On the Gibbs states of the Potts model

Yvan Velenik

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based on a joint work with
Loren Coquille, **Hugo Duminil-Copin** et **Dmitry Ioffe**

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Box: $\Lambda \Subset \mathbb{Z}^2$

Bound. cond.: $\omega \in \Omega \equiv \{1,\ldots,q\}^{\mathbb{Z}^2}$

$\Omega_\Lambda^\omega = \{\sigma \in \Omega \,:\, \sigma_i = \omega_i,\, \forall i \notin \Lambda\}$

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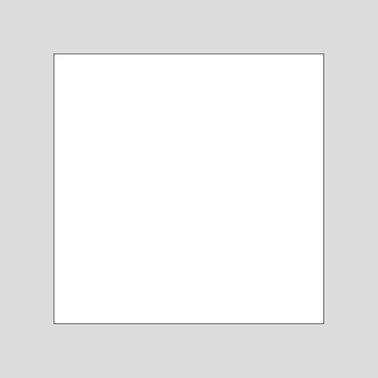
$$H_\Lambda(\sigma) = \sum_{\substack{\{i,j\}\cap\Lambda\neq\varnothing \\ i\sim j}} \mathbf{1}_{\{\sigma_i\neq\sigma_j\}}$$



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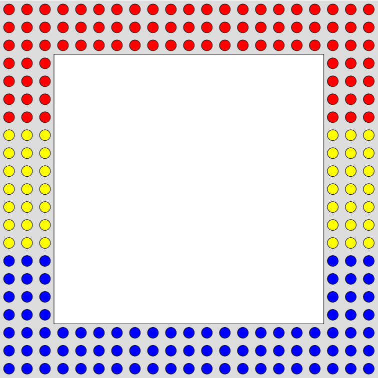

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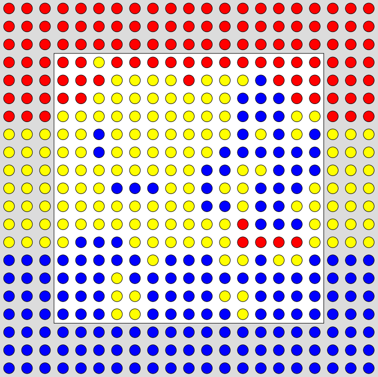

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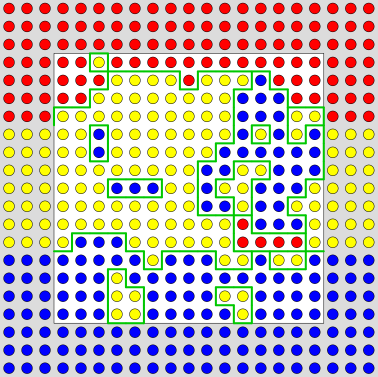

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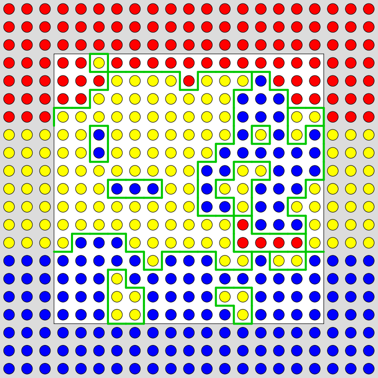

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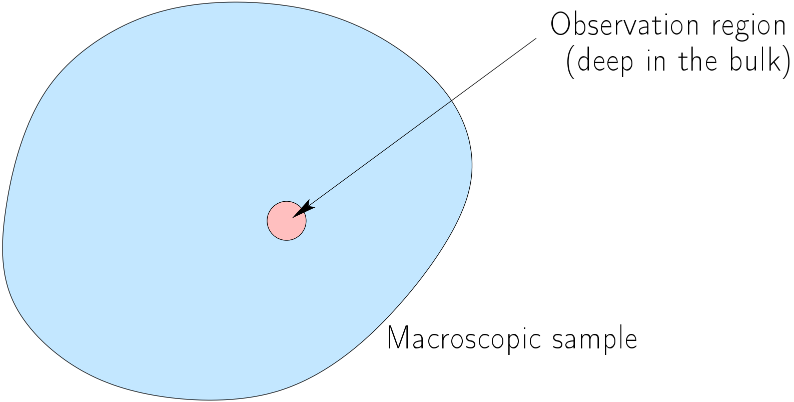
Bulk behavior



Basic question: What possible behaviors can be observed in a (small) subregion deep in the bulk?

