Planar spatial random permutations

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Spatial random permutations:

the model

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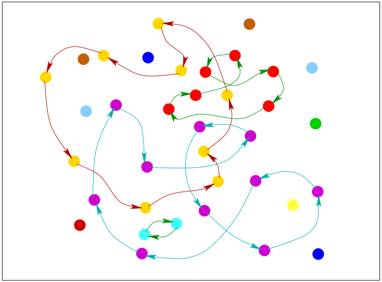

- $\Lambda \subset \mathbb{R}^d$, finite volmue V.
  $\boldsymbol{x} = \{x_1, \dots, x_N\} \subset \Lambda \subset \mathbb{R}^d$
- $\mathcal{S}_N =$ set of permutations on $\pi : \{1, \dots, N\} \to \{1, \dots, N\}$.
- Typical example for a measure on $\mathcal{S}_N$:

$$\mathbb{P}_{\boldsymbol{x}}(\{\pi\}) = \frac{1}{Z(\boldsymbol{x})} \exp\left(-\beta \sum_{i=1}^{N} |x_i - x_{\pi(i)}|^2 \right).$$

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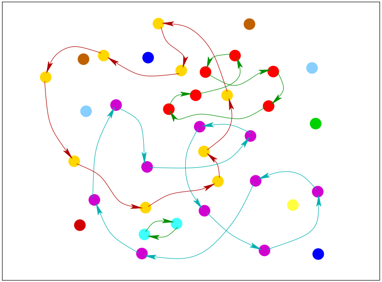

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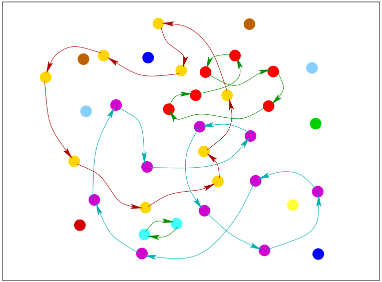

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- Periodic boundary conditions.
- Question: Existence, distribution, geometry and evolution (under Glauber dynamics) of long cycles.

Three (or more) dimensions: phase transition

$$\mathbb{P}_{\boldsymbol{x}}(\{\pi\}) = \frac{1}{Z(\boldsymbol{x})} \exp\left(-\beta \sum_{i=1}^{N} |x_i - x_{\pi(i)}|^2\right).$$

- Conjecture: For $d \geqslant 3$ there exists $\beta_{\text{crit}} > 0$ such that for $\beta < \beta_{\text{crit}}$ there are macroscopic cycles:

$$\mathbb{P}\Big((\text{length of cycle containing } x_1) > \varepsilon N\Big) > c(\beta, \varepsilon) > 0$$

for some $\varepsilon > 0$, uniformly in N.

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  - Phase transition - connections with the free Bose gas
  - Poisson-Dirichlet distribution of long cycles.
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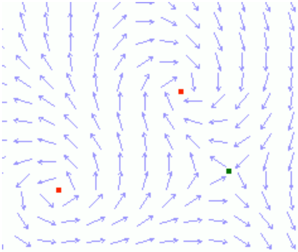
  - Poisson-Dirichlet distribution of long cycles.
  - No mesoscopic cycles.
- MCMC for the cubic lattice: [Grosskinsky-Lovisolo-Ueltschi 2012]
  - Numerical support for all the above statements in the lattice case. See also [Gandolfo-Ruiz-Ueltschi 2007]
  - Geometry: points in long cycles are equidistributed.
  - Dynamics: split-merge process.

Kosterlitz-Thouless phase transition

Two dimensions: Kosterlitz-Thouless transition

- Standard example for a KT-transition: XY-model with nearest neighbour interaction.
- $X := \Lambda \cap \mathbb{Z}^2$ Spins $(S_j)_{j \in X}$ with $S_j \in \mathbb{R}^2, |S_j| = 1$.
- Hamiltonian $H = -\sum_{x_i \sim x_j} S_i \cdot S_j$. Inverse Temperature β.



- KT-transition: $\big|E(S_0)\big| = 0$ for all β, but decay of correlations goes from exponential to algebraic. [Fröhlich, Spencer 1981]
- Reason: formation of vertices.

