EXTREMAL GEOMETRY OF A BROWNIAN POROUS MEDIUM

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QUESTION

Run a Brownian motion on a d-dimensional torus for a long time. What does its complement look like?

GOAL OF THIS TALK

Describe the geometry of the complement for $d \geq 3$ and reflect on what happens for $d = 2$.

Let $W = (W(t))_{t \geq 0}$ denote standard Brownian motion wrapped around the unit torus $\mathbb{T}^d = \mathbb{R}^d \,(\mathrm{mod}\, \mathbb{Z}^d)$. Write $W[0,t]$ for the path of W up to time t.

Simulation of $W[0,t]$ for $t = 15$ in $d = 2$.

§ TWO FUN APPLICATIONS

- The Brownian mosquito

You and a mosquito are trapped in a room for one hour. How should you position yourself to minimize the chance of being bitten?

- The Brownian heating coil

A deranged builder runs a heating coil through your house in an erratic manner and switches it on. How fast does the temperature equilibrate?

§ PROPERTIES OF INTEREST

Local properties:

Fix $E \subset \mathbb{R}^d$ compact. For suitably chosen $\phi = \phi_{\mathsf{local}}(t)$, consider the event

$$\Big\{x + \phi E \subset \mathbb{T}^d \setminus W[0,t]\Big\}, \quad x \in \mathbb{T}^d \text{ fixed}.$$

Global properties:

Fix $E \subset \mathbb{R}^d$ compact. For suitably chosen $\phi = \phi_{\mathsf{global}}(t)$, consider the event

$$\Big\{\exists\, x \in \mathbb{T}^d \colon\ x + \phi E \subset \mathbb{T}^d \setminus W[0,t]\Big\}.$$

We will see that the proper scales for these events to be typical are

$$\phi_{\text{local}}(t) = \left(\frac{1}{t}\right)^{1/(d-2)},$$

$$\phi_{\text{global}}(t) = \left(\frac{\log t}{t}\right)^{1/(d-2)}.$$

Note that

$$\phi_{\text{local}}(t) \ll \phi_{\text{global}}(t),$$

i.e., the largest holes are much larger than the typical holes.

§ LITERATURE

The largest inradius at time t is defined as

$$R_{\text{in}}(t) = \sup_{x \in \mathbb{T}^d} d(x, W[0,t]).$$

THEOREM 1 Dembo, Peres, Rosen 2003

For $d \geq 3$*,*

$$\lim_{t \to \infty} \frac{R_{\text{in}}(t)}{\phi_d(t)} = 1 \qquad \text{a.s.}$$

with

$$\phi_d(t) = \left(\frac{d}{(d-2)\kappa_d} \frac{\log t}{t} \right)^{1/(d-2)},$$

where $\kappa_d = 2\pi^{d/2}/\Gamma(d/2-1)$ *is the Newtonian capacity of the unit ball.*

Theorem 1 can be reformulated in terms of the ϵ-cover time

$$\mathcal{C}(\epsilon) = \inf\left\{t \geq 0\colon\ \rho_{\text{in}}(t) \leq \epsilon\right\},$$

namely,

$$\lim_{\epsilon \downarrow 0} \frac{\mathcal{C}(\epsilon)}{\psi_d(\epsilon)} = d \qquad \text{a.s.}$$

with

$$\psi_d(\epsilon) = \frac{1}{\kappa_d} \frac{\log(1/\epsilon)}{\epsilon^{d-2}}.$$

A point $x \in \mathbb{T}^d$ is called α-late when

$$\limsup_{\epsilon \downarrow 0} \frac{\mathcal{T}(x,\epsilon)}{\psi_d(\epsilon)} = \alpha$$

with

$$\mathcal{T}(x,\epsilon) = \inf \Big\{ t \geq 0 \colon \; d(x, W[0,t]) \leq \epsilon \Big\}.$$

Since $\mathcal{C}(\epsilon) = \sup_{x \in \mathbb{T}^d} \mathcal{T}(x,\epsilon)$, Theorem 1 states that no point is α-late for any $\alpha > d$.

THEOREM 2 Dembo, Peres, Rosen 2003

For $d \geq 3$ *and* $0 \leq \alpha \leq d$,

$$\dim \Big\{ x \in \mathbb{T}^d \colon \; x \text{ is } \alpha\text{-late} \Big\} = d - \alpha \qquad \text{a.s.}$$

Theorem 2 identifies the fractal dimension of late points.

