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Random walks in a 1D Levy random environment

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Anomalous diffusions

Anomalous diffusions are stochastic processes $X(t) \in \mathbb{R}^d$ such that

$$\mathbb{E}(X^2(t)) = ct^\delta \,, \quad \delta \neq 1$$

This behavior of superdiffusive processes ($\delta > 1$) characterizes many different natural systems and is mainly connected to
motion in disorder media:

- light particle in an optical lattice;
- tracer in a turbolent flow;
- efficient routing in network;
- predator hunting for food

Main features

- long ballistic “flights“, where particle moves at constant velocity
- short disorder motion

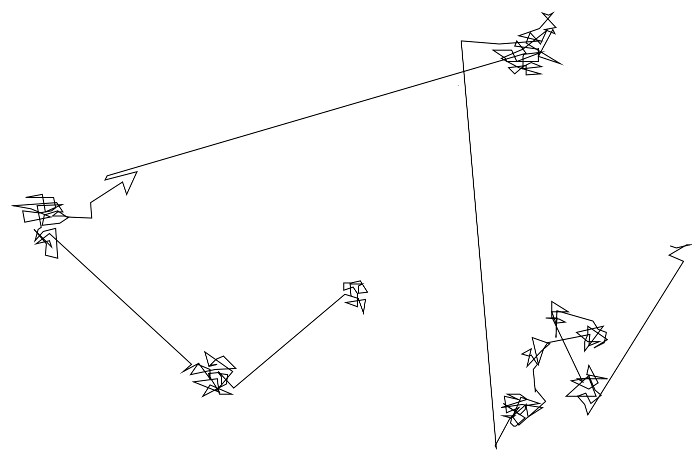

Figure 1: Typical Levy flight

Models for anomalous diffusions

LEVY FLIGHTS

Schlesinger, Klafter['85] , Blumen, Klafter, Schlesinger, Zumofen ['90] ,

Random walk $(X(n))_{n\in\mathbb{N}}$ on $\mathbb{R}^d$ with lenght steps given by a sequence of i.i.d. Levy α-stable distribution with $\alpha \in (0,2)$:

heavy-tailed distribution $\qquad \mathbb{P}(Z > x) \sim x^{-\alpha}$ for $x \to +\infty$

$$\longrightarrow \quad \text{Var}(Z) = +\infty \quad ; \quad \mathbb{E}(Z) \begin{cases} < \infty & \text{if } \alpha \in (1,2) \\ = \infty & \text{if } \alpha \in (0,1] \end{cases}$$

Formally:

Given $(\xi_k)_{k\in\mathbb{N}}$, i.i.d. U$[0;2\pi]$, independent of $(Z_k)_{k\in\mathbb{N}}$, i.i.d Levy α-stable

$$X(0) = 0 \quad , \quad X(n) = X(n-1) + Z_n \xi_n, \quad n \geq 0$$

LEVY WALKS

Stochastic process $(X(t))_{t\in\mathbb{R}^+}$ on $\mathbb{R}^d$ defined similarly to Levy flights but with jumps covered at constant velocity v_0.

Formally:
Given $(\xi_k)_{k\in\mathbb{N}}$, i.i.d. U$[0;2\pi]$, independent of $(Z_k)_{k\in\mathbb{N}}$, i.i.d Levy α-stable

$$X(0)=0 \quad , \quad X(t)=X(\frac{Z_{k-1}}{v_0})+\xi_k v_0 t \quad , \text{for } t\in(\frac{Z_{k-1}}{v_0},\frac{Z_k}{v_0}]$$

Notice: in both processes **increments are independent**, $\longrightarrow$ scatterers are removed after each collision event.

Annealed results on the second moments

Levy flights and walks give rise to superdiffusive anomalous motion and in particular

$$\mathbb{E}(X^2(t)) \sim \begin{cases} t^{3-\alpha} & \text{if } \alpha \in (0,1] \\ t^2 & \text{if } \alpha \in (1,2) \end{cases} \qquad \text{for } t \to \infty$$

This suggests to model the transport in inhomogeneous material with the motion of a particle in a "Levy randon environment".

LEVY-LORENTZ GAS
Barkai, Fleurov,Klafter['00]

Motion of a particle in a fixed array of scatterers arranged randomly in such a way that the interdistances between them are i.i.d. α-stable Levy random variables.