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We want to consider limits of the type $\lim_{\Lambda\uparrow\mathbb{Z}^2} \mu^{\omega}_{\Lambda;T}$.

(Relevant topology: $\mu^{\omega}_{\Lambda;T} \to \mu$ iff $\mu^{\omega}_{\Lambda;T}(f) \to \mu(f)$, $\forall f$ local)

Important particular case: pure boundary conditions, $\omega \equiv i$, $i \in \{1,\dots,q\}$. In that case, the limits

$$\mu^{i}_{T} = \lim_{\Lambda\uparrow\mathbb{Z}^2} \mu^{i}_{\Lambda;T}$$

exist and are translation invariant.

Problems with this definition:

- In general difficult to establish convergence and determine the limit.
- Not very convenient for an abstract study of infinite-volume Gibbs measures.

(Easy) observation: any limit μ satisfy

$$\mu(f) = \mu\big(\mu^{\cdot}_{\Lambda;T}(f)\big), \qquad \forall \Lambda \Subset \mathbb{Z}^2. \qquad \text{(DLR)}$$

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Let $\mathcal{G}_{T,q}$ be the set of all such random fields.

Some properties of the set $\mathcal{G}_{T,q}$:

- $\exists T_c(q) \in (0,\infty)$ such that $|\mathcal{G}_{T,q}| = 1$ when $T > T_c(q)$ but $|\mathcal{G}_{T,q}| > 1$ when $T < T_c(q)$.
- $\mathcal{G}_{T,q}$ is a simplex.
- $\forall \mu \in \operatorname{ex}\mathcal{G}_{T,q}$, $\lim_{\Lambda\uparrow\mathbb{Z}^2} \mu^{\omega}_{\Lambda;T} = \mu$, for μ-a.e. ω.

Natural question: determine the set $\operatorname{ex}\mathcal{G}_{T,q}$.

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Determination of $\mathcal{G}_{T,q}$

Unfortunately, determining the set of all infinite-volume Gibbs measures associated to a given model is usually *very* difficult.

- In dimension 2, when $T \ll 1$, there are general results by Dobrushin & Shlosman (1985).
- Essentially no results in dimensions $d \geq 3$, even at $T \ll 1$.
- In any dimension, there are general results about the set of all *translation invariant* Gibbs measures (e.g., via Pirogov-Sinai theory).

To be addressed now:

What about non-perturbative results in 2d?

What makes $d = 2$ simpler than $d \geq 3$?

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Earlier non-perturbative results for 2d Ising/Potts model

Two-dimensional 2-states Potts model (i.e., Ising model)

- Messager & Miracle-Sole '75: all *translation invariant* elements of $\mathcal{G}_{T,2}$ are convex combinations of μ_T^1 and μ_T^2.
- Russo '79: Any measure $\mu \in \mathcal{G}_T$ invariant under translations in direction $\vec{e}_1$ is invariant under all translations.
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$$\mathcal{G}_{T,2} = \{\alpha\mu_T^1 + (1-\alpha)\mu_T^2 \,:\, 0 \le \alpha \le 1\}.$$

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Our results

Our results (first obtained for the case $q = 2$ in a joint work with Loren Coquille), take the following form:

Theorem [Coquille, Duminil-Copin, Ioffe, V.]

Let $T < T_c(q)$, $\Lambda_n = \{-n, \ldots, n\}^2$, $\omega \in \Omega$ et $R = o(n^{1/2})$.
$\exists \alpha_i^{n,\omega}(T) \in [0, 1]$, $i = 1, \ldots, q$, such that $\sum_{i=1}^{q} \alpha_i^{n,\omega}(T) = 1$ and

$$\mu^\omega_{\Lambda_n;T}(f) = \sum_{i=1}^{q} \alpha_i^{n,\omega} \mu^i_T(f) + O_T(\|f\|_\infty \, R \, n^{-1/2}),$$

for all $n > n_0(T)$, uniformly in functions f with support included inside Λ_R.