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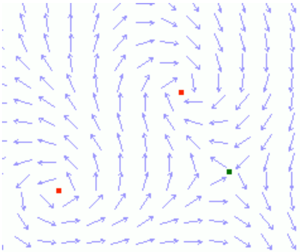


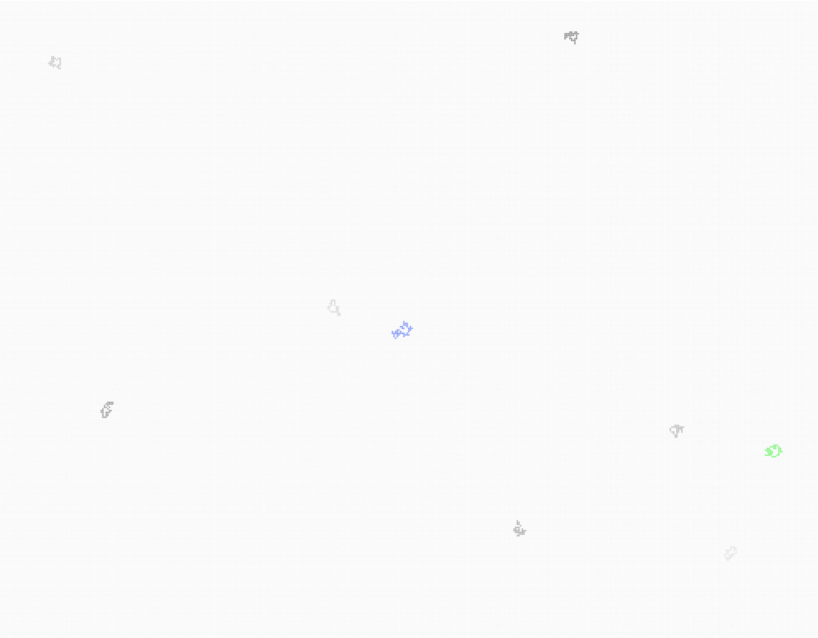
- KT-transition: $\big|E(S_0)\big| = 0$ for all β, but decay of correlations goes from exponential to algebraic. [Fröhlich, Spencer 1981]
- Reason: formation of vertices.
- Analogue in spatial random permutations: bubbles.

[see also Sütö 1993]

