = \int_{C_{\varepsilon,\delta}} u(w)\,\mathrm{d}w. \tag{9.9}$$

As before, the integral over the contour with infinite length is to be interpreted as a limit of suitable approximations. The integral converges for all $z \in \mathbb{C}$ and constitutes an entire function. Indeed, the term e^{-w} enforces rapid convergence to zero "at ∞" and there is no convergence problem near zero, as the Hankel contour stays away from the singularity of u near 0.

The value of $H_{\varepsilon,\delta}$ does not depend on the precise value of ε, δ indeed, by changing them (as long as $0 < \delta < \varepsilon < 2\pi$) the contour does not cross any singularities and hence the value of the integral does not change. Hence, we can and will drop the indices ε and δ in what follows.

Proposition 9.21 *For* $\Re(z) > 1$ *we have*

$$\zeta(z) = \frac{-H(z)}{2\mathrm{i}\,\sin(\pi z)\,\Gamma(z)}\,. \tag{9.10}$$

Proof. For $\Re(z) > 1$ and fixed ε, δ we can write

$$H(z) = \lim_{\varepsilon,\delta\to 0} \Bigg(\int_{\tilde{\varepsilon}}^{\infty} \frac{\exp\left((z-1)\log(-t-\mathrm{i}\delta)\right)\mathrm{e}^{-(t+\mathrm{i}\delta)}}{1-\mathrm{e}^{-(t+\mathrm{i}\delta)}}\,\mathrm{d}t$$

$$+ \int_{\tilde{\varepsilon}}^{\infty} \frac{\exp\left((z-1)\log(-t+\mathrm{i}\delta)\right)\mathrm{e}^{-(t-\mathrm{i}\delta)}}{1-\mathrm{e}^{-(t-\mathrm{i}\delta)}}\,\mathrm{d}t + \int_{\tilde{\delta}}^{2\pi-\tilde{\delta}} \frac{(-\varepsilon\mathrm{e}^{\mathrm{i}\theta})^{z-1}}{1-\mathrm{e}^{-\varepsilon\mathrm{e}^{\mathrm{i}\theta}}}\,\mathrm{i}\varepsilon\mathrm{e}^{-\varepsilon\mathrm{e}^{\mathrm{i}\theta}}\,\mathrm{d}\theta \Bigg)$$

$$=: I + II + III. \tag{9.11}$$

Here in the third term III $\tilde{\delta}$ is an appropriate angle (depending on ε, δ). The exact value does not matter.

We have observed above that the value of the Hankel function does not depend on the precise choice of ε, δ as long as $\varepsilon < 2\pi$. We can thus let ε and δ go to zero without changing the value of $H(z)$.

Using the assumption that $\Re(z) > 1$ we see that the term III goes to zero as ε goes to zero (uniformly in δ). Indeed, we have

$$|\,III\,| \leq C2\pi\,\frac{\varepsilon^{\Re(z)-1}}{\varepsilon}\varepsilon$$

for a suitable constant C. In this estimate we make use of the fact that $e^{-\varepsilon e^{i\theta}}$ is bounded for ε small enough and that for $\varepsilon > 0$ small enough $|1-\mathrm{e}^{-\varepsilon\mathrm{e}^{\mathrm{i}\theta}}|$ can be bounded from below uniformly in θ by $\hat{C}\varepsilon$ for a suitable constant $\hat{C}$. The fact that III disappears as we let ε go to zero corresponds exactly to the convergence of the integral in (9.8) for $\Re(z) > 1$.

Let us now consider the terms I and II. At first glance, the integrands look the same as δ goes to zero, and one might think that the integrals cancel in this case. This impression is wrong because of the jump of the principal branch of the logarithm which was used to interpret the term $(-w)^{z-1}$. Indeed, for the log-term in I we get

$$\log(-t-\mathrm{i}\delta) = \log(\sqrt{t^2+\delta^2}) + \mathrm{i}(-\pi+\hat{\delta})$$

where $\hat{\delta}$ is a suitable angle (depending on t, δ) - again the precise value does not matter. In the same way the log-term in II reads

$$\log(-t+\mathrm{i}\delta) = \log(\sqrt{t^2+\delta^2}) + \mathrm{i}(+\pi-\hat{\delta}).$$

Both expressions have the same real part, but there is a jump of $2\pi - 2\hat{\delta}$ in the imaginary part.