Corollary

For all $T < T_c(q)$, all Gibbs measures are translation invariant and

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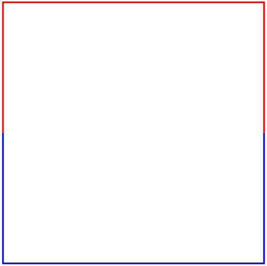
$d = 2$ vs $d \geq 3$, translation invariance

A natural candidate when trying to generate a non translation invariant Gibbs measure is to consider the Dobrushin boundary condition:

$$\omega_i^{1,2} = \begin{cases} 1 & \text{si } \langle i, \mathbf{e}_2 \rangle \geq 0, \\ 2 & \text{si } \langle i, \mathbf{e}_2 \rangle < 0. \end{cases}$$



After diffusive scaling, the interface induced by this b.c. in $\Lambda_L = \{-L, \dots, L\}^2$ weakly converges, as $L \to \infty$, toward a Brownian bridge [Higuchi '79, Greenberg & Ioffe '05, Campanino, Ioffe, V. '08].

In particular, this interface has fluctuations of size $O(\sqrt{L})$, and the expectation of a local function f thus satisfies

$$\lim_{L \to \infty} \mu^{1,2}_{\Lambda_L;T}(f) = \tfrac{1}{2}\mu^1_T(f) + \tfrac{1}{2}\mu^2_T(f),$$

since the support of this function is either far above or far below the interface with equal probability 1/2.

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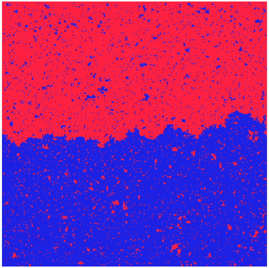

After diffusive scaling, the interface induced by this b.c. in $\Lambda_L = \{-L, \dots, L\}^2$ weakly converges, as $L \to \infty$, toward a Brownian bridge [Higuchi '79, Greenberg & Ioffe '05, Campanino, Ioffe, V. '08].

In particular, this interface has fluctuations of size $O(\sqrt{L})$, and the expectation of a local function f thus satisfies

$$\lim_{L\to\infty} \mu^{1,2}_{\Lambda_L;T}(f) = \tfrac{1}{2}\mu^1_T(f) + \tfrac{1}{2}\mu^2_T(f),$$

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$d = 2$ vs $d \geq 3$, translation invariance

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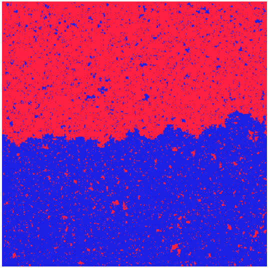

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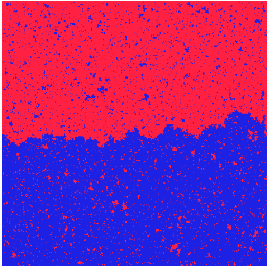

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$d = 2$ vs $d \geq 3$, translation invariance

This is **not true when $d \geq 3$ and $T \ll 1$**: $\mu^{1,2}_{\Lambda_L;T}$ gives rise to an extremal, translation non-invariant, Gibbs state [Dobrushin '72].

Crucial difference:

- When $d = 2$, interfaces are "one-dimensional" objects, which undergo unbounded fluctuations at any $T < T_c$.
- When $d \geq 3$ and $T \ll 1$, the horizontal interface is **rigid**.

Interface fluctuations are responsible for the absence of translation non-invariant Gibbs measures in two-dimensional systems and are central to our proof.

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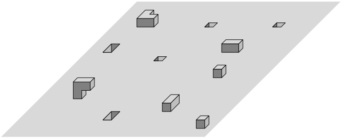

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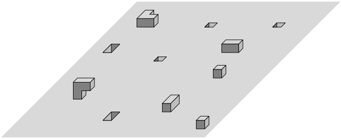

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Contents

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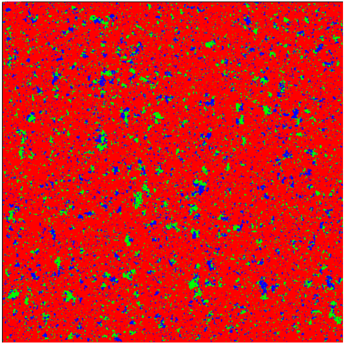
Ingredient #1: Exponential relaxation in pure phases

