Long range order for planar Potts antiferromagnets

Roman Kotecký, Warwick/Prague

6th May, 2013

Potts antiferromagnet

Potts antiferromagnet on a planar lattice (graph) G

- $\sigma \in \{1, 2, \ldots, q\}^{V(G)}$ (we consider $q = 3$)
- $H(\sigma) = \sum_{\{x,y\} \in G} \delta_{\sigma_x, \sigma_y}$
- G infinite, planar, biparite: $G = G_0 \cup G_1$, quasi-transitive, one end

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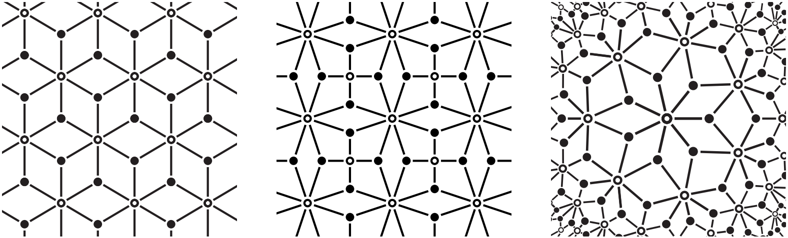

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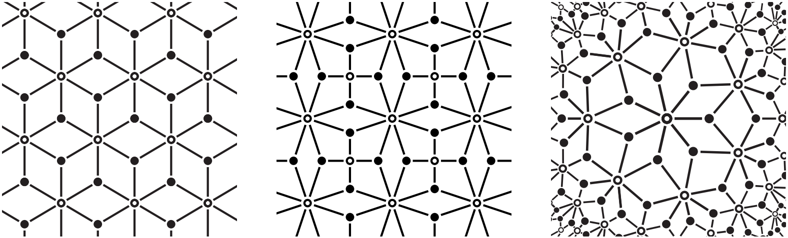

R. K., Alan Sokal, Jan Swart, arXiv 2012

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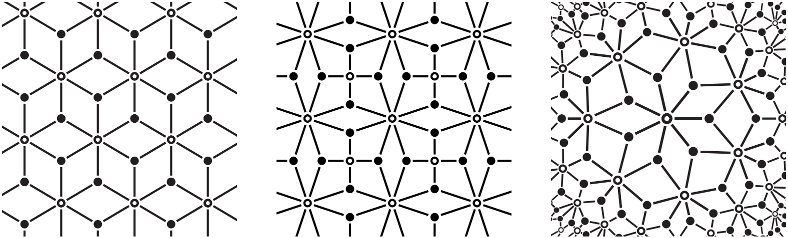

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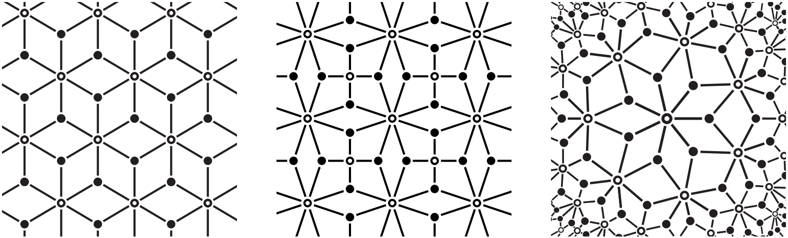

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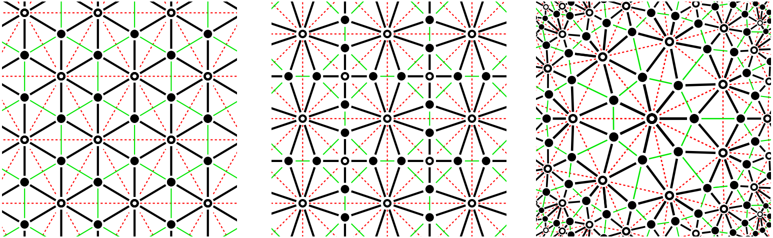
Existence of entropic long-range order for Potts antiferromagnet at low temperatures.

