How many eigenvalues of a truncated orthogonal matrix are real?

Boris Khoruzhenko (QMUL)

Workshop Random Combinatorial Structures and Statistical Mechanics

Venice 6-10 May 2013

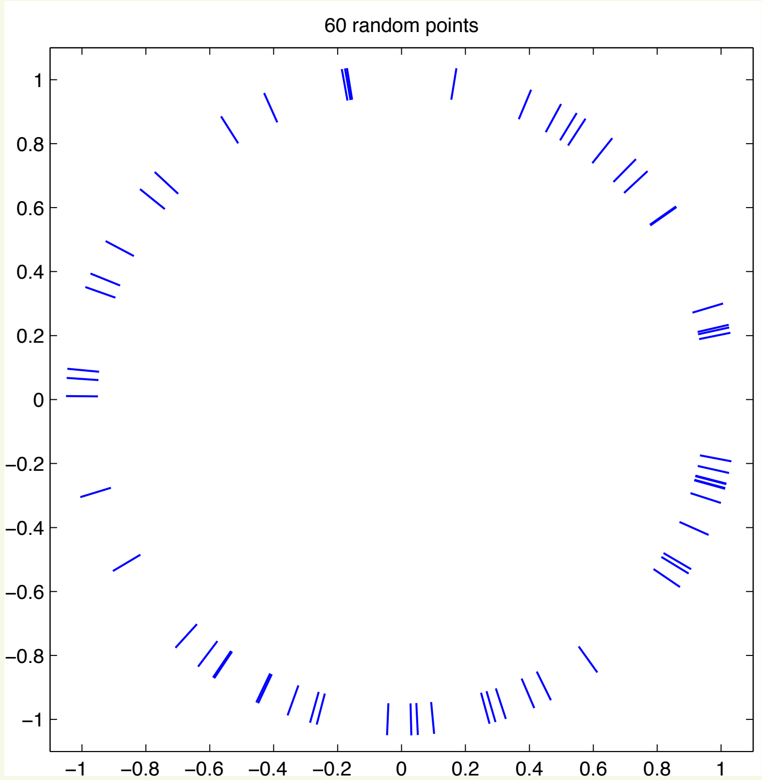
Collaborators (Phys. Rev E. **82** 040106(R) (2010))