MODEL

1D random walk in a Levy Random environment

• Let $(Z_k)_{k\in\mathbb{Z}}$ i.i.d. random variables taking value on $\mathbb{N}^+$ and with law P s.t.

$$P(Z > k) \sim k^{-\alpha} \quad \text{for} k \ll 1 \qquad \text{(heavy tails)}$$

• Construct a (non-equilibrium) Renewal Point Process on $\mathbb{Z}$, denoted by $\text{PP}(Z) = \{\dots Y_{-1} < Y_0 < Y_1 < \dots\}$, s.t.

1. $Y_0 = 0$
2. $|Y_k - Y_{k-1}| = Z_k$

so that $Y_k = \text{sgn}(k) \sum_{j=1}^{|k|} Z_{\text{sgn}(k)j}$, $\qquad k \neq 0$

Levy Random environment$\equiv$PPZ , i.e., scatterers are placed at points Y_k.

• Let $(\xi_k)_{k\in\mathbb{Z}}$ i.i.d. symmetric random variables taking value on $\{-1,+1\}$ with law Q.

Definition 1. *$X(t)$, $t \in \mathbb{N}$ is the process on $\mathbb{Z}$ such that*

$X(0) = 0$
$X(t+1) = X(t) + \xi_{n(t)}$, *for $t > 0$*

with $n(t) = |\{s \le t : X(s) \le PP(Z)\}|$ = number of collisions up to t.

For a given realization $z \in (\mathbb{N}^+)^{\mathbb{Z}}$ of the variables $(Z_k)_{k\in Z}$, let $\mathbb{P}$ and P_z denote respectively the annealed and quenched law of $X(t)$, $t \in \mathbb{N}$, so that

$$\mathbb{P} = P \times P_z$$

NOTE: Scatterers are now fixed by the environment and the increments have no trivial correlation.

Goal: Study of the quenched behavior of $X(t)$, $t \in \mathbb{N}$.

Previous results

Barkay, Fleurer, Klafter ['00] provide upper bounds on the (annealed) second moment

THM 1. *For* $\alpha \in (1, 2)$

(i) $\mathbb{E}(X^2(t)) \geq c(\alpha)t^{2-\alpha}$ *for non-equilibrium PP*(Z)
(ii) $\mathbb{E}(X^2(t)) \geq c(\alpha)t^{3-\alpha}$ *for equilibrium PP*(Z)

where in equilibrium PP(Z), $P(Y_1 = \ell) = \frac{\ell \mathbb{P}(Z=\ell)}{\mathbb{E}(Z)}$

Main tools: Laplace trasform and Tauberiam theorem.

The results is compatible with a Levy flight scheme but not much informative in the non-equilibrium scheme. Nothing is known about the quenched process.

Process at collision times

For $n \in \mathbb{N}$, let $t(n)$ =time of the nth collision
and set

$$\tilde{X}(n) \equiv X(t(n)), \quad n \in \mathbb{N}$$

- $\tilde{X}(n)$ is a SSRW on PP(Z).
- Letting $S_n = \sum_{k=1}^{n} \xi_k$ the coupled SSRW on $\mathbb{Z}$, it holds

$$\tilde{X}(n) = Y_{S_n}$$

that is, $\tilde{X}(n)$ is the position of scatter label by S_n.

Quenched law of $\tilde{X}(n)$

Proposition 1. *For $\alpha \in (1,2)$ and a non-equilbrium PP(Z), it holds*

$$P_z\left(\frac{\tilde{X}(n)}{\mu\sqrt{n}}\right) \xrightarrow{n\to\infty} \int_x^{+\infty} \frac{e^{-t^2/2}}{\sqrt{2\pi}}dt \qquad P\text{-a.s.}$$

where $\mu = \mathbb{E}(Z_k)$.

