Branching Brownian motion: extremal process and ergodic theorems

Anton Bovier
with Louis-Pierre Arguin and Nicola Kistler

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Plan

1. BBM
2. Maximum of BBM
3. The Lalley-Sellke conjecture
4. The extremal process of BBM
5. Ergodic theorems
6. Universality

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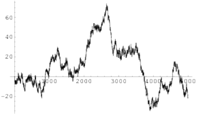

Pure random motion

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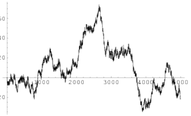
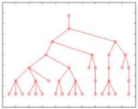
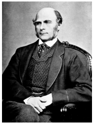
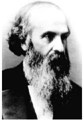


Pure random motion

Galton-Watson process







Pure random genealogy

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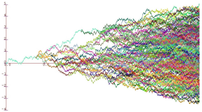
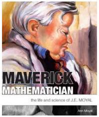

Picture by **Matt Roberts**, Bath


Maury Bramson


H. McKean

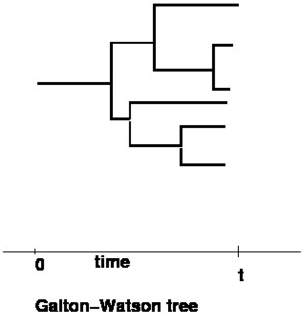

A.V. Skorokhod



J.E. Moyal

BBM is the canonical model of a spatial branching process.

Galton-Watson tree and corresponding BBM



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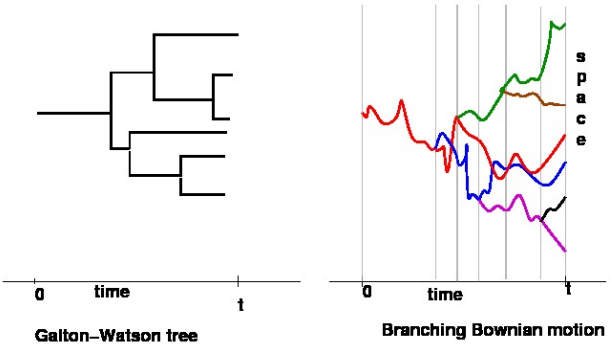

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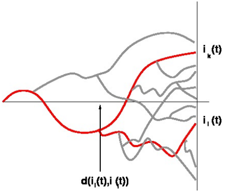

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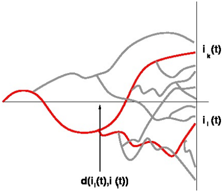

- BBM is Gaussian process with covariance

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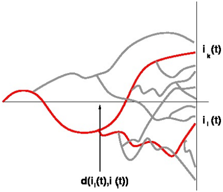

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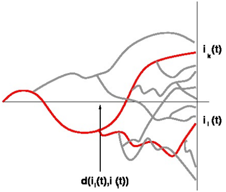

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$\Rightarrow$ GREM models of spin-glasses.

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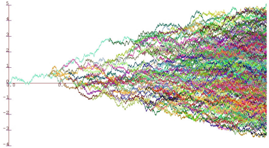
- $e^t = \mathbb{E}n(t)$ independent Brownian motions:

$$\mathbb{P}\left[\max_{k=1,\dots,e^t} x_k(t) \leq t\sqrt{2} - \frac{1}{2\sqrt{2}} \ln t + x\right] \to e^{-\sqrt{4\pi}e^{-\sqrt{2}x}}$$



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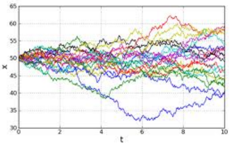

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Many BMs

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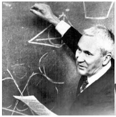

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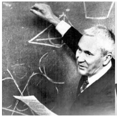

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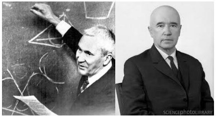