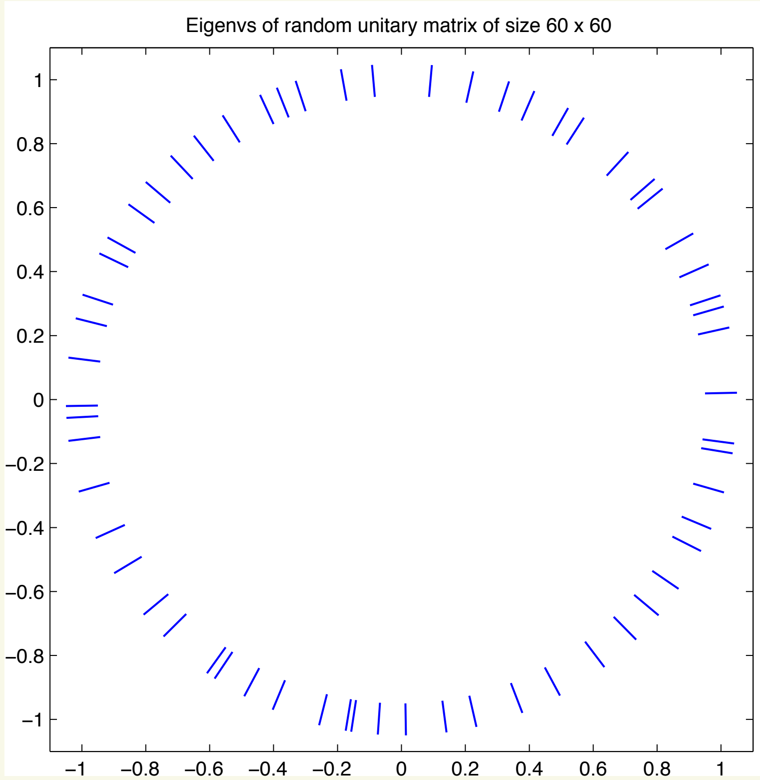
Hans-Jürgen Sommers (Duisburg-Essen),

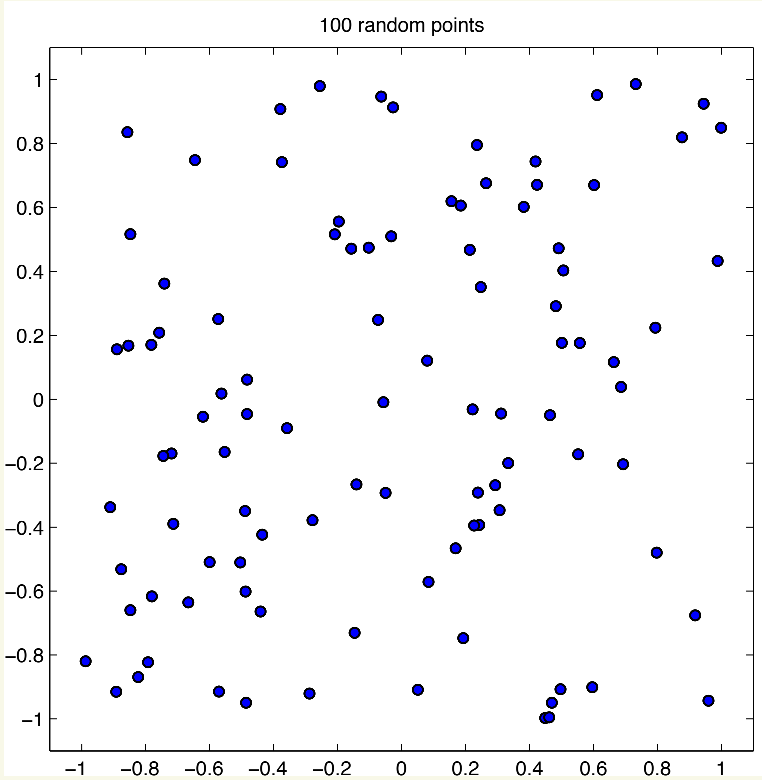
Karel Życzkowski (Kraków & Warszawa)

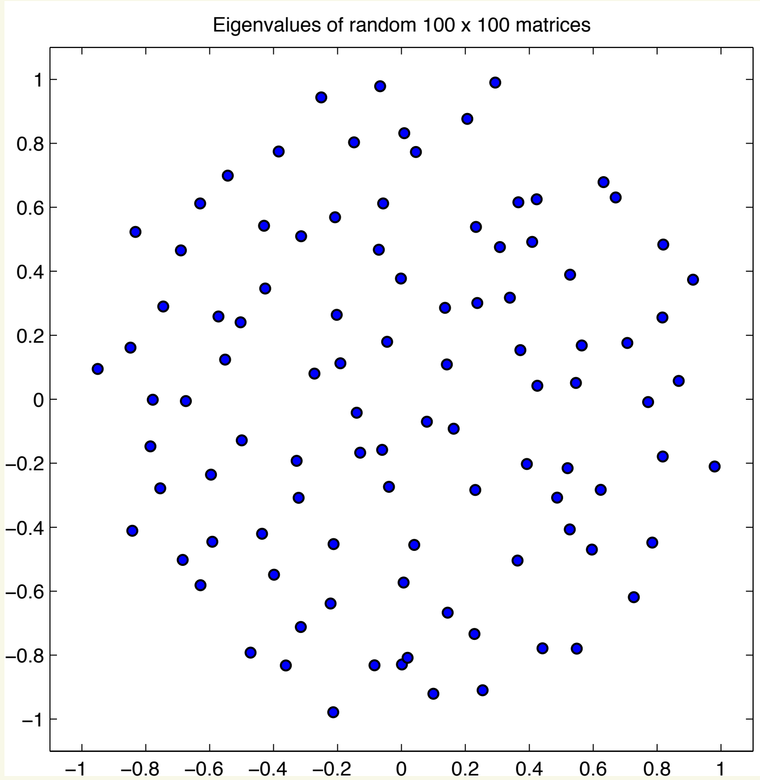
Random matrices as a probability machine

Recall Galton's quincunx ...