SRP for parameter $\beta = 1.3$

SRP for parameter $\beta = 1.2$

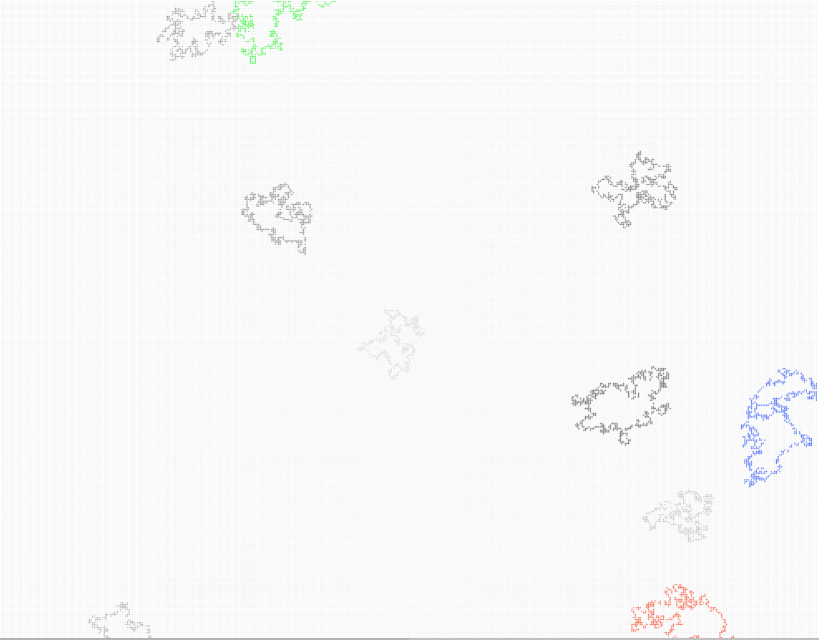
SRP for parameter $\beta = 1.1$



SRP for parameter $\beta = 1.0$

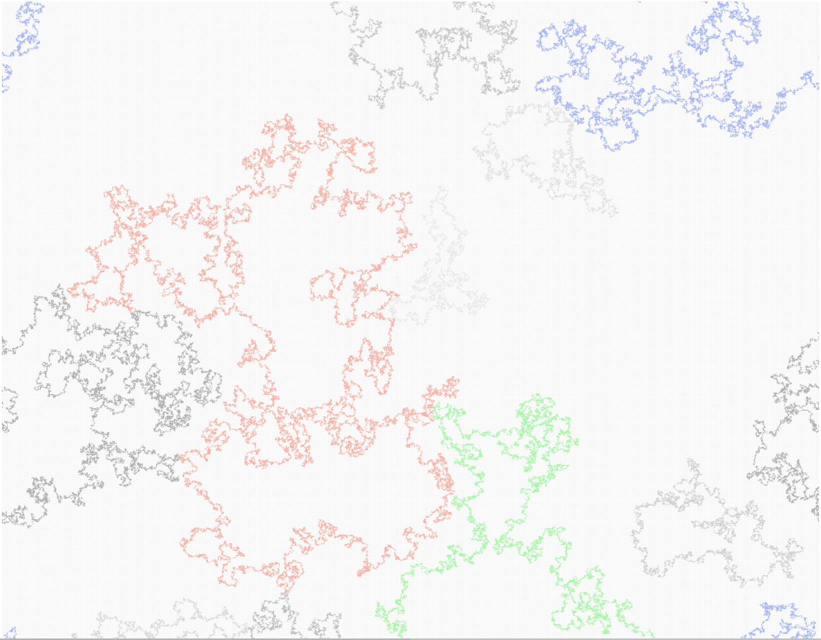
SRP for parameter $\beta = 0.9$

SRP for parameter $\beta = 0.8$



SRP for parameter $\beta = 0.75$

SRP for parameter $\beta = 0.7$



KT phase transition in SRP

- Qunatity to observe: $\mathbb{P}(x_i \sim x_j)$ as a function of $|x_i - x_j|$.
- $x_i \sim x_j$ means that they are in the same cycle.
- Let $\ell(x_i)$ denote the length of the cycle containing x_i.
- For large β: High temperature estimate leads to

$$\mathbb{P}(|x_i - x_j| > n) \leqslant \mathbb{P}(\ell(x_i) > cn) \leqslant \exp(-\alpha n) \qquad (c, \alpha > 0).$$

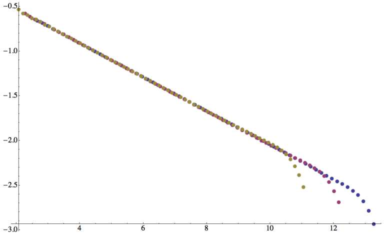
- Reasonable assumption: For all $\beta > 0$ there exist $C > 0$ such that whenever $|x_i = x_j| = n$

$$\mathbb{P}(\ell(x_i) > cn) \geqslant \mathbb{P}(x_i \sim x_j) \geqslant \mathbb{P}(\ell(x_i) > Cn^2).$$

- $\mathbb{P}(\ell(x_i) > c) =$ the fraction of points in cycles longer than c and thus numerically easy to observe.

KT phase transition: numerics

- Clear power law decay of $\mathbb{P}(\ell(x_i) > c)$ for $\beta = 0.5$.



Log-log-plot of $\mathbb{P}(\ell(x_i) > c)$
for $n = 1000^2, 2000^2, 4000^2$.

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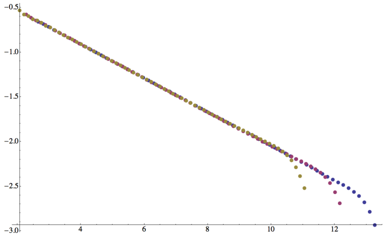
- Numerical evidence suggests $0.7 < \beta_c < 0.75$.



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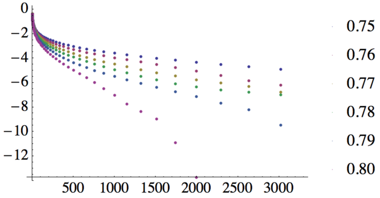
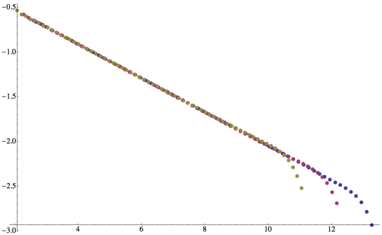


Logplot of $\mathbb{P}(\ell(x_i) > c)$ for $n = 1000$ and different β.

KT phase transition: numerics

- Clear power law decay of $\mathbb{P}(\ell(x_i) > c)$ for $\beta = 0.5$.
- Numerical evidence suggests $0.7 < \beta_c < 0.75$.
- Determining β_c precisely is very difficult due to large flucutuations in cycle lengths over time.