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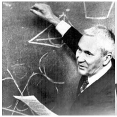

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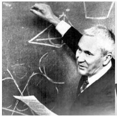

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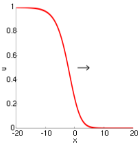
$\frac{1}{2}\partial_x^2\omega + \sqrt{2}\partial_x\omega + \omega^2 - \omega = 0$



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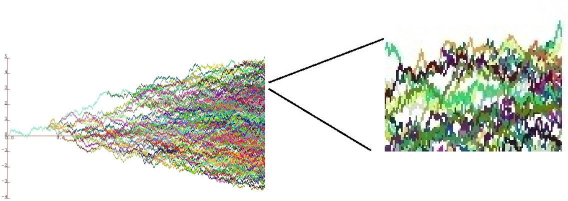
Lalley-Sellke conjecture: $\mathbb{P}$-a.s., for any $x \in \mathbb{R}$,

$$\lim_{T\uparrow\infty} \frac{1}{T}\int_0^T \mathbb{1}_{\left\{\max_{k=1}^{n(t)} x_k(t) - m(t) \leq x\right\}} = \exp\left(-CZe^{-\sqrt{2}x}\right)$$

Looking at BBM from the top

Closer look at the extremes: Zooming into the top



Can we describe the asymptotic structure of the largest points, and their genealogical structure?

Classical Poisson convergence for many BMs

From classical extreme values statistics one knows:

Let $X_i(t)$, $i \in \mathbb{N}$, iid Brownian motions. Then, the point process

$$\mathcal{P}_t \equiv \sum_{i=1}^{e^t} \delta_{X_i(t) - \sqrt{2}t + \frac{1}{2\sqrt{2}} \ln t} \to \mathrm{PPP}\left(\sqrt{4\pi} e^{-x} dx\right),$$

where $\mathrm{PPP}(\mu)$ is Poisson point process with intensity measure μ.

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Borderline: If A takes only finitely many values, and $A(x) \leq x$, for all $x \in [0,1]$, but $A(x) = x$, for some $x \in (0,1)$, the extremal process is again Poisson, but with reduced intensity (B-Kurkova).

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What happens at the natural border $A(x) = x$??

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There are three phases with distinct properties and effects:

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Let us look at them.......

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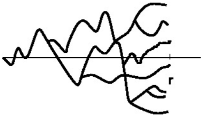
and 

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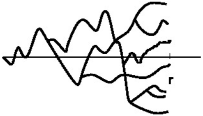
and 

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Two consequences:

- the random variable Z, the "derivative martingale"
- particles near the maximum at time t can have common ancestors at finite, t-independent times (when $t \uparrow \infty$).

...Midlife...

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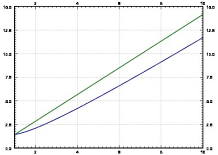
Key fact: The function $m(t)$ is convex:



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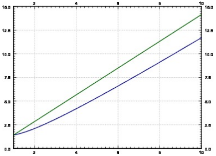

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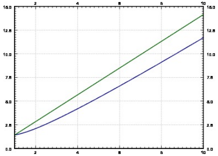

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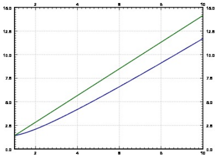
- Offspring of the selected particles is atypical!



The function $m(t)$

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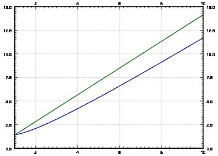
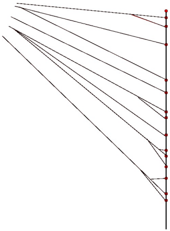
...Midlife...