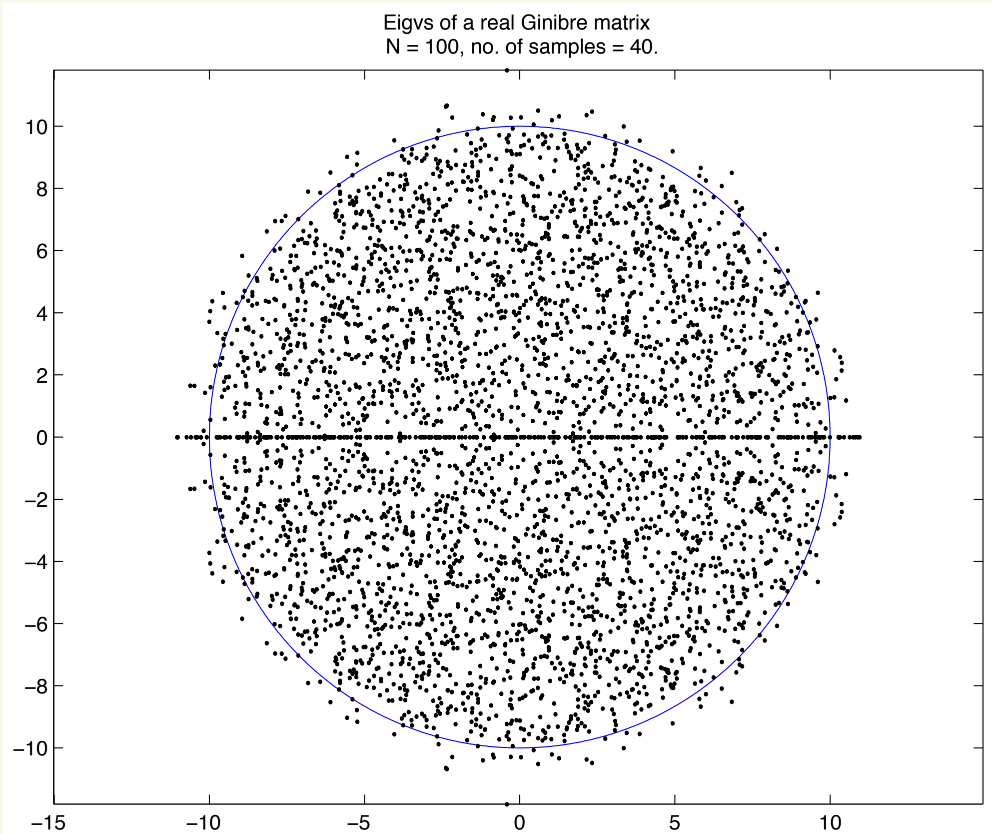




Ginibre ensembles

Quite a lot of interest to EV statistics in the complex plane in the last 20 years or so. Mostly Gaussian ensembles studies and a few beyond.

Real matrices with no symmetry conditions imposed – non-zero prob of having real EVs – interesting, though challenging. Recent progress for the real Ginibre [Kanzieper & Akemann 2005, Forrester & Nagao 2007, Sommers 2007] and for its chiral counterpart [Akemann et al 2010] building on foundations laid by Lehmann & Sommers 1991, Edelman 1998 and Edelman, Kostlan & Shub 1994.



Truncations of unitary matrices

What truncations of unitary matrices are good for?

(1) **Quantum transport problems** (Beenakker '97) Additive stats of EVs of $TT^\dagger$ describe phys quantities of interest, i.e. $\operatorname{tr} TT^\dagger$ for conductance of quasi one-dimensional wires

(2) **Open chaotic sys** (Fyodorov & Sommers, '97 Życzkowski & S. '00) Eigenvalues of T are used to model resonances

(3) **Combinatorics of vicious walkers**(Novak '09) $< |\operatorname{tr} T|^N >_T$ enumerates configs of random-turn vicious walkers

Singular values of T (1); eigenvalues of T (2,3)

Why truncation of orthogonal matrices? To explore the degree of universality of EVs statistics in the complex plane (no mathematical theory known).