Letting ε, δ go to zero (we leave the justification of this to the reader) we obtain

$$H(z) = \lim_{\varepsilon,\delta\to 0} I + II = (-\mathrm{e}^{-\pi(z-1)} + \mathrm{e}^{\mathrm{i}\pi(z-1)}) \int_0^{\infty} \frac{t^{z-1}\mathrm{e}^{-t}}{1-\mathrm{e}^{-t}}\,\mathrm{d}t.$$

Noting that $(-\mathrm{e}^{-\pi(z-1)} + \mathrm{e}^{\mathrm{i}\pi(z-1)}) = -2\mathrm{i}\sin(\pi z)$ and recalling (9.8) we obtain the desired conclusion. $\square$

Corollary 9.22 *The ζ function can be extended to a holomorphic function on $\mathbb{C}\setminus\{1\}$. At 1 it has a simple pole with residue 1.*

Proof. We already know that ζ is holomorphic for $\Re(z) > 1$. Then equation (9.10) gives a way to extend ζ. Indeed, as observed above $H(z)$ is an entire function and so is sin. The only zeros of $\sin(\pi z)$ are the integers. The Γ function is holomorphic on $\mathbb{C}\setminus\{0,-1,-2,-3,\ldots\}$ and does not have any zeros. Hence (9.10) defines a holomorphic extension of ζ to $\mathbb{C}\setminus\{1,0,-1,-2,-3,-4,\ldots\}$.

For the non-negative integers $-n$ for $n\in\mathbb{N}_0$ we note that Γ has a pole of order one and $\sin(\pi z)$ has a zero of order one. Hence, the product, has a nonzero extension to those points. This implies that the singularities of ζ at these points are removable.

It remains to treat the singularity at 1. For $z=1$ consider the description of the Hankel function given in (9.11). If $z=1$, the term $(-w)^{z-1}$ in the definition of u is constant. In particular, there is no discontinuity for u on the positive real axis. Then (using the notation from (9.11)) it is easy to see that $\lim_{\varepsilon,\delta\to 0} I + II = 0$. But for $z=1$ the term III does not vanish as ε,δ go to zero. Indeed, for fixed ε and $\delta\to 0$ we get

$$III = \int_{\partial B_\delta(0)} \frac{\mathrm{e}^{-w}}{1-\mathrm{e}^{-w}} = 2\pi\mathrm{i}$$

by the *Residue theorem.*

Hence, we can conclude that

$$\mathrm{res}(\zeta,1) = \lim_{z\to 1} \frac{(z-1)}{2\mathrm{i}\sin(\pi z)} \frac{-H(z)}{\Gamma(z)} = \frac{1}{-2\pi\mathrm{i}} \frac{-2\pi\mathrm{i}}{1} = 1.$$

□

We close the discussion of the ζ function by stating without proof that, like the Γ function, it satisfies a functional equation. Indeed for any $z\in\mathbb{C}\setminus\{0,1\}$

$$\zeta(1-z) = 2\zeta(z)\,\Gamma(z)\cos\left(\frac{\pi}{2}z\right)(2\pi)^{-z}. \tag{9.12}$$

The proof of this formula is a nice application of the residue theorem which the reader should have seen on the exercise sheets. A complete proof can be found, for example, in [Ahl78] or in [GK06]. It follows immediately from this formula that the only zeros of ζ outside of the strip $\{z : 0\le \Re(z)\le 1\}$ are $-2n$ for any $n\in\mathbb{N}$. These are the so called *trivial zeros* of the zeta function. Another consequence of (9.12) is that $\zeta(-1) = \frac{-1}{12}$. Plugging this into the definition of the ζ- function (9.7) (which is of course not at all justified) formally gives the nice (and of course not to be taken literally) identity

$$1+2+3+4+5+6+7+8+9+10+\ldots = \frac{-1}{12}.$$

The question about different zeros leads is one of the most famous open problems in all of mathematics.

Conjecture 9.23 (Riemann hypothesis) All nontrivial zeros z of the Riemann zeta function have $\Re(z) = \frac{1}{2}$.

10 The Riemann mapping theorem

In this chapter we will study one of the deepest and important results in complex analysis, the *Riemann mapping theorem*. This requires a more general version of connectedness when we define *simply-connected* sets below. We first recall the following definition on conformal mappings.