Key fact: The function $m(t)$ is convex:



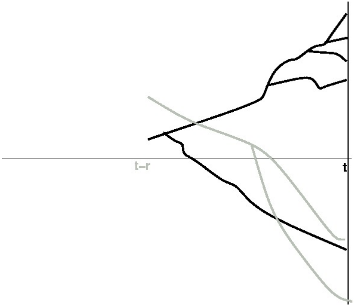
The function $m(t)$

- Descendants of a particle maximal at time $0 \ll s \ll t$ cannot be maximal at time t!
- Particles realising the maximum at time t have ancestors at times s that are selected from the very many particles that are a lot below the maximum at time s.
- Offspring of the selected particles is atypical!
- Only one descendant of a selected particle at times $0 \ll s \ll t$ can be at finite distance from the maximum at time t.



...just before the end

Any particle the arrives close to the maximum at time t can have produced offspring shortly before. These will be only a finite amount smaller then their brothers.



Hence, particles near the maximum come in small families.

The extremal process

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Let
$$\mathcal{E}_t \equiv \sum_{i=1}^{n(t)} \delta_{x_i(t)-m(t)}$$

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$$\mathcal{P}_Z = \sum_{i\in\mathbb{N}} \delta_{p_i} \equiv \mathrm{PPP}\left(CZe^{-\sqrt{2}x}dx\right)$$

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Law of $\Delta(t)$ under $\mathbb{P}\left(\cdot|\mathcal{L}(t)\right)$ converges to law of point process, Δ. Let $\Delta^{(i)}$ be iid copies of Δ, with atoms $\Delta_j^{(i)}$.

The extremal process

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Theorem (Arguin-B-Kistler, 2011 (PTRF 2013))

With the notation above, the point process $\mathcal{E}_t$ converges in law to a point process $\mathcal{E}$, given by

$$\mathcal{E} \equiv \sum_{i,j\in\mathbb{N}} \delta_{p_i+\Delta_j^{(i)}}$$

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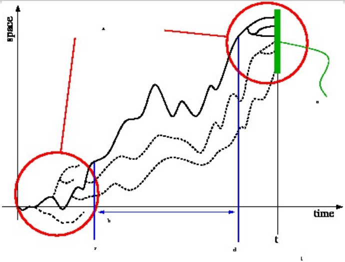

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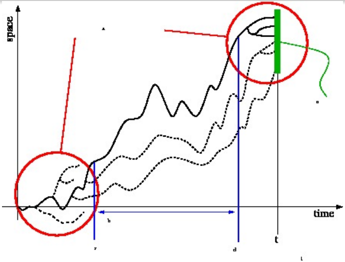

Similar result obtained independently by Aïdékon, Brunet, Berestycki, and Shi.

Ergodic theorem for the max

Alternative look: what happens if we consider time averages?
Naively one might expect a law of large numbers:

$$\lim_{T\uparrow\infty} \frac{1}{T}\int_0^T \mathbb{1}_{\left\{\max_{k=1}^{n(t)} x_k(t)-m(t)\leq x\right\}} = \mathbb{E}\exp\left(-CZe^{-\sqrt{2}x}\right) \quad \text{a.s.}$$

But this cannot be true!

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But this cannot be true! Lalley and Sellke conjectured a random version:

Theorem (Arguin, B, Kistler, 2012 (EJP 18))

$\mathbb{P}$-a.s., for any $x \in \mathbb{R}$,

$$\lim_{T\uparrow\infty}\frac{1}{T}\int_0^T \mathbb{1}_{\left\{\max_{k=1}^{n(t)} x_k(t)-m(t)\leq x\right\}} = \exp\left(-CZe^{-\sqrt{2}x}\right)$$

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Ergodic theorem for the extremal process

We prove this conjecture and extend it to the entire extremal process:

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Theorem (Arguin, B, Kistler (2012))

$\mathcal{E}_t$ converges $\mathbb{P}$-almost surely weakly under time-average to the Poisson cluster process $\mathcal{E}_Z$. That is, $\mathbb{P}$-a.s., $\forall f \in \mathcal{C}_c^+(\mathbb{R})$,

$$\frac{1}{T}\int \exp\left(-\int f(y)\mathcal{E}_{t,\omega}(dy)\right) dt \to E\left[\exp\left(-\int f(y)\mathcal{E}_Z(dy)\right)\right]$$

Here $\mathcal{E}_Z$ is the process $\mathcal{E}$ for given value Z of the derivative martingale, E is w.r.t. the law of that process, given Z.