Plan:

- State the result
- Mention main ideas of the proof

Main claim



Theorem

$G = (V, E)$: *a quadrangulation of the plane*

$G_0 = (V_0, E_0), G_1 = (V_1, E_1)$: *its sublattices with edges drawn the diagonals of quadrilaterals.*

Assume that G_0 *is a locally finite 3-connected quasi-transitive triangulation with one end.*

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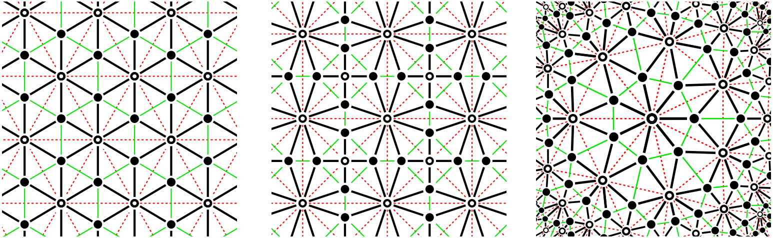

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Then there exist $\beta_0, C < \infty$ *and* $\varepsilon > 0$ *such that for each inverse temperature* $\beta \in [\beta_0, \infty]$ *and each* $k \in \{1, 2, 3\}$, *there exists an infinite-volume Gibbs measure* $\mu_{k,\beta}$ *for the 3-state Potts antiferromagnet on* G *satisfying:*

(a) *For all* $v_0 \in V_0$, *we have* $\mu_{k,\beta}(\sigma_{v_0} = k) \geq \frac{1}{3} + \varepsilon$.

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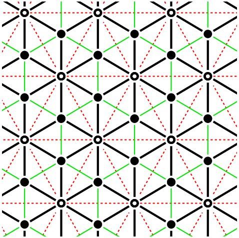

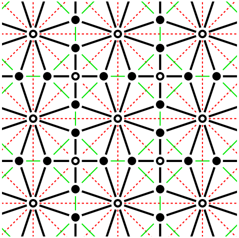

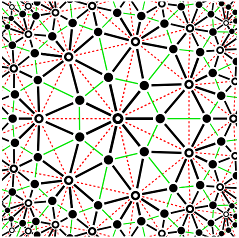

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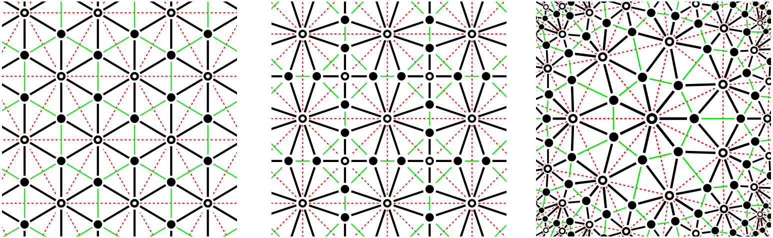

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In particular, for each inverse temperature $\beta \in [\beta_0, \infty]$*, the 3-state Potts antiferromagnet on* G *has at least three distinct extremal infinite-volume Gibbs measures.*

Walking through the proof

To simplify, notice that:

- The problem makes sense (it is nontrivial and well defined) even at zero temperature
- At zero temperature, the configurations are perfect colourings with 3 colours and the Gibbs states in finite volume are just the uniform distributions on those that are consistent with boundary conditions

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For uniformly distributed perfect colourings of finite $\Lambda \subset G$ with a fixed colour (say "$1 =$ red") on $G_0 \setminus \Lambda$, we have

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PHYSICAL REVIEW B VOLUME 31, NUMBER 5 1 MARCH 1985

Long-range order for antiferromagnetic Potts models

Roman Kotecký
Department of Mathematical Physics, Charles University, V Holešovičkách 2, 180 00 Praha 8, Czechoslovakia
(Received 29 March 1984; revised manuscript received 30 July 1984)

Long-range order for the three-state antiferromagnetic Potts model may appear at zero temperature as an instability with respect to boundary conditions. It is studied using an approximate correspondence, reminiscent of duality, which links this model with the ferromagnetic Ising model at a particular temperature. The basic idea is to represent entropy constraints in the former in terms of energy increase in the latter. The correspondence can be made exact by modifying the Ising model. The (non)existence of long-range order is then linked to the location of the critical temperature of the modified Ising model with respect to the particular value given by the correspondence.