Definition 10.1 Let U and V be to open subsets of $\mathbb{C}$. We say that U and V are *conformally equivalent* if there exists a $\varphi\colon U \to V$ with the following properties:

- φ is holomorphic,
- φ is a bijection from U to V,
- the inverse mapping $\varphi^{-1}\colon V \to U$ is holomorphic.

We call such a map φ conformal.

Actually, the third assumption follows from the first two. Indeed, if $\varphi\colon U \to V$ is injective, then φ' cannot vanish on U (see the discussion in Section 5.2). Then the inverse function theorem implies that automatically φ^{-1} is holomorphic.

Remark 10.2 The reader has certainly already encountered similar concepts of *equivalence* in geometry. In school, one learns that two subsets U and V (e.g. triangles) of $\mathbb{C} = \mathbb{R}^2$ are *congruent* if there exists a mapping of the type $\varphi(z) = \alpha z + \beta$ for $\alpha, \beta \in \mathbb{C}$ with $|\alpha| = 1$ such that $U = \varphi(V)$. In topology there is a much less rigid concept: Two sets U and V are *homeomorphic* if there exists a *continuous* bijective mapping with continuous inverse from U to V. Conformal equivalence is in between these two concepts in the sense that any two congruent sets are conformally equivalent, and any two conformally equivalent sets are homeomorphic. $\diamond$

Example 10.3 The unit disc $\mathbf{\Delta} = \{z \in \mathbb{C}\colon |z| < 1\}$ and all of $\mathbb{C}$ are *not* conformally equivalent. This follows from Liouville's theorem, 5.8, because any holomorphic mapping from $\mathbb{C} \to \mathbf{\Delta}$ is bounded and hence constant. ♣

Example 10.4 Liouville's theorem does *not* imply that unbounded sets cannot be conformally equivalent to bounded sets. Consider for example the sets $D = \{z \in \mathbb{C}\colon \Im(z) > 0\}$ and $V = \mathbf{\Delta}$. The Möbius transform $\varphi(z) = \frac{z-i}{z+i}$ is a conformal map from D to $\mathbf{\Delta}$. ♣

Example 10.5 (The Koebe function) Consider the function

$$k(z) = \frac{z}{(1-z)^2}.$$

We have

$$k(z) = z\,\partial_z\left(\frac{1}{1-z}\right) = z\partial_z \sum_{k=0}^{\infty} z^k = z + \sum_{k=2}^{\infty} k z^k.$$

On the other hand, we can write

$$k(z) = \frac{1}{4}\left(\frac{1+z}{1-z}\right)^2 - \frac{1}{4}. \tag{10.1}$$

Equation (10.1) shows that k is a conformal mapping onto its image. Indeed, k is the composition of the mappings $f_1\colon z \mapsto \frac{1+z}{1-z}$, $f_2\colon z \mapsto \frac{1}{4}z^2$, and $f_3\colon z \mapsto z - \frac{1}{4}$, that is, $k(z) = f_3(f_2(f_1(z)))$. The function f_1 is a Möbius transformation and it maps the unit disc conformally to the half space $\mathrm{H} = \{z \in \mathbb{C}\colon \Re(z) > 0\}$. The function f_2 is of course not injective on all of $\mathbb{C}$ but restricted to H it is, and it maps H to the “slit plane” $\mathbb{C}_- = \mathbb{C} \setminus \{\lambda \in \mathbb{R}\colon \lambda \leq 0\}$. Finally, f_3 is a shift. This shows that k is injective on $\boldsymbol{\Delta}$, and the image of the unit disc is the slit plane $\mathbb{C} \setminus \{\lambda \in \mathbb{R}\colon \lambda \leq -\frac{1}{4}\}$. ♣

Before we continue our discussion of conformal equivalence, let us recall a notion from topology.

We start by observing that we can integrate holomorphic functions over arbitrary continuous curves without any further regularity assumption. We give a brief sketch this, in order to avoid carrying unnecessary regularity assumptions through the topological discussion.