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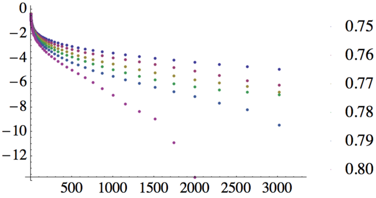

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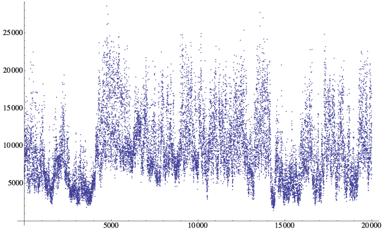

Time series of longest cycle length at $n = 1000$ with 10 full sweeps between each measurement.

Curve shortening flow

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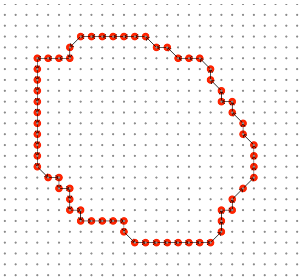
- Start with one circular cycle.
- Run Glauber dynamics with $\beta \gg 1$.
- At zero temperature: Connection to zero temperature Ising model.
- Adapting techniques from [Sphon 93], [Lacoin, Simenhaus, Toninelli 2012]:



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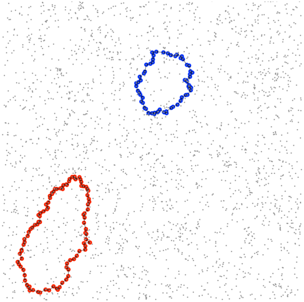
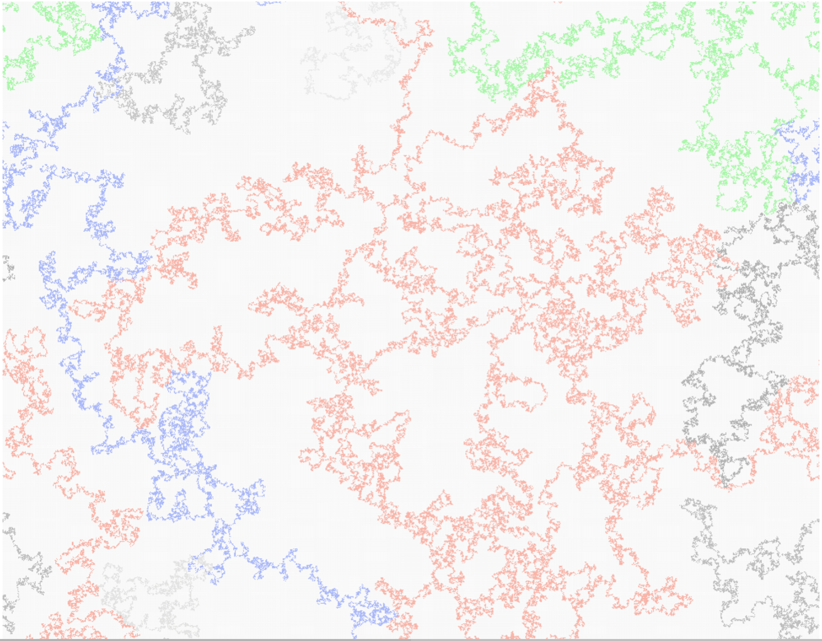


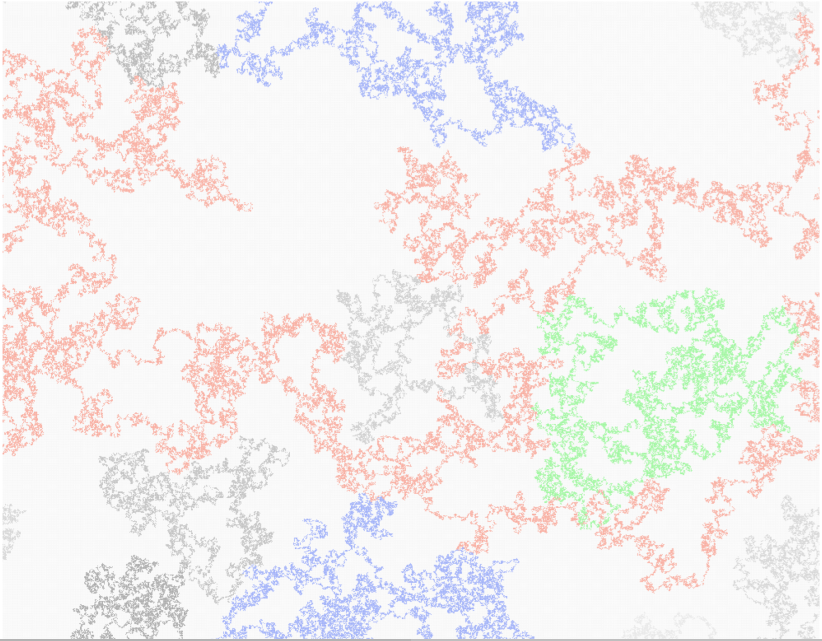
- Map to a SSEP with 'range-2-blocking' lattice points. Study hydrodynamic limit. (Joint project with Stefan Walter, Darmstadt).
- Added flexibility: More hope of doing the $\beta < \infty$ case, or different point configurations.