Let $\Lambda \subset \mathbb{Z}^2$. Then [Beffara & Duminil-Copin '12]: For any $T < T_c(q)$, there exists $C(T) > 0$ such that

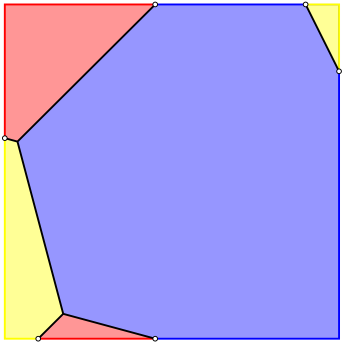
$$\left|\mu^i_{\Lambda;T}(f) - \mu^i_T(f)\right| \leq \|f\|_\infty \, |S(f)| \, e^{-C\, d(S(f),\Lambda^c)},$$

uniformly for all local functions f with support $S(f) \subset \Lambda$.



Ingredient #2: Macroscopic interfaces

Consider a large box $\Lambda \Subset \mathbb{Z}^2$, with a "macroscopic" boundary condition. Then the interfaces concentrate on the solution of a variational problem ("minimize total surface tension, taking into account the constraints induced by the b.c."). The solution consists in a **finite family of "well-separated" trees**, with **inner nodes of degree** 3.



(Of course, when $q = 2$ there are no inner nodes.)

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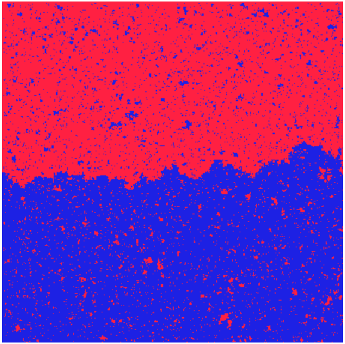
Ingredient #3: Gaussian fluctuations of interfaces

Open contours corresponding to linear macroscopic interfaces have **Gaussian fluctuations** (convergence to a **Brownian bridge** after diffusive scaling).



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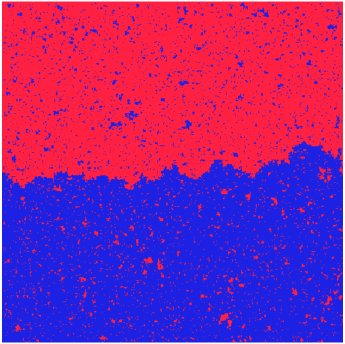

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Proof: Step #1

Let $\Lambda_n = \{-n, \dots, n\}^2$.

The boundary condition is generically *not* macroscopic (there can be $O(n)$ changes of colors along the boundary of Λ_n).

Claim: at most $M(T) < \infty$ interfaces reach the sub-box $\Lambda_{n/2}$.

Indeed

- Each interface "costs" e^{-cn}, $c > 0$, so that K interfaces cost e^{-cKn}.
- The "cost" of having 0 interfaces is at most $e^{-c'n}$, $c' > 0$ (just force all spins along the inner boundary of Λ_n to have color 1).

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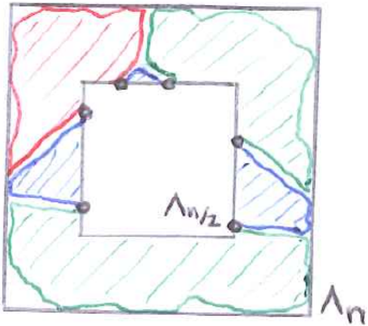
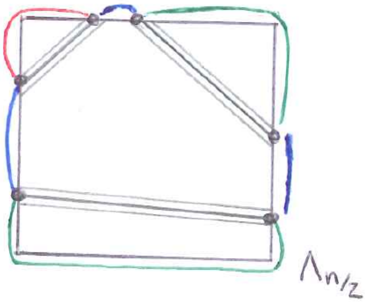
Proof: Step #1

This creates a random box consisting of the inner box $\Lambda_{n/2}$ with random "petals" attached. The boundary condition on this random box changes color only at most $M'(T)$ times (at points of $\partial\Lambda_{n/2}$).