Indeed, let $D \subseteq \mathbb{C}$ be open and let $\gamma\colon [t_0, t_1] \to D$ be a continuous curve. Then a simple compactness argument shows that we can always find a partition $t_0 = s_0 < \ldots < s_N = t_1$ and a radius $r > 0$ such that for each s_i the ball $B_r(\gamma(s_i))$ is fully contained in D and such that each ball $B_r(\gamma(s_i))$ contains the full path $\gamma([s_i, s_{i+1}])$. Let now $f\colon D \to \mathbb{C}$ be a holomorphic function. Of course, we cannot in general assume that f has an antiderivative on all of D. But the balls $B_r(\gamma(s_i))$ are convex (and in particular star shaped), so f does have an antiderivative F_i on each of these balls. Then we define

$$\int_\gamma f(z)\,\mathrm{d}z = \sum_{i=0}^{N-1} F_i(z_{i+1}) - F_i(z_i). \tag{10.2}$$

Each of the F_i is only determined uniquely up to an additive constant, but the difference that appears in (10.2) does not depend on the specific choice. According to the chain rule, this definition coincides with the original definition if γ is piecewise $\mathcal{C}^1$. We leave it to the reader check that the value of (10.2) does not depend on the specific choice of the s_i and $r > 0$. Furthermore, the reader can convince himself that the Cauchy theorem and the Residue theorem are still valid if one removes the $\mathcal{C}^1$ assumption on the curves.

Definition 10.6 (Homotopic) Let D be a subset of $\mathbb{C}$ and let $\gamma_1, \gamma_2\colon [t_0, t_1] \to D$ be two continuous curves in D with the same endpoints, i.e. $\gamma_1(t_0) = \gamma_2(t_0)$ and $\gamma_1(t_1) = \gamma_2(t_1)$. Then γ_1 and γ_2 are called *homotopic* if there exists a continuous mapping $h\colon [0,1] \times [t_0, t_1] \to D$ such that for all $t \in [t_0, t_1]$ and all $s \in [0,1]$

$$h(0,t) = \gamma_1(t), \quad h(1,t) = \gamma_2(t), \quad h(s,t_0) = \gamma_1(t_0), \quad \text{and} \quad h(s,t_1) = \gamma_1(t_1).$$

Such a maping h is called a *homotopy*.

Remark 10.7 The concept of *homotopy* is related to *homology* that we already encountered above. Indeed, we will see below that if γ_1 and γ_2 are homotopic in D, then the concatenation of γ_1 and the inverse of γ_2 does not wind about any point in the complement of D. The inverse implication is not true. $\diamond$

Theorem 10.8 *Let γ_1 and γ_2 be homotopic in an open set D and let $f\colon D \to \mathbb{C}$ be holomorphic in D, i.e. $f \in \mathcal{H}(D)$. Then*

$$\int_{\gamma_1} f(z)\,\mathrm{d}z = \int_{\gamma_2} f(z)\,\mathrm{d}z.$$

Proof. Let $h\colon [0,1] \times [0,1] \to D$ be a homotopy of γ_1 and γ_2. Then for $\tau \in [0,1]$ we define a closed curve

$$\sigma_\tau(t) = \begin{cases} \gamma_1(t) & \text{if } t \in [0,1], \\ h(\tau, 2-t) & \text{if } t \in [1,2]. \end{cases}$$

We want to show that $\int_{\sigma_1} f(z)\,\mathrm{d}z = 0$. It is obvious that $\int_{\sigma_0} f(z)\,\mathrm{d}z = 0$ - the curve σ_1 follows along γ_1 once and then again with opposite orientation.

We claim that the mapping $\tau \mapsto \int_{\sigma_\tau} f(z)\,\mathrm{d}z$ is locally constant. More precisely, we will establish that for every $\tau \in [0,1]$ there exists an $\varepsilon > 0$ such that for $\hat{\tau} \in [0,1]$ with $|\tau - \hat{\tau}| < \varepsilon$ we get

$$\int_{\sigma_\tau} f(z)\,\mathrm{d}z = \int_{\sigma_{\hat{\tau}}} f(z)\,\mathrm{d}z.$$

To see this we observe that for any subdivision $1 = s_0 < s_1 < \ldots < s_N = 2$ we get

$$\int_{\sigma_\tau} f(z)\,dz - \int_{\sigma_{\hat{\tau}}} f(z)\,\mathrm{d}z = \sum_{n=0}^{N-1} \int_{\gamma_n} f(z)\,\mathrm{d}z,$$

where γ_n is the boundary of the image under h of the square with corner (τ, s_n), (τ, s_{n+1}), $(\hat{\tau}, s_{n+1})$, and $(\hat{\tau}, s_n)$. The continuity of h implies that for ε small enough the subdivision can always be made fine enough to ensure that each of the γ_i is fully contained in a ball that is fully contained in D. For such a choice the integrals over the γ_i are all equal to 0. $\square$

Definition 10.9 A connected subset D of $\mathbb{C}$ is called *simply connected* if any two continuous curves γ_1 and γ_2 with the same endpoints are homotopic.