Also, truncations and Kac polynomials [Krishnapour 2008, Forrester 2010]

How Many Eigenvalues of a Random Matrix are Real?

Alan Edelman; Eric Kostlan; Michael Shub

Journal of the American Mathematical Society, Vol. 7, No. 1. (Jan., 1994), pp. 247-267.

$\rho_n(\lambda) = \frac{e^{-\lambda^2/2}}{2^{n/2}\Gamma(n/2)} D_{n-1}(\lambda)$, $D_{n-1}(\lambda) = \mathbf{E}_{A_0}|\det(A_0 - \lambda I)|$, where the expected value is

taken over all $(n-1)$-by-$(n-1)$ matrices A_0

$$D_n(\lambda) = \mathbf{E}_A(|\det(A - \lambda I)|) = \frac{2^{n/2}\Gamma((n+1)/2)}{\sqrt{\pi}} {}_1F_1\left(-\frac{1}{2}; \frac{n}{2}; -\frac{\lambda^2}{2} I_n\right).$$

Corollary 4.3. *If* λ_n *denotes a real eigenvalue of an* n*-by-*n *random matrix, then its marginal probability density* $f_n(\lambda)$ *is given by*

$$f_n(\lambda) = \frac{1}{E_n}\left(\frac{1}{\sqrt{2\pi}}\left[\frac{\Gamma(n-1, \lambda^2)}{\Gamma(n-1)}\right] + \frac{|\lambda^{n-1}|e^{-\lambda^2/2}}{\Gamma(n/2)2^{n/2}}\left[\frac{\gamma((n-1)/2, \lambda^2/2)}{\Gamma((n-1)/2)}\right]\right).$$

$$E_n = \int_{-\infty}^{\infty} \rho_n(\lambda)\, d\lambda,$$

Truncations of orthogonal matrices

Choose $U \in O(n)$ at random, and then truncate:

$U = \left(\begin{smallmatrix} T & S \\ Q & R \end{smallmatrix}\right) \mapsto T$, where T is $m \times m$

Haar measure on $O(n)$ induces a probability distribution $\mathrm{d}\rho_{n,m\times m}(T)$ on truncated orthogonals. No dependence on the block's position.

Matrix measure:

- If $n \geq 2m$ then $\mathrm{d}\rho_{n,m\times m}(T)$ is supp by the matrix ball $TT^\dagger \leq I$ and

$$\mathrm{d}\rho_{n,m\times m}(T) = \frac{V_{n-m}^2}{V_n V_{n-2m}} \det(I - TT^\dagger)^{\frac{1}{2}(n-2m-1)} \prod_{j,k=1}^{m} \mathrm{d}T_{jk}$$

  where $V_n = Vol\ O(n)$ [Friedman&Mello '85, Fyodorov&Sommers '03, Forrester '06]

- If $n < 2m$ then $\mathrm{d}\rho_{n,m\times m}(T)$ is singular (supported on the boundary of $TT^\dagger \leq I$; no useful expression for matrix measure is known.

This is similar to truncated unitaries.

Gaussian approximation and beyond

Gaussian Approximation (small size truncs of large orth matrices):

If $m \ll n$ then suitably scaled T becomes Gaussian (entries are independent normals), known as the Borel Thm [Borel 1906, Gallardo 1982, Yor 1985, Diaconis et al 1987, 1992, Jiang 2009]. Not surprising as

$$\mathrm{d}\rho_{n,m\times m}/\mathrm{d}T \propto \det(I - TT^{\dagger})^{\frac{1}{2}(n-2m-1)}, \quad n \geq 2m$$

Thus expect Ginibre EVs stats in Gaussian regime.