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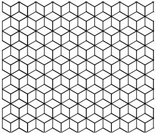
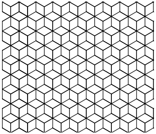
- And I will do it in pictures with minimum of formulas



G diced lattice, $q = 3$



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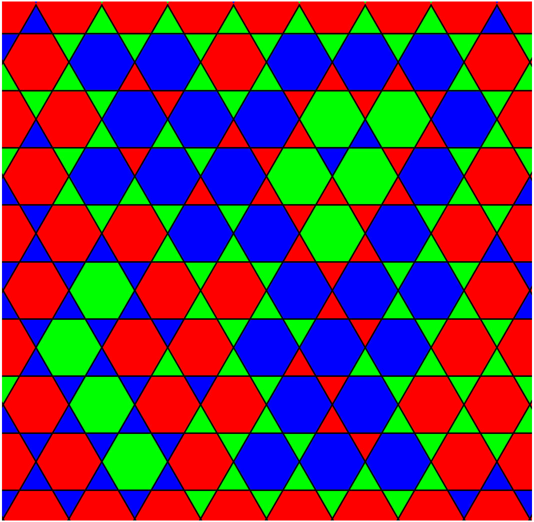

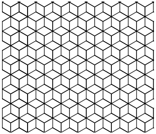

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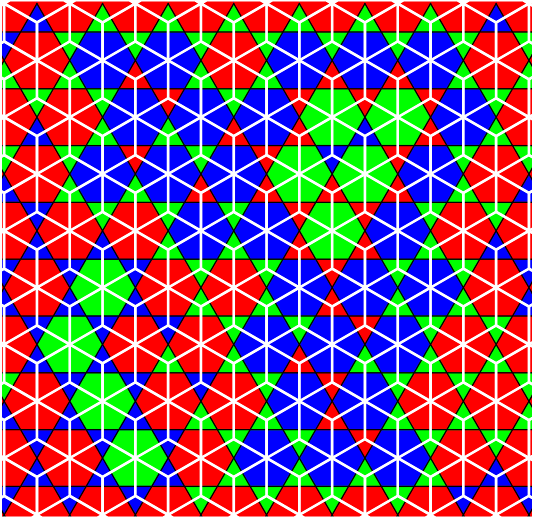
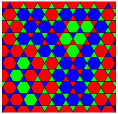
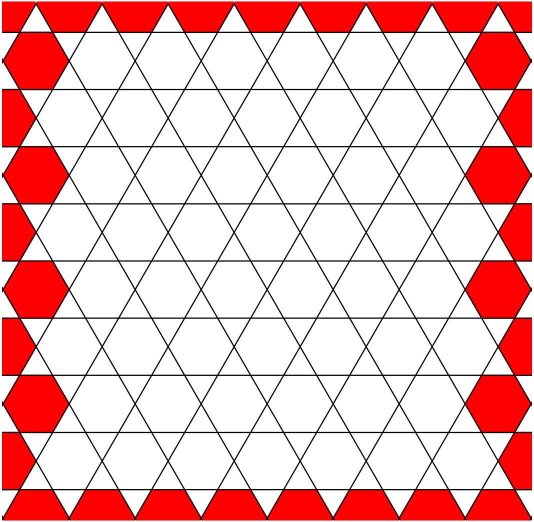
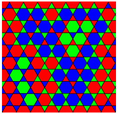
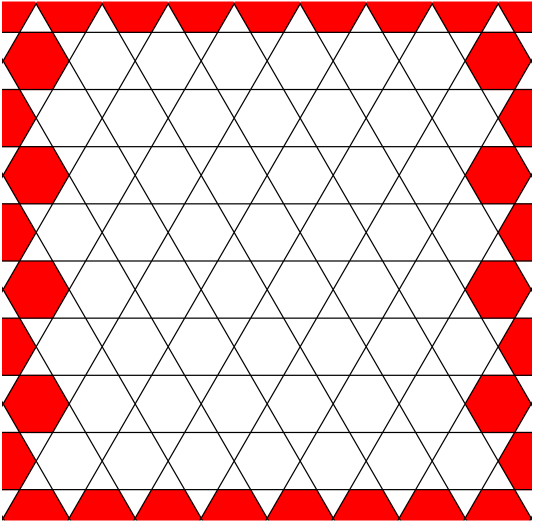






Which colour is in the centre?