Remark 10.10 (a) Intuitively speaking, D is simply connected if it has no holes.

(b) Convex sets and more generally star shaped sets are simply connected.

(c) There is an equivalent characterisation of simply connected sets in terms of closed paths: A set D is simply connected if and only if every closed path is homotopic to a constant path.

(d) In particular, if D is simply connected and γ is a closed curve in D, then γ is homotopic to a constant curve. For any $z_0 \in \mathbb{C} \setminus D$ the function $z \mapsto \frac{1}{z-z_0}$ is holomorphic in D. Hence, the winding number about any point in the complement of D is 0. This yields the following version of Cauchy's theorem.

(e) If D is simply connected and V is homeomorphic to D, then V is also simply connected. This follows immediately from the definition. One has to be a bit careful though: It is in general not true that the continuous image of a simply connected set is simply connected. Consider for example the set $\{z \in \mathbb{C} \colon \Re(z) > 0\}$. This set is convex and hence simply connected. But the image under the mapping, say $z \mapsto z^3$ is $\mathbb{C} \setminus \{0\}$ which is not simply connected.

$\diamond$

Theorem 10.8 immediately allows to state the following version of Cauchy's theorem.

Theorem 10.11 (Cauchy's theorem for simply connected domains) *Let $D \subset \mathbb{C}$ be open and simply connected. Then for any closed curve γ in D and for any holomorphic $f \colon D \to \mathbb{C}$ we have*

$$\int_\gamma f(z)\,\mathrm{d}z = 0.$$

Proof. It suffices to observe that every closed curve is homotopic to a constant curve. $\square$

In particular, $D = \mathbb{C} \setminus \{0\}$ is not simply connected. To see this, it suffices to observe that the $z \mapsto \frac{1}{z}$ is holomorphic in D but

$$\int_{\partial\boldsymbol{\Delta}} \frac{1}{z}\,\mathrm{d}z = 2\pi\mathrm{i} \neq 0\,.$$

Lemma 10.12 *Let $D \subset \mathbb{C}$ be open and simply connected and let $f \colon D \to \mathbb{C} \setminus \{0\}$ be holomorphic. Then there exists a holomorphic function $g \colon D \to \mathbb{C}$ such that*

$$f(z) = \mathrm{e}^{g(z)} \qquad \text{for } z \in D.$$

The function g is unique up to an additive constant $2\pi i n$ for all $n \in \mathbb{Z}$. We say that f has a holomorphic logarithm.

Before we prove the lemma, let us collect some useful facts.

Remark 10.13 (a) Lemma 10.12 implies the existence of a holomorphic square root of f. Indeed, set

$$h(z) = \mathrm{e}^{\frac{1}{2}g(z)}\,.$$

Then $h(z)^2 = f(z)$ on D.

(b) If $0 \notin D$ the statement can in particular be applied to the holomorphic function $f(z) = z$.

(c) Note that this property is not true for the not simply connected set $\mathbb{C} \setminus \{0\}$.

(d) It turns out that this seemingly harmless property already characterises simply connected sets. Indeed, the existence of square roots is the only property of a simply connected set that we use in the proof of the Riemann mapping theorem.

$\diamond$

Proof. Fix an arbitrary $z_0 \in D$. As $f(z_0) \neq 0$ by assumption, it is always possible to find a $w_0 \in \mathbb{C}$ such that $\exp(w_0) = f(z_0)$. This w_0 is uniquely determined up to a multiple of $2\pi\mathrm{i}$. Let us fix such an w_0 and set $g(z_0) = w_0$. Now for any other point $z \in D$, let γ be a curve in D that connects z_0 to z (such a curve always exists because D is open and connected) and set

$$g(z) = w_0 + \int_{f\circ\gamma} \frac{1}{z}\,\mathrm{d}z = w_0 + \int_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z.$$

Of course the choice of γ is by no means unique, but due to the simple connectedness of D, any two different curves give the same value for $g(z)$. The function g one obtains in this way is holomorphic and its derivative is $\frac{f'(z)}{f(z)}$.