Proof idea: From $\tilde{X}(n) = Y_{S_n}$, we used

- CLT for S_n
- LLN for Y_k

Quenched law of $X(t)$

THM 2. *For $\alpha \in (1,2)$ and a non-equilbrium PP(Z), it holds*

$$P_z\left(\frac{X(t)}{\sqrt{\mu t}}\right) \xrightarrow{t\to\infty} \int_x^{+\infty} \frac{e^{-t^2/2}}{\sqrt{2\pi}}dt \qquad P\text{-a.s.}$$

Proof idea: Write $\frac{X(t)}{\sqrt{\mu t}} = \frac{X(t)-\tilde{X}(n(t))}{\sqrt{\mu t}} + \frac{\tilde{X}(n(t))}{\mu\sqrt{n(t)}}\sqrt{\frac{\mu n(t)}{t}}$

- $\mathbb{E}_z\left(|\frac{X(t)-\tilde{X}(n(t))}{\sqrt{\mu t}}|\right) \xrightarrow{t\to\infty} 0\,, \quad P-$ a.s
- By the ergodicity of the annealed process for the PVP

$$\frac{n(t)}{t} \xrightarrow{t\to\infty} \frac{1}{\mu}\,, \quad \mathbb{P}-\text{a.s}$$

Quenched Moments of $\tilde{X}(n)$

Proposition 2. *For $\alpha \in (1,2)$ and a non-equilbrium PP(Z), it holds*

$$E_z\left(\frac{\tilde{X}^m(n)}{n^{\frac{m}{2}}}\right) \xrightarrow{n\to\infty} \begin{cases} 0 & \text{for } m = 2k-1 \\ \mu^m(m-1)!! & \text{for } m = 2k \end{cases} , \quad P\text{-a.s.}$$

i.e., to the moments of $N(0,\mu^2)$.

Proof idea: From $\tilde{X}(n) = Y_{S_n}$, we used

- Moments convergence for S_n
- LLN for Y_k

Quenched Moments of $X(t)$

THM 3. *For $\alpha \in (1,2)$ and a non-equilbrium PP(Z), it holds*

$$E_z\left(\frac{X^m(t)}{t^{\frac{m}{2}}}\right) \xrightarrow{n\to\infty} \begin{cases} 0 & \text{for } m = 2k-1 \\ \mu^{\frac{m}{2}}(m-1)!! & \text{for } m = 2k \end{cases}, \quad P\text{-a.s.}$$

i.e., to the moments of $N(0,\mu)$.

Proof idea: Write $\frac{X^m(t)}{t^{\frac{m}{2}}} = \frac{X^m(t)-\tilde{X}^m(n(t))}{t^{\frac{m}{2}}} + \frac{\tilde{X}^m(n(t))}{n(t)^{\frac{m}{2}}}\left(\frac{n(t)}{t}\right)^{\frac{m}{2}}$

- Define the event $E = \left\{|X(t)| \vee |\tilde{X}(n(t))| < t^{\gamma}\right\}$ with $P_z(E^c) \leq e^{-t^{\gamma}}$ P-a.s

Then

- $\mathbb{E}_z\left(|\frac{X^m(t)-\tilde{X}^m(n(t))}{t^{\frac{m}{2}}}|\,\Big|\, E^c\right) P_z(E^c) \leq 2t^{\frac{m}{2}}e^{-t^{\gamma}}$

- $\mathbb{E}_z\left(|\frac{X^m(t)-\tilde{X}^m(n(t))}{t^{\frac{m}{2}}}|\,\Big|\, E\right) P_z(E)$

$$\leq \mathbb{E}_z\left(|X(t) - X(n(t))||E\right) \cdot mt^{\gamma(m-1)-\frac{m}{2}}$$

Choosing $\frac{1}{2} < \gamma < \frac{m}{2(m-1)}$ and from $\mathbb{E}_z\left(|X(t) - X(n(t))|\right) \xrightarrow{t\to\infty} \mu$
we conclude.

Corollary 1. *For $\alpha \in (1,2)$ and a non-equilbrium PP(Z), it holds*

$$\mathbb{E}\left(X^2(t)\right) \geq t\,, \quad \text{for } t \ll 1$$

This improves the annealed bound on the second moment given by BFK['00].

Conclusion, work in progress, open problems

• The quenched behavior of the 1 D Levy Lorentz gas with non-equilibrium initial condition do not displays anomalous diffusive behavior.

• Under the equilibrium initial condition, we expect to find a similar behavior (work in progress)

• Improved bound on the annealed second moment. Its exact behavior has still to be determined (work in progress).

• Provide a similar construction for a 2 D Levy Lorentz gas (open problem), where we expect a quenched anomalous diffusive behavior.

Thank you for your attention!