Which colour is in the centre? Any is compatible with the boundary conditions, but there is some subtle obstacle that makes it less likely to differ from the boundary colour.

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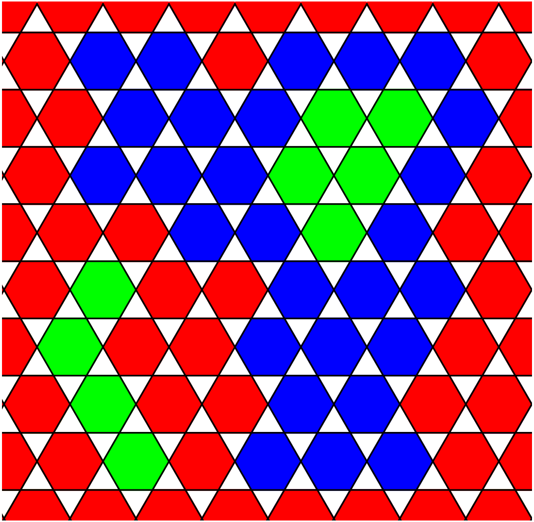
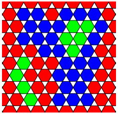
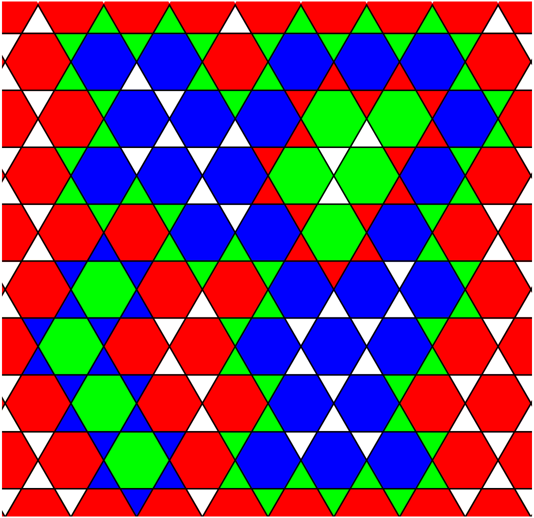
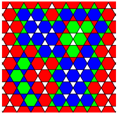
- Condition on a particular configuration of colours on the even sublattice:



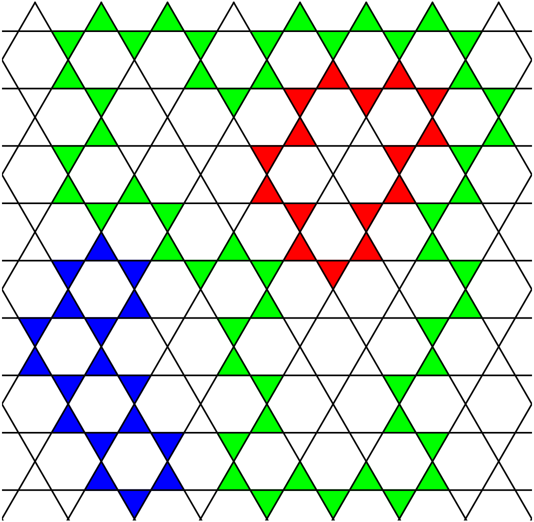


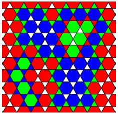
- Fix the obligatory colours:

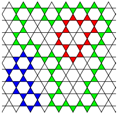




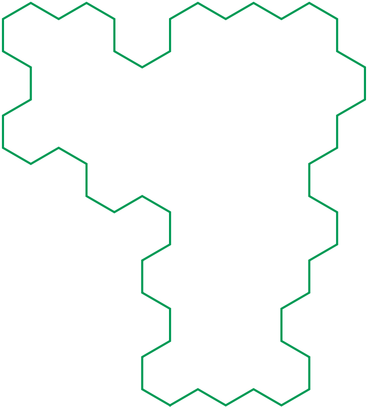
- Count the number of remaining configurations consistent with what we have:

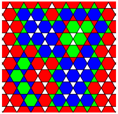
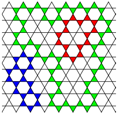






- For the colour in the centre to differ from the boundary, it has to be surrounded by a contour γ:



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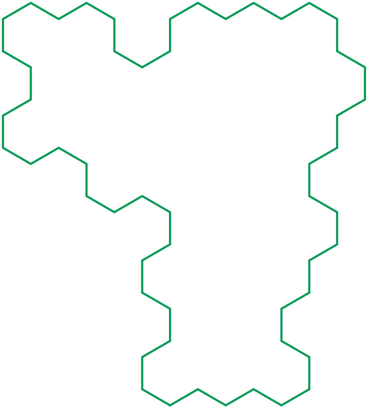
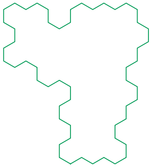
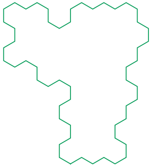
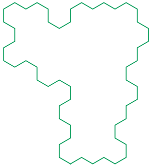


$$\mathbb{P}(\gamma) = 2^{-|\gamma|}$$





$\mathbb{P}(\text{the centre is not red}) \leq \sum_{\gamma \text{ surrounding centre}} \mathbb{P}(\gamma) \leq$



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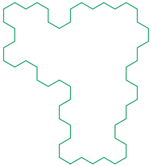

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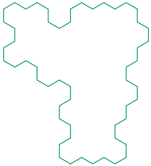
$q_n \sim \left(\sqrt{2+\sqrt{2}}\right)^n \sim 1.848^n$ (Smirnov, Duminil-Copin)



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ALMOST, BUT NOT YET SUFFICIENT

$\sum_{n=6}^{\infty} (\mu/2)^n = \frac{(\mu/2)^6}{1-\mu/2} \sim 8.17$

Nevertheless, it works!

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R. K., Jesús Salas, and Alan D Sokal, *Phase Transition in the Three-State Potts Antiferromagnet on the Diced Lattice*, Phys. Rev. Lett. **101**, 2008

What we had to do for evaluating q_n was rather nasty:

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Iwan Jensen, *Honeycomb lattice polygons and walks as a test of series analysis techniques*, Journal of Physics, 2006

Enumerating q_n exactly:

$q_6 = 1$

...

...

$q_{140} =$ 12 203 494 959 311 144 967 485 193 175 739 454

$$\sum_{n=6}^{\infty} q_n 2^{-n} = \sum_{n=6}^{140} q_n 2^{-n} + \sum_{n=142}^{\infty} \frac{n^2}{36} 2^{-n} \left(\sqrt{2+\sqrt{2}}\right)^{n-2}$$

$$= 0.03168 + 0.01731$$

Notice that we needed diced lattice ($\mu < 2$), it would not work for square lattice!

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Sideremark:
Similar long range order is expected to occur for hypercubic lattice with $d \geq 3$.

$$\sum_{n=6}^{\infty} q_n 2^{-n} = \sum_{n=6}^{140} q_n 2^{-n} + \sum_{n=142}^{\infty} \frac{n^2}{36} 2^{-n} \left(\sqrt{2+\sqrt{2}}\right)^{n-2}$$

$$= 0.03168 + 0.01731$$

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Indeed, it was proven for hypercubic lattice of high dimension d:

Ron Peled, *High-Dimensional Lipschitz Functions are Typically Flat*
arXiv 2010

David Galvin, Jeff Kahn, Dana Randall, Gregory B. Sorkin,
Phase coexistence and torpid mixing in the 3-coloring model on $\mathbb{Z}^d$,
arXiv 2012

In our paper with Alan Sokal and Jan Swart we avoided the use of explicit values of q_n by employing two tricks:

- Long range order by the bound using only the tail of the series
- Percolation (random cluster) reformulation

Long range order by the bound using only the tail of the series

Consider the events $A_{k,\Delta}$ that a big area Δ of the even sublattice (G_0) around the centre is covered by the colour k and define

$$A_\Delta = A_{1,\Delta} \cup A_{2,\Delta} \cup A_{3,\Delta}$$

the event that this area is monocolour.