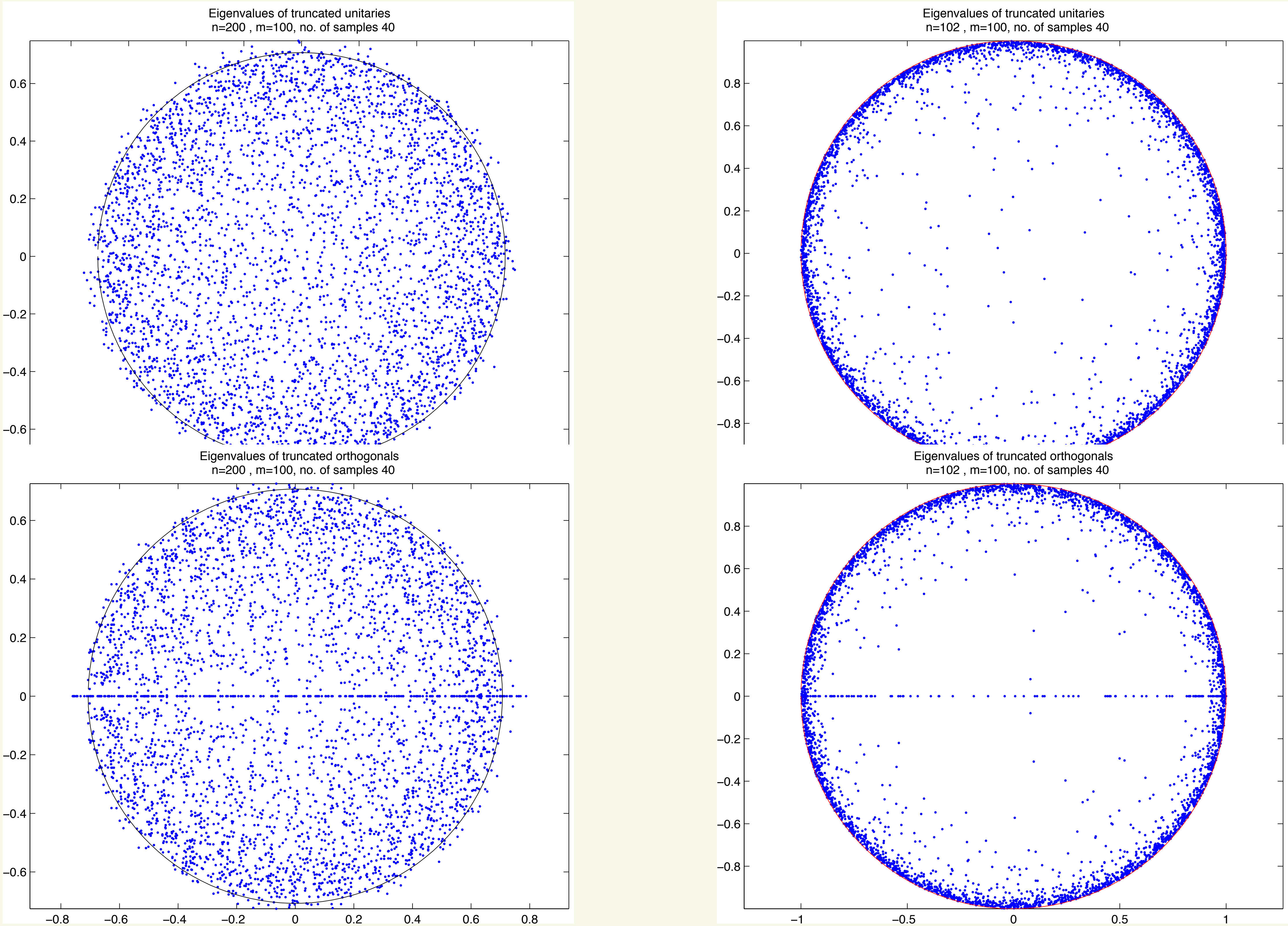
Two other interesting regimes beyond Gaussian approximation:

(i) $n \to \infty$, $l := n - m = const.$ **(weak non-orthogonality)**

(ii) $n \to \infty$, $\frac{m}{n} = const.$ **(strong non-orthogonality)**

Similar to weak/strong non-Hermiticity $X + ivY$ [Fyodorov, Kh. & Sommers 1997], and weak/strong non-unitarity [Życzkowski & Sommers 2000, Fyodorov &Sommers, 2003]

