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Two Routes to
Superdiffusivity

(Based on joint work
with Jens Marklof
and Benedek Valkó)

$t \longmapsto X(t) \in \mathbb{R}^d$ 1

random motion with
stationary / ergodic increments

diffusivity:

$$\mathbb{E}\left(X(t)^2\right) \sim t \quad \text{diffusive}$$
$$\gg t \quad \text{superdiff.}$$
$$\ll t \quad \text{subdiffusive}$$

Similarly in discrete
times

X_n

Typically: ~~later~~

$$X(t) = \int_0^t V(s)\,ds + M(t)$$

$$\mathbb{E}\left(M(t)^2\right) \sim t$$

superdiffusivity may come from the integral.

Similarly: in discrete time

$$X_n = \sum_{k=1}^{n} \xi_k$$

~~In classical probab~~

A classroom example:

(ξ_n) iid.

$$P(\xi_n > x) = P(\xi_n < -x) \sim \frac{1}{x^2}$$

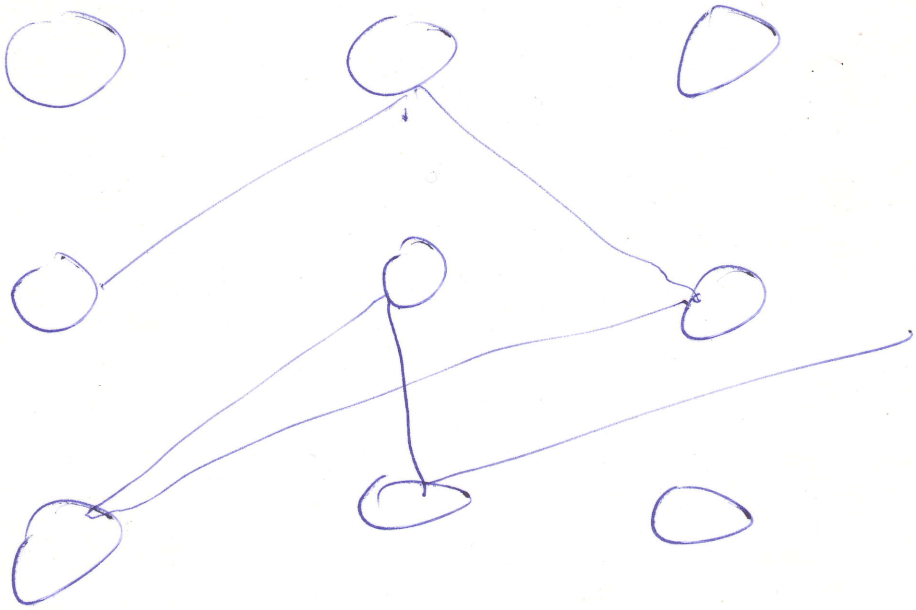
$$E(\xi_n) = 0 \; ; \; E(\xi_n^2) = \infty \text{ (marg)}$$

Thm $$\frac{\xi_1 + \cdots + \xi_n}{\sqrt{n \lg n}} \Rightarrow N(0,1)$$

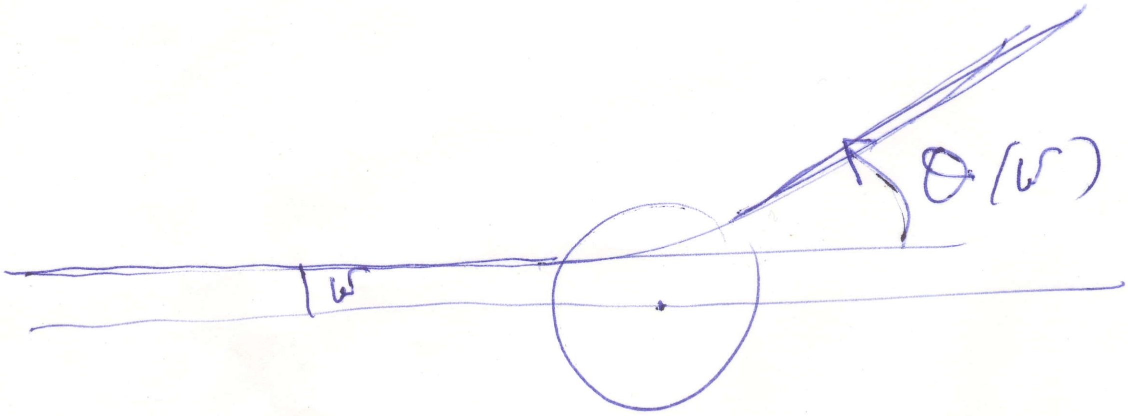
Another route:

$$E\big(V(0) V(t)\big) = C(t) \quad \text{noniateg}$$

B-G limit for periodic Lorentz Gas



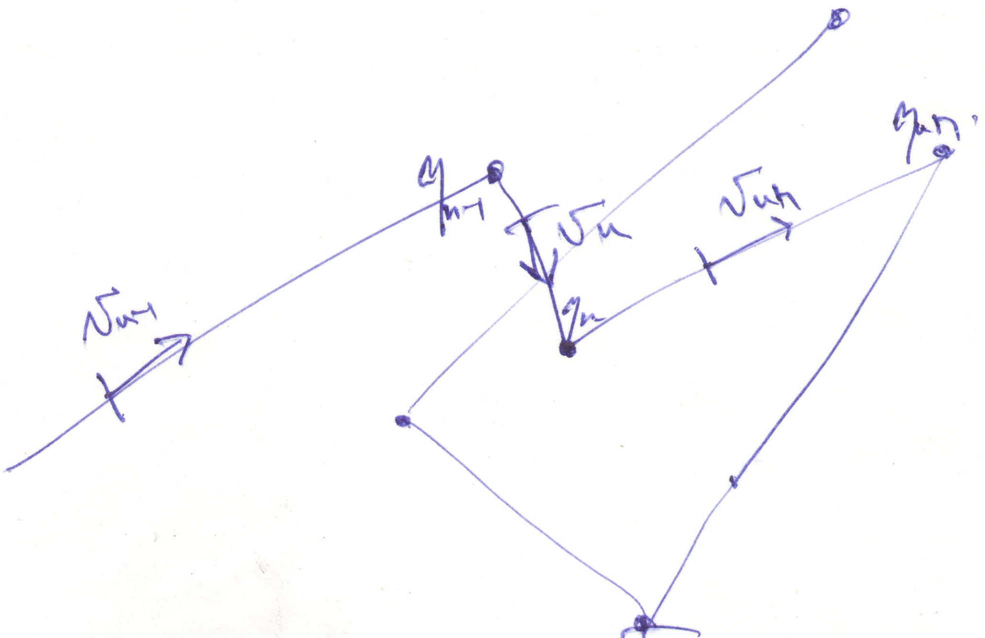
$\boxed{\text{B-G}}$



B-G lim $\quad g \to \infty ;\ r \to 0$

$g \cdot r^{d-1} \to 1$

$\boxed{\text{free flight} \sim 1}$



Bourgain / Golse / Caglioti / Marklof /

Strömbergsson

1998 — 2011.

(η_n, ξ_n) MC on $[-1,1] \times [0,\infty)$

$$\mathbb{P}\Big(\eta_{n+1} \in (z, z+dz),\ \xi_{n+1} \in (x, x+dx) \,\Big|\, \eta_n = w,\ \xi_n = y\Big) =$$

$$\Phi(w, z; x)\, dz\, dx$$

$$v_n = U(\eta_{n-1})\, v_{n-1} = U(\eta_{n-1}) \cdots U(\eta_0)\, v_0$$

$$\mathbb{P}\big(\eta_{n+1} \in (z, z+dz) \,\big|\, \eta_n = w\big) = K(w, z)\, dw$$

$((\xi_i) \mid \eta_n)$ indep.

$$\mathbb{E}(\xi_n \mid \eta_{n-1}, \eta_n) = m_n$$

$$\mathrm{Var}(\xi_n \mid \eta_{n-1}, \eta_n) = a_n^2$$

$$\mathbb{E}(m_n) < \infty \qquad \mathbb{E}(m_n^2) = \infty$$

$$\mathbb{E}(a_n^2) = \infty$$

$$\mathbb{P}(m_n > u) \sim \frac{1}{u^2 \log u}$$

$$\mathbb{P}(a_n > u) \sim \frac{1}{u^2}$$

$$X_n = \sum_{k=1}^{n} \nu_k \xi_k =$$

$$\sum_{k=1}^{n} \nu_k (\xi_k - m_k) + \boxed{\sum_{k=1}^{n} \nu_k m_k}$$

Apply Conditionally Lindeberg

$$\dots \frac{1}{\sqrt{n \log n}} \sum_{k=1}^{n} \nu_k (\xi_k - m_k) \Rightarrow N(0,1)$$

$$\boxed{\frac{1}{\sqrt{n \log\log n}} \sum_{k=1}^{n} \nu_k m_k}$$

tight

The other route

$$dX(t) = dB(t) - (\nabla * l_t)(X(t))\,dt$$

$$d\cancel{X}(t) = dB(t) + F(\cancel{X}(t))\,dt$$

Where

$$F : \mathbb{R}^2 \to \mathbb{R}^2$$

U

$$F(\cancel{x}) = \mathrm{rot}\left(\cancel{\nabla} * FGF\right)$$

$$X(t) = \cancel{x} \int_0^t V(s)\,ds + M(t)$$

$$V(s) = \dots$$

Superdiff comes from $\mathbb{E}\left(V(0)\,V(s)\right)$

non integr

$\boxed{\text{Actually}}$

$$V(\Delta) = \varphi(\eta(s)) \qquad \boxed{\varphi(\underline{\omega}) = \omega(0)}$$

$$\eta(s) = \eta(s, X) = F(X(s) + x)$$

MP on $\Omega = \{ \text{—} \}$

stationary / ergodic

$$\Pi : \ \mathbb{E}_\pi \left(\omega_i(x)\, \omega_j(y) \right) =$$

$$C_{ij}(y - x)$$

$$\hat{C}_{ij}(p) = \hat{V}(p) \frac{\tilde{\hat{p}}_i \tilde{\hat{p}}_j}{|p|^2}$$

$$\left[\omega(x) = \sqrt{\sigma} \left(V * FGF \right) \right]$$

$$L^2(\Omega, \pi) = \overline{\bigoplus_{k=0} \mathcal{H}_k}$$

$$L = \frac{1}{2}\Delta + \left(a^*\nabla + \nabla a\right)$$

Variational formula for the limiting ~~variance~~

$$\boxed{R_\lambda = (\lambda 1 - L)^{-1}}$$

$$(\varphi, R_\lambda \varphi) =$$

$$\sup_{\psi \in \mathcal{H}} \left\{ 2(\varphi, \psi) - (\psi, (\lambda + |\Delta|)\psi) - A\psi, (\lambda + |\Delta|)^{-1} A\psi \right\} \geq$$

$$\sup_{\psi \in \mathcal{H}_1} \left\{ \rule{5cm}{0.4pt} \right\}$$

....

Thm

$$t \log\log t \ll \mathbb{E}(X(t)^2) \ll t \log t$$

Conj

$$\boxed{t\sqrt{\log t}}$$

(formally similar for the polymer)

Outlook