It remains to establish that g satisfies $\exp(g(z)) = f(z)$. We calculate

$$\left(f(z)\mathrm{e}^{-g(z)}\right)' = f'(z)\mathrm{e}^{-g(z)} - f(z)\mathrm{e}^{-g(z)}\frac{f'(z)}{f(z)} = 0.$$

Hence $f(z)\mathrm{e}^{-g(z)}$ is constant on D. If we denote the value by α we get

$$\alpha = f(z_0)\mathrm{e}^{-g(z_0)} = f(z_0)\mathrm{e}^{-w_0} = 1.$$

In order to define the square root, it suffices to set $r(z) = \exp\left(\frac{1}{2}l(z)\right)$. $\square$

The following surprising theorem is the main result of this chapter.

Theorem 10.14 (Riemann mapping theorem) *Let $D \subset \mathbb{C}$ be open, simply connected and $D \neq \mathbb{C}$. Then D is conformally equivalent to the unit disc $\mathbf{\Delta} = \{z \in \mathbb{C} \colon |z| < 1\}$.*

Remark 10.15 Already the topological implication of the Riemann mapping theorem is not trivial. It implies that any simply connected $D \subset \mathbb{C}$ is homeomorphic to the unit disc. Note that here $D = \mathbb{C}$ is allowed as can be checked easily. $\diamond$

Proof. We split the proof into two main steps.

Step 1. Let $D \neq \mathbb{C}$ be open and simply connected. We start by establishing that there exists an injective holomorphic mapping $\varphi \colon D \to \mathbf{\Delta}$. Of course, such a mapping is automatically a conformal transformation onto its image, so that this will imply that D is conformally equivalent to a subset of $\mathbf{\Delta}$. This claim is trivial if D is fully contained in $B_R(0)$ for any R – indeed in that situation $\varphi(z) = \frac{z}{R}$ does the job. It is also easy to find such a φ, if there is only a single $w_0 \in \mathbb{C}$ and a $\delta > 0$ such that $B_\delta(w_0) \cap D = \varnothing$. In this case one can simply set $\varphi(z) = \frac{\delta}{z-w_0}$. Hence, by composing with this such a map it is sufficient to show that there exists a φ such that there exists a ball $B_\delta(w_0)$ such that $\varphi(D)$ does not intersect $B_\delta(w_0)$.

In order to establish this, we can assume without loss of generality that $0 \notin D$ (this is the only place, where we need that $D \neq \mathbb{C}$). If this is not the case we can always translate D. Then according to Lemma 10.12 we can find an holomorphic branch of the square root r on D. We claim that r has the desired properties. First of all r is injective, because, $r(z_1) = r(z_2)$ implies $z_1 = r(z_1)^2 = r(z_2)^2 = z_2$. Also, if $w = r(z)$ for some $z \in D$ than $-w$ cannot be in the image of D. Actually, assume that $-w = r(z')$ for some z'. Then $z' = (-w)^2 = w^2 = z$ which is a contradiction, because z cannot be mapped both to w and $-w$. But then we are done: by the open mapping theorem, the image of D under r must contain at least some ball $B_\delta(z_0)$ which implies that the ball $B_\delta(-z_0)$ is disjoint from the image of D. (Exercise: Check that we could have taken the logarithm instead of the square root.)

By Step 1 we can and will assume from now on that $D \subset \mathbf{\Delta}$ and that $0 \in D$ (if 0 is not in D, we can always contract D by a small factor and then translate it). We consider the following set of functions

$$\mathcal{F} = \{f\colon D \to \mathbf{\Delta},\ f \text{ holomorphic and injective},\ f(0) = 0\}.$$

It is clear that $\mathcal{F}$ is not empty, because the mapping $f(z) = z$ always satisfies the required assumptions. In Steps 2 and 3 we will establish that $\mathcal{F}$ contains at least one surjective function.

Step 2. We will establish the following claim: Assume that $f \in \mathcal{F}$ is *not* surjective. Then there exists another function $F \in \mathcal{F}$ with $|f'(0)| < |F'(0)|$.

