Interacting Particle Systems

Problem sheet

1. Consider the TASEP on the lattice $\Lambda_L = \{1, \ldots, L\}$ with open boundaries and generator

$$\begin{aligned}\mathcal{L}f(\eta) \;=\; &\sum_{x=1}^{L-1} \eta(x)\big(1-\eta(x+1)\big)\big(f(\eta^{x,x+1}) - f(\eta)\big) + \\ &+\rho_l(1-\eta(1))\big(f(\eta^1) - f(\eta)\big) + \eta(L)(1-\rho_r)\big(f(\eta^L) - f(\eta)\big)\ .\end{aligned}$$

(a) For which values of the boundary densities ρ_l and ρ_r is this process irreducible?

(b) For a given $\eta \in \{0,1\}^{\Lambda_L}$ write the stationary probability $\mu_L(\eta)$ in terms of a product of matrices, and give the quadratic algebra that has to be fulfilled.

(c) Compute the stationary current and the density profile for a system of size $L = 2$ in terms of ρ_l and ρ_r. (Hint: Use the recursion relations and normalize the boundary vectors to $w^T v = 1$.)

2. Suppose the stationary current of a lattice gas on $\Lambda = \mathbb{Z}$ is given by $j(\rho) = \rho^2(1-\rho)$ and its density obeys the macroscopic conservation law

$$\frac{\partial}{\partial t}\rho(x,t) + \frac{\partial}{\partial t} j\big(\rho(x,t)\big) = 0\ .$$

(a) Starting with a step initial measure

$$\mu = \nu_{\rho_l,\rho_r} \quad \text{product measure with} \quad \nu_{\rho_l,\rho_r}(1_x) = \begin{cases} \rho_l & , \ x \le 10000 \\ \rho_r & , \ x > 10000 \end{cases}\ ,$$

what density $\mu_t(1_0)$ do you expect in the origin as $t \to \infty$?

(b) Summarize your results in a phase diagram for the corresponding dynamic phase transition.

(c) What density $\mu_t(1_{[t]})$ do you expect to see at site $[t]$ as $t \to \infty$?

3. Consider the totally asymmetric simple exclusion process (TASEP) $(\eta_t : t \ge 0)$ on the finite periodic lattice $\tilde{\Lambda}_L = \mathbb{Z}/L\mathbb{Z}$ with jump rates $p = 1$ and $q = 0$. Suppose that one of the particles is special, i.e. it jumps with rate $\alpha > 0$.

(a) Explain how a TASEP configuration η can be mapped onto a configuration ζ of a zero-range process (ZRP) with a special site in the origin.
Define the ZRP $(\zeta_t : t \ge 0)$ corresponding to the TASEP $(\eta_t : t \ge 0)$ by giving the lattice Λ, the state space and the generator.

(b) Compute the grand-canonical stationary product measures ν_ϕ of the ZRP and give the marginals on site 0 and any other site $x \ne 0$.

(c) Compute the stationary density $\rho_x(\phi) = \nu_\phi(\eta(x))$ as a function of the fugacity for $x = 0$ and $x \ne 0$.

(d) The total density is given by

$$\rho(\phi) = \frac{1}{|\Lambda|} \sum_{x\in\Lambda} \rho_x(\phi)\ .$$

Use this to derive the current-density relation $j(\rho)$, i.e. the stationary current as a function of ρ. Do not attempt to solve the quadratic equation yourself but use something like MATLAB or MAPLE, then plot the function $j(\rho)$ for $L = 100$ and $\alpha = 0.5$ and 2.

(e) Explain why the total density ρ in the ZRP is related to the TASEP density $\tilde{\rho}$ via

$$\rho = \frac{1-\tilde{\rho}}{\tilde{\rho}} \,,$$

and the stationary currents are related via $\tilde{j} = \tilde{\rho}\, j$.
Use this to plot the current-density relation $\tilde{j}(\tilde{\rho})$ of the TASEP for $L = 100$ and $\alpha = 0.5$ and 2. Interpret your results.

4. Are there boundary induced phase transitions for ZRPs analogous to the ones for the ASEP? Justify your answer.

5. The single server queue (M/M/1):
Let $(\eta_t : t \geq 0)$ be a continuous time Markov chain with state space $\mathbb{N} = \{0, 1, \ldots\}$ and jump rates

$$c(\eta, \eta+1) = \alpha \,, \quad c(\eta, \eta-1) = \beta(1-\delta_{0,\eta}) \,.$$

η_t can be interpreted as the number of customers at time t, arriving at rate $\alpha > 0$ and being served at rate $\beta > 0$.

(a) Write down the master equation for this process.

(b) Show that for $\alpha > \beta$ the process is transient (transience and recurrence are defined in the lecture notes on Markov chains section1.4).
Hint: Derive from the master equation that $\mathbb{E}(\eta_t) \to \infty$ as $t \to \infty$.

(c) Show that for $\alpha < \beta$ the process is positive recurrent by giving its stationary distribution μ. Is the distribution reversible?

(d) What do you think happens for $\alpha = \beta$?

(e) Let $A \sim PP(\alpha)$ be the arrival process of customers. Show that for $\alpha < \beta$ the departure process D is also Poisson $D \sim PP(\alpha)$ given that the process is stationary (this is called **Burke's theorem**).
Hint: Conditioning on the value of η_t, show that

$$\mathbb{P}\big(\text{at least one departure in } [t, t+\Delta t)\big) = 1 - e^{-\alpha \Delta t} \,.$$

What do you think happens for $\alpha \geq \beta$?

6. Random sequential update:
Claim: To simulate the (continuous-time) TASEP with $p = 1$, $q = 0$ on $\Lambda_L = \mathbb{Z}/L\mathbb{Z}$ for large L do the following:

- Pick a site $x \in \Lambda_L$ uniformly at random;
- update your time counter $t \mapsto t + \Delta t$ by $\Delta t = 1/L$;
- if $\eta(x) = 1$ and $\eta(x+1) = 0$ move the particle, i.e. put $\eta(x) = 0$, $\eta(x+1) = 1$;

then start over again.

(a) Show that the number of timesteps k it takes for a given particle to attempt a jump is a geometric random variable with parameter $1/L$, i.e. mean L.

(b) Show that for $L \to \infty$, the waiting time $t = k * \Delta t$ for a given particle to attempt a jump is an exponential with rate 1.

(c) How does this algorithm have to be modified to simulate the ASEP with $p, q > 0$?

7. (a) Consider a general ZRP and define $p_x(\phi) = \log z_x(\phi)$ (called pressure in statistical mechanics). Show that you can write $\rho_x(\phi) = \phi \frac{\partial}{\partial \phi} p_x(\phi)$. Use this to show that $\rho_x(\phi)$ is monotone increasing, which is equivalent to $p_x(\phi)$ being convex. To do this, compute the derivative of $\rho_x(\phi)$ and write it as the variance of $\eta(x)$ (which is positive).

(b) Consider now a ZRP on $\Lambda_L = \mathbb{Z}/L\mathbb{Z}$ with $g_x(k) = \alpha k$ and translation invariant jump probabilities $p(x, y) = p(x - y)$. Show that the stationary measure is homogeneous and Poisson and compute $\rho(\phi)$ and the current.