Proof: Step #2

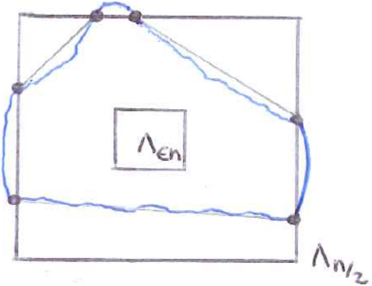
We restrict now our attention to the inner box $\Lambda_{n/2}$. Since the latter has a "macroscopic" boundary condition, the induced interfaces inside $\Lambda_{n/2}$ concentrate into tubes along the solution of the corresponding variational problem.



Consider a new small macroscopic box $\Lambda_{\epsilon n}$.
Since the macroscpic interfaces are "well-separated", there three possibilites:

1. None of the tubes intersect $\Lambda_{\epsilon n}$.
2. Exactly one tube cuts through $\Lambda_{\epsilon n}$.
3. Exactle one "tripod" has its node inside $\Lambda_{\epsilon n}$.

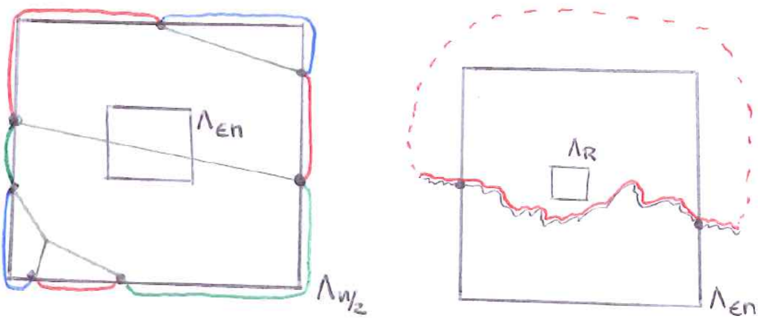
First case: None of the tubes intersect $\Lambda_{\epsilon n}$.



In that case, $\Lambda_{\epsilon n}$ (and thus also Λ_R) lies deeply in a pure phase.
We conclude using exponential relaxation in pure phases.

Proof: Step #3

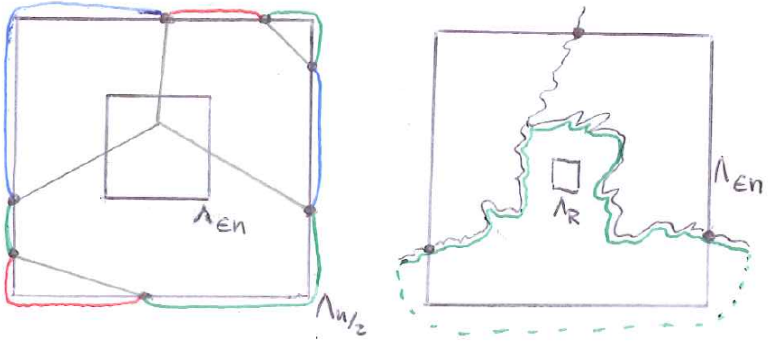
Second case: Exactly one tube cuts through $\Lambda_{\epsilon n}$.



In that case, because of its Gaussian fluctuations, the corresponding open contour inside $\Lambda_{\epsilon n}$ stays far from Λ_R with high probability. Consequently Λ_R lies again deeply in a pure phase.
We conclude using exponential relaxation in pure phases.

Proof: Step #3

Third case: Exactle one "tripod" has its node inside $\Lambda_{\epsilon n}$.