Elements of the proof

For $\varepsilon > 0$ and $R_T \ll T$, decompose

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$$+ \underbrace{\frac{1}{T}\int_{\varepsilon T}^T Y_t(\omega) dt}_{(III)\text{: needs LLN}}$$

Elements of the proof: the LLN

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$$(III) \equiv \exp\left(-\int_{\epsilon T}^{T} f(y)\mathcal{E}_{t,\omega}(dy)\right) - \mathbb{E}\left[\exp\left(-\int_{\epsilon T}^{T} f(y)\mathcal{E}_{t,\omega}(dy)\right) \Big| \mathcal{F}_{R_T}\right]$$

should vanish by a law of large numbers.

Elements of the proof: the LLN

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should vanish by a law of large numbers.
We use a criterion which is an adaptation of the theorem due to Lyons:

Lemma

Let $\{Y_s\}_{s\in\mathbb{R}_+}$ *be a.s. uniformly bounded and* $\mathbb{E}[Y_s] = 0$ *for all* s*. If*

$$\sum_{T=1}^{\infty} \frac{1}{T}\mathbb{E}\left[\left|\frac{1}{T}\int_0^T Y_s ds\right|^2\right] < \infty,$$

then

$$\frac{1}{T}\int_0^T Y_s \, ds \to 0, \quad \text{a.s.}$$

LLN

Requires covariance estimate:

Lemma

Let Y_s from (III). For $R_T = o(\sqrt{T})$ with $\lim_{T\to\infty} R_T = +\infty$, there exists $\kappa > 0$, s.t.

$$\mathbb{E}[Y_s Y_{s'}] \leq C e^{-R_T^\kappa} \text{ for any } s, s' \in [\varepsilon T, T] \text{ with } |s - s'| \geq R_T.$$

Universality

The new extremal process of BBM should not be limited to BBM:

- Branching random walk (Aïdekon, Madaule)
- Gaussian free field in $d = 2$ [Bolthausen, Deuschel, Giacomin, Bramson, Zeitouni,Biskup and Louisdor (!) ....]
- Cover times of random walks [Ding, Zeitouni, Sznitman,....]
- Spin glasses with log-correlated potentials [Fyodorov, Bouchaud,..]

and building block for further models:

- Extensions to stronger correlations: beyond the borderline [Fang-Zeitouni '12]...
- Extension back to spin glasses: some of the observations made give hope.... see Louis-Pierre's talk

References

- L.-P. Arguin, A. Bovier, and N. Kistler, *The genealogy of extremal particles of branching Brownian motion*, Commun. Pure Appl. Math. **64**, 1647–1676 (2011).
- L.-P. Arguin, A. Bovier, and N. Kistler, *Poissonian statistics in the extremal process of braching Brownian motion*, Ann. Appl. Probab. **22**, 1693–1711 (2012).
- L.-P. Arguin, A. Bovier, and N. Kistler, *The extremal process of branching Brownian motion*, to appear in Probab. Theor. Rel. Fields (2012).
- L.-P. Arguin, A. Bovier, and N. Kistler, *An ergodic theorem for the frontier of branching Brownian motion*, EJP 18 (2013).
- L.-P. Arguin, A, Bovier, and N. Kistler, *An ergodic theorem for the extremal process of of branching Brownian motion*, `arXiv:1209.6027`, (2012).
- E. Aïdékon, J. Berestyzki, É. Brunet, Z. Shi, *Branching Brownian motion seen from its tip*, to appear in Probab. Theor. Rel. Fields (2012).

Thank you for your attention!

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