Fractal dimension

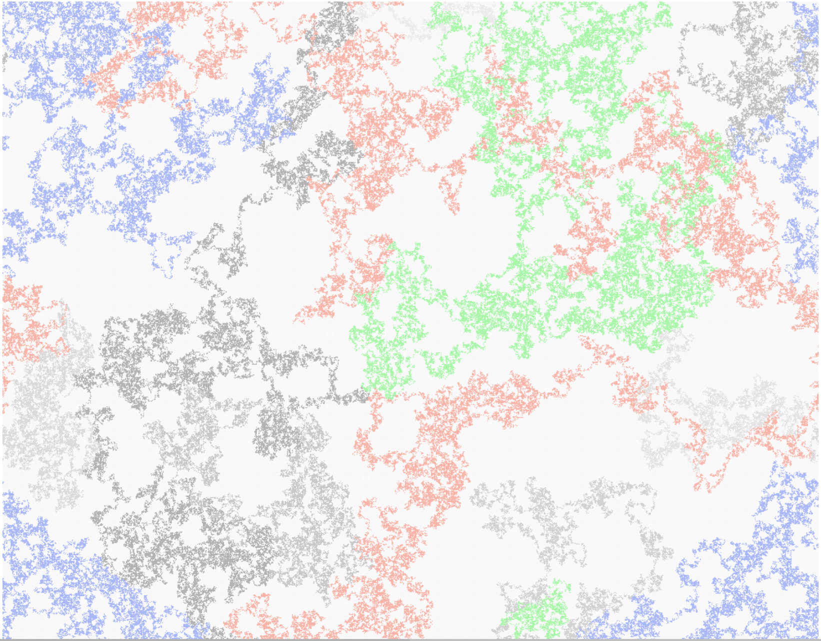
SRP for parameter $\beta = 0.6$



SRP for parameter $\beta = 0.5$



SRP for parameter $\beta = 0.4$



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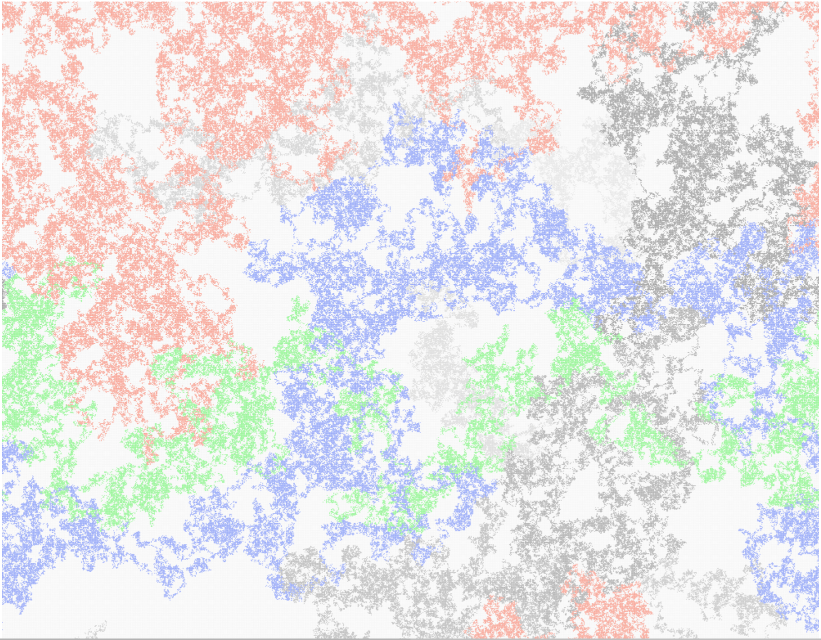

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- Compute the box-counting dimension:

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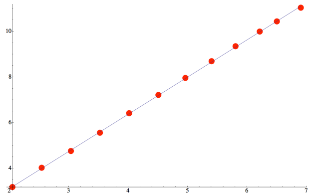

Loglog plot of the number of boxes needed to cover the longest cycle vs the box side length

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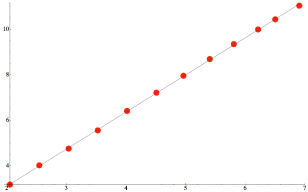
- Sample with 2000×2000 points in $\Lambda = [0,1]^2$, with $1/1000 \leqslant \varepsilon \leqslant 1/10$:
- Linear fitting gives $d_{\text{box}}(\beta) \approx 2 - \frac{7}{10}\beta$.