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$$\mathbb{P}_{\Lambda,1}(A_{1,\Delta}) - \mathbb{P}_{\Lambda,2}(A_{1,\Delta}) = \mathbb{P}_{\Lambda,1}(A_{1,\Delta}) - \mathbb{P}_{\Lambda,1}(A_{2,\Delta})$$

$$= \big(\mathbb{P}_{\Lambda,1}(A_{1,\Delta} \mid A_\Delta) - \mathbb{P}_{\Lambda,1}(A_{2,\Delta} \mid A_\Delta)\big)\mathbb{P}_{\Lambda,1}(A_\Delta) \geq \epsilon$$

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We also employ a fixed lower bound, uniform in Λ, on $\mathbb{P}_{\Lambda,1}(A_\Delta)$ (say, $\mathbb{P}_{\Lambda,1}(A_\Delta) \geq (1/3)^{|\Delta|}$).

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while, in the standard Peierls argument, to get

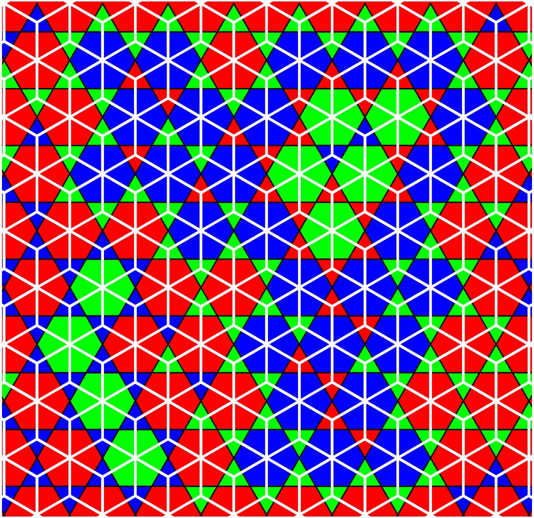
$$\sum_{n=4}^{\infty} \frac{n^2}{16}(e^{-2\beta}\mu)^n \ldots \frac{(e^{-2\beta}\mu)^4}{1-e^{-2\beta}\mu} < 1/2,$$

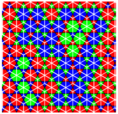
we need at least $\beta > \beta_1$ with

$$\beta_1 = 0.702.$$

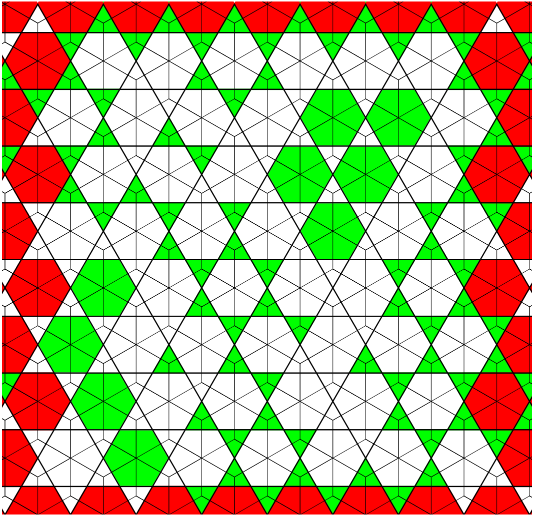
Argument using a random cluster reformulation

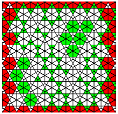
The difference $\mathbb{P}_{\Lambda,1}(A_{1,\Delta}) - \mathbb{P}_{\Lambda,1}(A_{2,\Delta})$ equals the probability that a properly defined percolative cluster reaching from boundary to Δ exists.