To see this, assume that $f \in \mathcal{F}$ and that $w_0 \notin f(U)$. The Möbius transformation $\varphi_0(z) = \frac{z-w_0}{1-\bar{w}_0 z}$ is a bijection from $\mathbf{\Delta}$ to $\mathbf{\Delta}$ and w_0 is the only point in $\mathbf{\Delta}$ that is mapped to 0 under φ_0. Hence, the function $\varphi_0 \circ f\colon U \to \mathbf{\Delta}$ is injective and does not attain the value 0. Also $\varphi \circ f$ is a conformal mapping onto its image which is in particular simply connected. By Lemma 10.12 there exists an holomorphic branch of the square root on $\varphi \circ f(U)$ which we denote by r. It follows as in Step 1 that r is injective. Denote by $w_1 = r(-w_0) = r(\varphi_0(f(0)))$ and define one more Möbius transform, $\varphi_1(z) = \frac{z-w_1}{1-\bar{w}_1 z}$. Then the mapping $F := \varphi_1 \circ r \circ \varphi_0 \circ f$ is holomorphic and injective, it maps U to $\mathbf{\Delta}$ and $F(0) = 0$, i.e. $F \in \mathcal{F}$. We claim that $|F'(0)| > |f'(0|$. Actually, to see this denote by $h(z) = \varphi_0^{-1}((\varphi_1^{-1}(z))^2)$. Note that h maps $\mathbf{\Delta}$ to $\mathbf{\Delta}$ with $h(0) = 0$. Also h cannot be a rotation because it is not injective (the Möbius transforms are bijections and the map $z \mapsto z^2$ is two to one). Hence by Schwarz' lemma, Theorem 5.26, $|h'(0)| < 1$. Therefore, we have $|f'(0)| = |h'(0)|\,|F'(0)| < |F'(0)|$, which shows the claim.

Step 3. To finish the prove of the Riemann mapping theorem, it is left to show that there exists an $F \in \mathcal{F}$ with maximal $|F'(0)|$. More precisely, we will establish, that there exists an $F \in \mathcal{F}$ such that

$$|F'(0)| = \sup\{|f'(0)|\colon f \in \mathcal{F}\}. \tag{10.3}$$

Here we will be able to make use of several facts that we have discussed throughout the course.

First we need to show that the set on the right hand side of (10.3) is indeed bounded. To see this, let δ be small enough to ensure that $\bar{B}_\delta(0) \subseteq U$. Then by Cauchy's integral formula (or the explicit expression for the Taylor coefficients of f in (5.1)) we have

$$|f'(0)| = \left| \frac{1}{2\pi \mathrm{i}} \int_{\partial B_\delta(0)} \frac{f(z)}{z^2}\, dz \right| \leq \frac{1}{\delta}.$$

Hence, the supremum in (10.3) is finite. Let us denote it by S.

Let f_n be a sequence in $\mathcal{F}$ with $|f_n'(0)| \uparrow S$. By Montel's theorem, Theorem 8.9, there exists a locally uniformly convergent subsequence. After relabelling we will still denote it by f_n and its limit by f. We claim that $f \in \mathcal{F}$ with $|f'(0)| = S$.

First of all, f is holomorphic and $f(0) = 0$. By Theorem 8.2 f_n' converge locally uniformly to f' which implies the $|f'(0)| = S$. This shows that f cannot be constant and hence, by Hurwitz' theorem, Theorem 8.3 and Corollary 8.5, f is injective. Finally, as the uniform limit of the f_n we have $|f(z)| \leq 1$ for all $z \in \boldsymbol{\Delta}$. But by the open mapping theorem, Theorem 5.23, $f(\boldsymbol{\Delta}) \subseteq \boldsymbol{\Delta}$. Hence, $f \in \mathcal{F}$ with maximal first derivative in 0. This finishes the proof of the claim, and hence the proof of the Riemann mapping theorem. □

Remark 10.16 We cannot expect the map φ in the Riemann mapping theorem to be unique. But if we fix any z_0 in D, there exists a unique conformal mapping $\varphi\colon D \to \boldsymbol{\Delta}$ with $\varphi(z_0) = 0$ such that $\varphi'(z_0)$ is real and positive. (Exercise!) ◇

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Index