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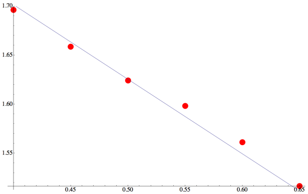

Box counting dimension as function of the temperature

Fractal dimension and (possibly) SLE

- It seems that

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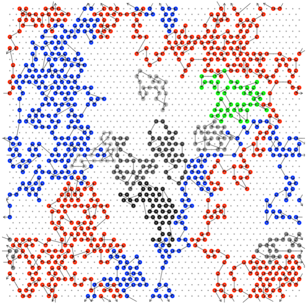

- The same result can be obtained for the triangular lattice.
- For an SLE(κ)-curve it is known that almost surely

$$d_{\mathrm{H}}(\kappa) = \min\left(2, 1 - \tfrac{\kappa}{8}\right)$$

[Rohde and Schramm 2005, Beffara 2008]

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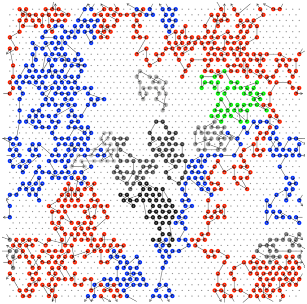

[Rohde and Schramm 2005, Beffara 2008]

- Assuming that SRP cycles are SLE curves, we get

$$\kappa(\beta) = 8(d_{\text{H}} - 1) = 8(1 - \tfrac{7}{10}\beta) = 8 - \tfrac{28}{5}\beta;$$

for $\kappa = 4$ (transition from simple to non-simple curves) we find $\beta = \frac{5}{7} \approx 0.71$. This fits well with the KT-Transition!

Double dimer model, SLE and SRP

Double dimer model, conformal invariance, SLE

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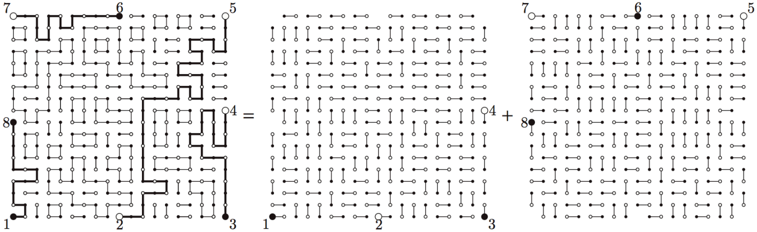
- The double dimer model is known to be conformally invariant [Kenyon 2011].



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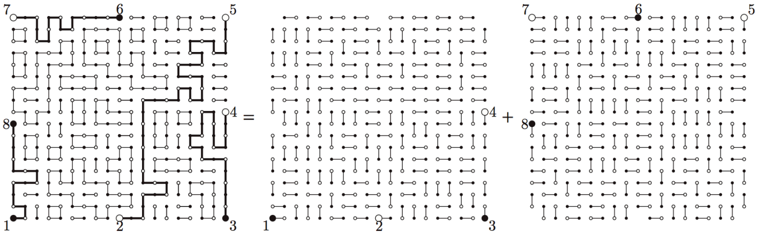

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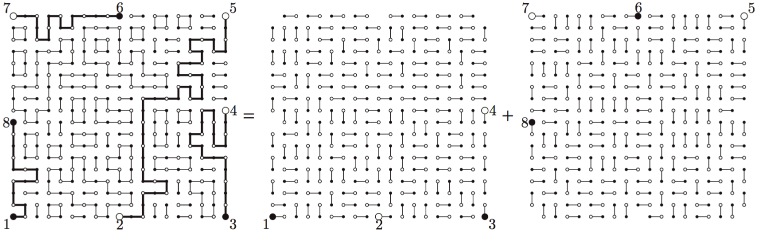
- With slightly less good reason we conjecture that $\kappa = 8 - \frac{28}{5}\beta$.



Thank you for your attention!