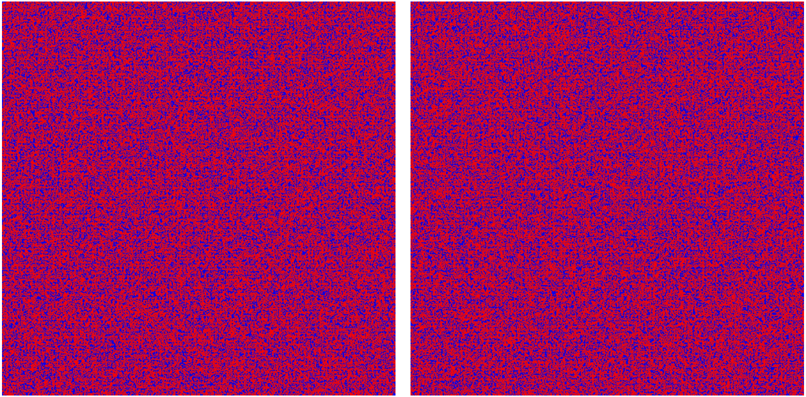


Similarly, because of the Gaussian fluctuations of the "center of the tripod" and of its "branches", the corresponding contours stay far from Λ_R with high probability. Consequently Λ_R lies again deeply in a pure phase. We conclude using exponential relaxation in pure phases.

Thank you!

Typical configurations $q = 2$ Potts (Ising) model
(500 $\times$ 500 spins)

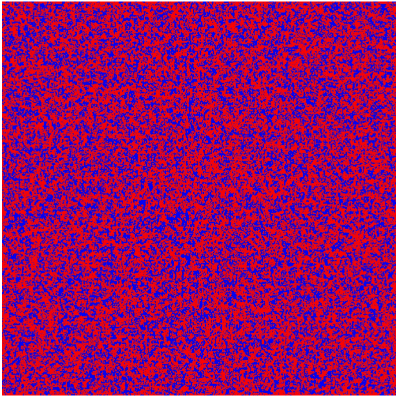
Blue b.c. Red b.c.



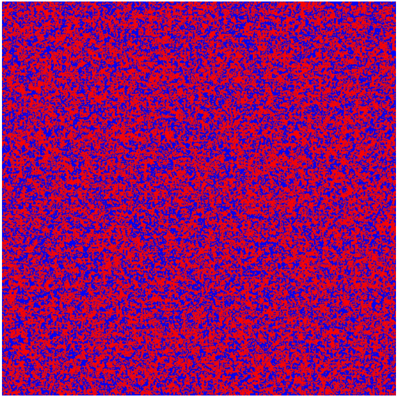
$T = \infty$

Typical configurations $q = 2$ Potts (Ising) model
(500×500 spins)

Blue b.c.



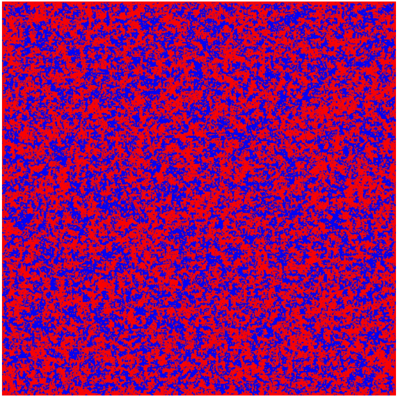
Red b.c.



$T = 1.96$

Typical configurations $q = 2$ Potts (Ising) model
$(500 \times 500$ spins$)$

Blue b.c.



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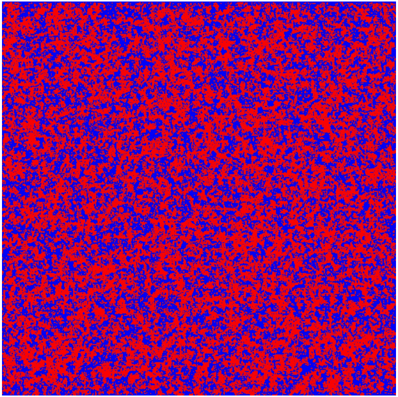
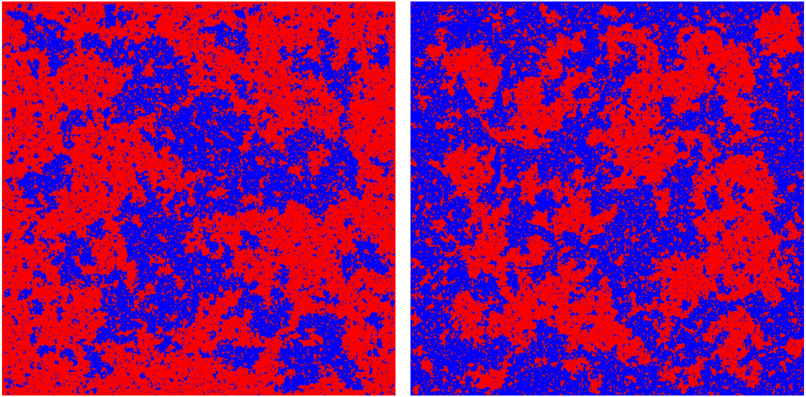


$$T = 1.44$$

Phase transition

Typical configurations $q = 2$ Potts (Ising) model
(500 × 500 spins)

Blue b.c.   Red b.c.