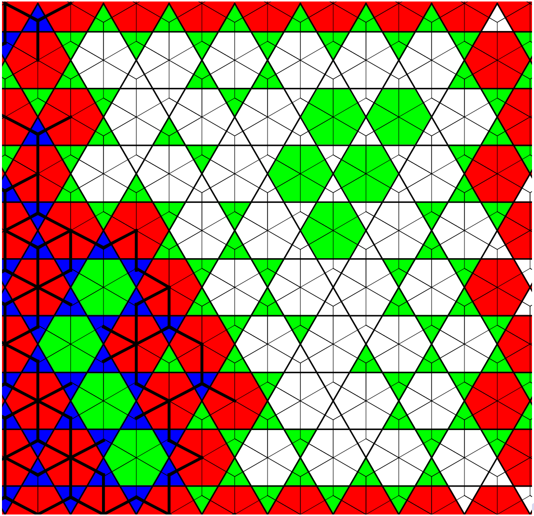


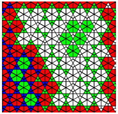
- Fix one colour:



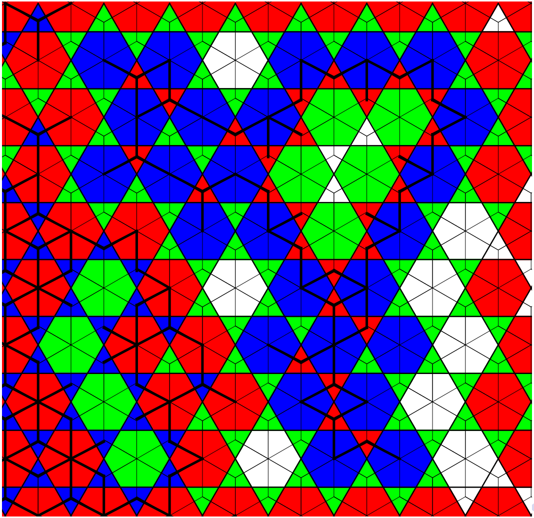


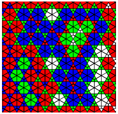
• The rest is covered by red and blue; for any configuration, consider the set of thick edges connecting pairs of red/blue neighbours. Given $\Lambda_3 \subset V$ coloured green, the set of those edges is automatically given (at zero temperature) and splits into several clusters (separated by green vertices):





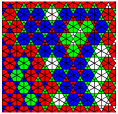
● Those clusters reaching the boundary have a uniquely determined colouring. Thus, if the centre is contained in a cluster of edges connected with the boundary, its colour is uniquely determined. If not, it can be coloured by red and blue with exactly the same probability:





$$\mathbb{P}_{\Lambda,1}(\sigma_{v_0} = 1 \,|\, \Lambda^3) \quad = \quad \begin{cases} 1 & \text{if } v_0 \leftrightarrow \partial\Lambda \\ \frac{1}{2} & \text{if } v_0 \in \Lambda^{12} \text{ and } v_0 \not\leftrightarrow \partial\Lambda \\ 0 & \text{if } v_0 \in \Lambda^3 \end{cases}$$

$$\mathbb{P}_{\Lambda,1}(\sigma_{v_0} = 2 \,|\, \Lambda^3) \quad = \quad \begin{cases} 0 & \text{if } v_0 \leftrightarrow \partial\Lambda \\ \frac{1}{2} & \text{if } v_0 \in \Lambda^{12} \text{ and } v_0 \not\leftrightarrow \partial\Lambda \\ 0 & \text{if } v_0 \in \Lambda^3 \end{cases}$$



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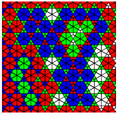

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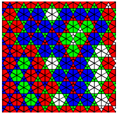
All is finished by finite energy argument (a bit of ingeneering on clusters with uniform bounds):

$$\mathbb{P}_{\Lambda,1}\big(v_0 \leftrightarrow \partial\Lambda) \geq \delta\mathbb{P}_{\Lambda,1}\big(A_\triangle \& \Delta \leftrightarrow \partial\Lambda)$$



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+ extensions to small temperatures