§ EXTREMAL GEOMETRY

van den Berg, Bolthausen & dH 2013

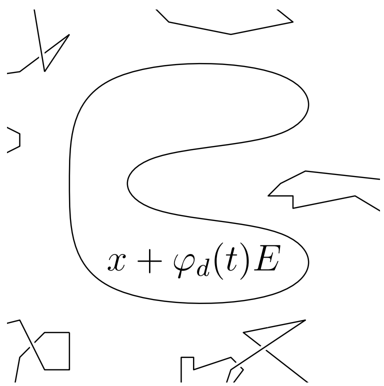
Goodman & dH 2013

In the remainder of the talk we will consider the following three quantitative questions:

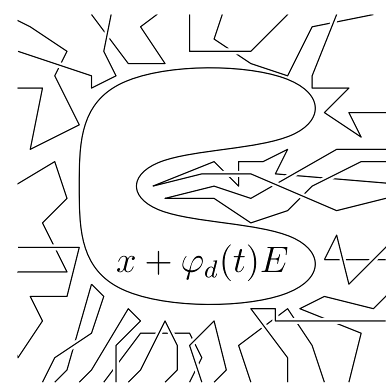
(1) Given $E \subset \mathbb{R}^d$ compact, does $\mathbb{T}^d \setminus W[0,t]$ contain a shifted scaled copy $x + \phi_d(t)E$ for some $x \in \mathbb{T}^d$?

(2) Is $W[0,t]$ sparse or dense in the neighborhood of this copy?

(3) Does $\mathbb{T}^d \setminus W[0,t]$ have some kind of component structure?



sparse picture



dense picture

Two possible strategies to avoid sets are:

- Temporal avoidance: $W[0,t]$ stays an atypically short amount of time near $x + \phi_d(t)E$ but moves in a typical manner.

- Spatial avoidance: $W[0,t]$ stays a typical amount of time near $x + \phi_d(t)E$ but moves in an atypical manner.

(1) SHAPE OF LARGE HOLES?

THEOREM 3

For $d \geq 3$*, let* $E \subset \mathbb{R}^d$ *be a compact set satisfying a certain regularity condition. Then*

$$\lim_{t\to\infty} \mathbb{P}\Big(\exists\, x \in \mathbb{T}^d\colon\ x + \phi_d(t)E \subset \mathbb{T}^d \setminus W[0,t]\Big) = \begin{cases} 1 & \text{if } \mathrm{Cap}(E) < \kappa_d, \\ 0 & \text{if } \mathrm{Cap}(E) > \kappa_d, \end{cases}$$

where $\mathrm{Cap}(E)$ *is the Newtonian capacity of* E.

(2) SPARSE OR DENSE?

THEOREM 4

For $d \geq 3$*, if* $\text{Cap}(E) < \kappa_d$*, then the maximal number* $\chi(t, E)$ *of disjoint shifts* $x + \phi_d(t)E$ *inside* $\mathbb{T}^d \setminus W[0, t]$ *satisfies*

$$\lim_{t\to\infty} \frac{\log \chi(t, E)}{\log t} = \frac{d}{d-2}\left(1 - \frac{\text{Cap}(E)}{\kappa_d}\right) \qquad \text{a.s.}$$

Pick any $\delta > 0$, and any $E \subset E'$ with

$$\text{Cap}(E') \geq \text{Cap}(E) + \delta.$$

Then subsets of the form $x + \phi_d(t)E'$ are much less numerous than subsets of the form $x + \phi_d(t)E$, and so dense picture with spatial avoidance applies.

(3) COMPONENT STRUCTURE?

The set $\mathbb{T}^d \setminus W[0,t]$ is connected a.s. However, large holes are rare and tend to be surrounded by $W[0,t]$. They can therefore be viewed as lakes connected by narrow channels, which can be sealed off by thickening up $W[0,t]$.

Let $\rho(t)$ be such that

$$\frac{\phi_d(t)}{(\log t)^{1/d}} \ll \rho(t) \ll \phi_d(t).$$

Let

$$W_{\rho(t)}[0,t] = \left\{x \in \mathbb{T}^d \colon\ d(x, W[0,t]) \leq \rho(t)\right\}$$

be the Wiener sausage of radius $\rho(t)$.

Write $(C_i)_{i\in I}$ to denote the components of $\mathbb{T}^d \setminus W_{\rho(t)}[0,t]$.

THEOREM 5

For $d \geq 3$,

$$\lim_{t\to\infty} \frac{\max_{i\in I} \mathrm{Cap}(C_i)}{\phi_d(t)^{d-2}} = \kappa_d.$$

Note that the almost-component structure of $\mathbb{T}^d \setminus W[0,t]$ is well-defined because the scaling does not depend on the fine details of the thickening radius $\rho(t)$.

COROLLARY 6

Let $V_{\mathrm{max}} = \max_{i \in I} \mathrm{Vol}(C_i)$ denote the maximal volume of the components of $\mathbb{T}^d \setminus W_{\rho(t)}[0,t]$. Then

$$\lim_{t \to \infty} \frac{V_{\mathrm{max}}}{\phi_d(t)^d} = \mathrm{Vol}(\text{unit ball}).$$

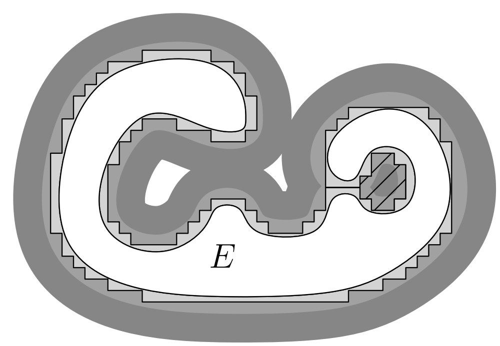
COROLLARY 7

Let $\lambda_{\mathrm{min}} = \min_{i \in I} \lambda(C_i)$ denote the minimal principal Dirichlet eigenvalue of the components of $\mathbb{T}^d \setminus W_{\rho(t)}[0,t]$. Then

$$\lim_{t \to \infty} \phi_d(t)^2 \lambda_{\mathrm{min}} = \lambda(\text{unit ball}).$$

