Interacting Particle Systems

Problem sheet 2

1. Consider the birth-death chain $(N(t) : t \geq 0)$ on the state space $X = \mathbb{N}$ with jump rates $c(n, n+1) = \alpha n$ for $n \geq 0$ and $c(n, n-1) = \beta n$ for $n \geq 1$, where $\alpha, \beta > 0$.

(a) Write down the master equation for this Markov chain.
Is the chain irreducible?

(b) Derive an equation for the expected value $e(t) = \mathbb{E}(N(t))$ and solve it with initial condition $N(0) = k \in \mathbb{N}$.
What does this imply for the asymptotic behaviour of $N(t)$ as $t \to \infty$?

2. On the finite complete graph $\Lambda_L = \{1, \ldots, L\}$ with $E = \Lambda_L \otimes \Lambda_L$ the CP $(\eta_t : t \geq 0)$ can be described by a single variable

$$N(t) = \sum_{x \in \Lambda} \eta_t(x) = \text{\# of infected sites at time } t \in \{0, \ldots L\} .$$

(a) Write down the jump rates and the master equation for $(N(t) : t \geq 0)$.

(b) For $N(0) = 1$ discuss the asymptotic behaviour of $N(t)$ as $t \to \infty$ depending on λ.

3. Consider the *pair contact process (PCP)* on the lattice $\Lambda = \mathbb{Z}$ with state space $X = \{0, 1\}^\Lambda$ and jump rates

$$110 \xrightarrow{\lambda} 111 , \quad 011 \xrightarrow{\lambda} 111 \quad \text{and} \quad 11 \xrightarrow{1} 00 .$$

(a) Write down the generator of the process.
Is the process irreducible? How many absorbing states does it have?

(b) Derive the mean field rate equation and analyze its stationary points depending on λ.
Give the mean field predition for the phase diagram and the stationary density $\rho(\lambda)$.

(c) Consider the PCP on the finite lattice $\lambda_4 = \mathbb{Z}/4\mathbb{Z}$ with periodic boundary conditions.
Provide a graphical construction that connects the initial configuration $\eta_0 = (1, 0, 1, 1)$ with $\eta_t = (0, 1, 0, 0)$. The construction should contain at least three of each basic event (infection to the left, to the right, and recovery).
How many absorbing states does this process have?

4. Consider the ZRP on the lattice $\Lambda_L = \mathbb{Z}$ with space-independent jump rates $g(n)$ and jump probabilities $p(x, y) = \delta_{y,x+1}$.

(a) Suppose that $g(n) \nearrow$ is non-decreasing in n. Use a basic coupling (particles jump together whenever possible) to show that the ZRP is attractive.

(b) Suppose that $g(\bar{n} + 1) < g(\bar{n})$ for some $\bar{n} \geq 1$. Find a suitable test function f and initial condition η to show that the ZRP is not attractive.
(Hint: Keep it as simple as possible.)