$T = 1.15$

Phase transition

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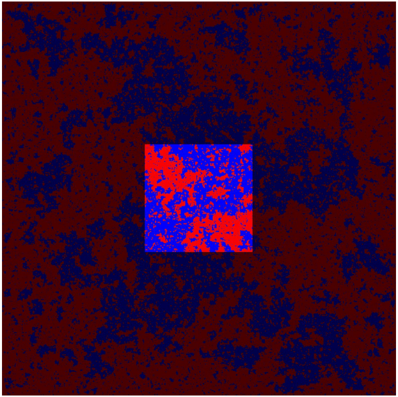
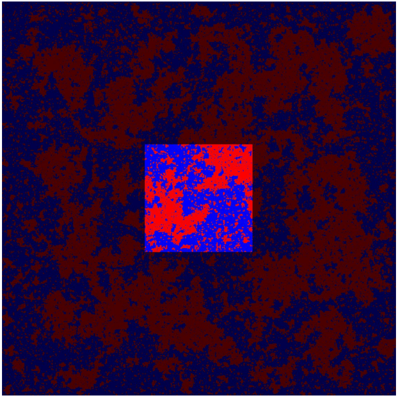
(500 × 500 spins)

Blue b.c.



Red b.c.



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Phase transition

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(500 × 500 spins)

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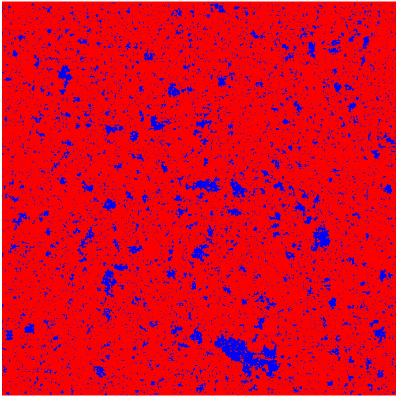

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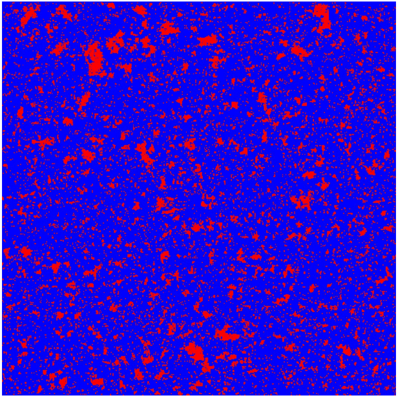
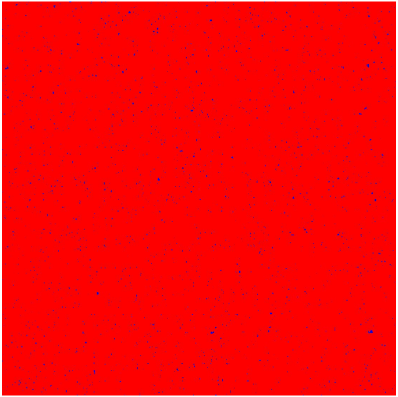


$T = 1.12$

Phase transition

Typical configurations $q = 2$ Potts (Ising) model
(500×500 spins)

Blue b.c.



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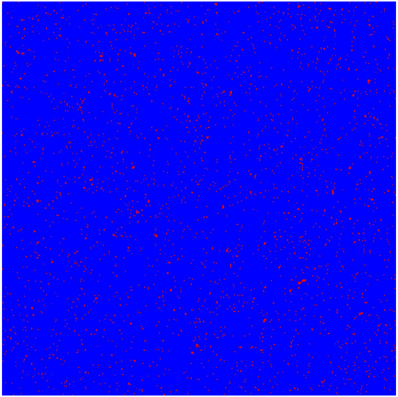

$T = 0.83$

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