§ KEY INGREDIENTS OF PROOFS

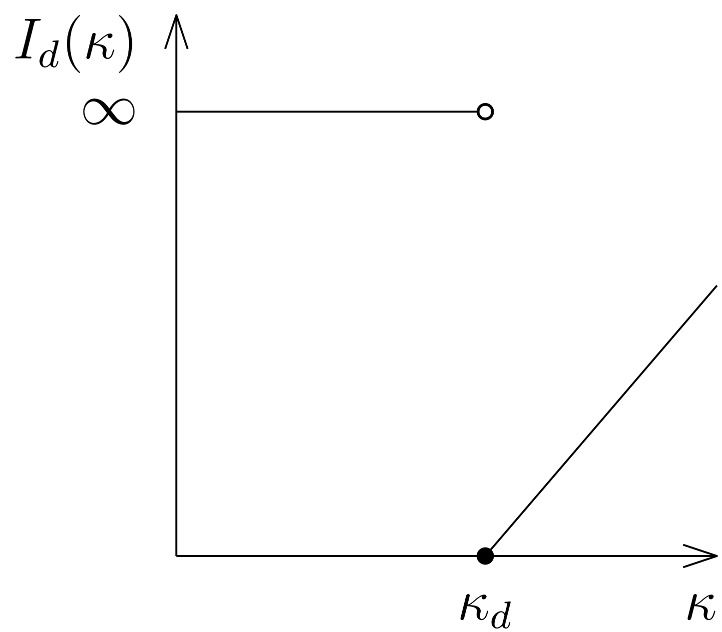
The proofs of Theorems 3–5 rely on a careful analysis of excursions between boundaries of small concentric balls that fill the torus, and on lattice approximations of compact sets that allow for a proper counting of shifts of scaled copies of sets.



The proofs of Corollaries 6–7 rely on the fact that, among all regions of a given volume, respectively, principal Dirichlet eigenvalue, the ball has the smallest capacity.

The proofs of Theorems 3–5 also imply large deviation principles. For instance, $R_{\text{in}}(t)/\phi_d(t)$ satisfies the LDP on $(0,\infty)$ with rate $\log t$ and]rate function I given by

$$I(r) = I_d(\kappa_d r^{d-2}), \qquad r \in (0,\infty).$$



§ RANDOM INTERLACEMENTS

Since

$$C(E) = 1 - e^{-\mathsf{Cap}(E)}$$

is a Choquet capacity with $0 \le C(E) \le 1$ and $C(\emptyset) = 0$, there is a random closed set $X \subset \mathbb{R}^d$ such that

$$\mathbb{P}(X \cap E = \emptyset) = e^{-\mathsf{Cap}(E)}$$

for all closed sets $E \subset \mathbb{R}^d$.

The discrete analogue of X is a random finite set $X_u \subset \mathbb{Z}^d$ such that

$$\mathbb{P}_u(X_u \cap E = \emptyset) = e^{-u\mathsf{Cap}_{\mathbb{Z}^d}(E)}$$

for all finite sets $E \subset \mathbb{Z}^d$, where $\mathsf{Cap}_{\mathbb{Z}^d}(E)$ is the discrete capacity of E and $u \in (0, \infty)$ is a parameter.

X_u is called the random interlacement with parameter u.

Sznitman & co-workers

Random interlacements arise from running a random walk $S = (S(n))_{n\in\mathbb{N}}$ on the N-torus $\mathbb{T}_N^d = \mathbb{Z}^d(\mathrm{mod} N)$ up to time uN^d, namely,

> $\mathbb{T}_N^d \setminus S[0, uN^d]$ seen locally from a uniformly chosen point of $\mathbb{T}_N^d$ converges in law to X_u as $N \to \infty$.

Interestingly, X_u undergoes a percolation transition at a critical threshold $u_* \in (0, \infty)$.

The link between the discrete and the continuum model is obtained by picking

$$N \approx t^{1/(d-2)}.$$

§ OPEN QUESTIONS

- Is the scale

$$\rho(t) \gg \frac{\phi_d(t)}{(\log t)^{1/d}}$$

  of the Wiener sausage optimal or can it be pushed down to $\rho(t) \gg t^{-1/(d-2)}$?

- Does the link $N \approx t^{1/(d-2)}$ for random interlacements carry over to large holes, i.e., is this the link for local and global properties?

- Is there a percolation transition in the continuum model for some critical choice $\rho_*(t)$ of the thickening radius?

§ WHAT HAPPENS IN $d = 2$?

Expect sparse picture and temporal avoidance, together with strong correlations between times spent near different points.

Goodman & dH work in progress