Eigenvalue scatter plots



Truncated Haar orthogonals: joint distribution of EVs

Consider truncations of size 2 – remove all but the top left 2x2 block. $d\mu(T) = Const.\det(I - T^\dagger T)^{(l-3)/2} dT$. The induced EV jpdf follows

$$d\mu(z_1, z_2) = (z_1 - z_2) f(z_1) f(z_2) dz_1 \wedge dz_2$$

with $$f^2(z) = \frac{l(l-1)}{2\pi} |1 - z^2|^{l-2} \int_{\frac{2|\operatorname{Im} z|}{|1-z^2|}}^{1} (1 - t^2)^{\frac{l-3}{2}} \, \mathrm{d}t.$$

Two real EVs ($z_1 = x_1, z_2 = x_2$): $(z_1 - z_2) dz_1 \wedge dz_2 = (x_1 - x_2) dx_1 dx_2$

Complex conj EVs($z_1 = z_2^* = z$): $(z - z^*) dz \wedge dz^* = 2iy(-2i) dx dy$

Thm 1 *For truncations of size m with l rows/columns removed:*

$$d\mu(z_1, \ldots, z_m) \propto \prod_{1 \le j < k \le m} (z_j - z_k) \prod_{j=1}^{m} f(z_j) \bigwedge_{j=1}^{m} dz_j, \quad |z_j| < 1.$$

Caveats: m even, sectors, ordering, $f^2(z) = (2\pi|1 - z^2|)^{-1}$ for $l = 1$.

Similarity to the Ginibre ens.

Incomplete Beta function

EV jpdf (Thm 1) and techniques developed for the real Ginibre ensemble yield the EV densities (and corr fncs) of truncations, for finite matrix sizes, in closed form in terms of the **incomplete Beta function**

$$I_x(a,b) = \frac{1}{B(a,b)} \int_0^x t^{a-1}(1-t)^{b-1}\,\mathrm{d}t = 1 - I_{1-x}(b,a)$$

For positive integer a, b, $I_x(a,b)$ is a truncated binomial series:

$$I_x(a,b) = 1 - (1-x)^b \sum_{j=0}^{a-1} \binom{b+j-1}{m} x^j$$

One essential ingredient of is determining the kernel/skew-orthogonal polynomials – a difficult task. The kernel can be expressed through averages of characteristic polynomials $\langle \det(z_1 - T)(z_2 - T) \rangle_T$ of matrices of smaller dimensions. The problem then can be reduced a Selberg type integral.

EV densities of truncated Haar orthogonals

As before, truncate $U \in O(n)$ to size m, so that l is the no. of columns (rows) removed, $l = n - m$.

Thm 2 *Assume m is even (technical). Then ($|x| \leq 1$)*

$$\rho_m^{(R)}(x) = \frac{I_{1-x^2}(l, m-1)}{B(\frac{l}{2}, \frac{1}{2})(1-x^2)} + \frac{(1-x^2)^{\frac{l}{2}-1}|x|^{m-1} I_{x^2}(\frac{m-1}{2}, \frac{l}{2})}{B(\frac{m}{2}, \frac{l}{2})}$$

and ($|z| \leq 1$)

$$\rho_m^{(C)}(z = x + \mathrm{i}y) = 4|y| \; f^2(z) \; \frac{I_{1-|z|^2}(l+1, m-1)}{(1-|z|^2)^{l+1}}$$

with $f^2(z) = \dfrac{l(l-1)}{2\pi} \; |1 - z^2|^{l-2} \displaystyle\int_{\frac{2|y|}{|1-z^2|}}^{1} (1-t^2)^{\frac{l-3}{2}} \, \mathrm{d}t$

These expressions become rather simple for m large!

Truncated Haar orthogonals: average no. of real EVs

$N_m^{(R)} = \int_{-1}^{1} \rho_m^{(R)}(x)dx$ = av. no. of real EVs of truncations of size m .

In the limit of **strong non-orthogonality,** $m, l \to \infty$**,** $\frac{l}{m} \to \alpha > 0$:

$$N_m^{(R)} \simeq \sqrt{\frac{l}{2\pi}} \ln \frac{\sqrt{m+l}+\sqrt{m}}{\sqrt{m+l}-\sqrt{m}} \propto \sqrt{m}$$

In the limit of **weak non-orthogonality,** $m \to \infty$**,** l **is finite**:

$$N_m^{(R)} \simeq \frac{\log m}{B\left(\frac{l}{2}, \frac{1}{2}\right)}$$

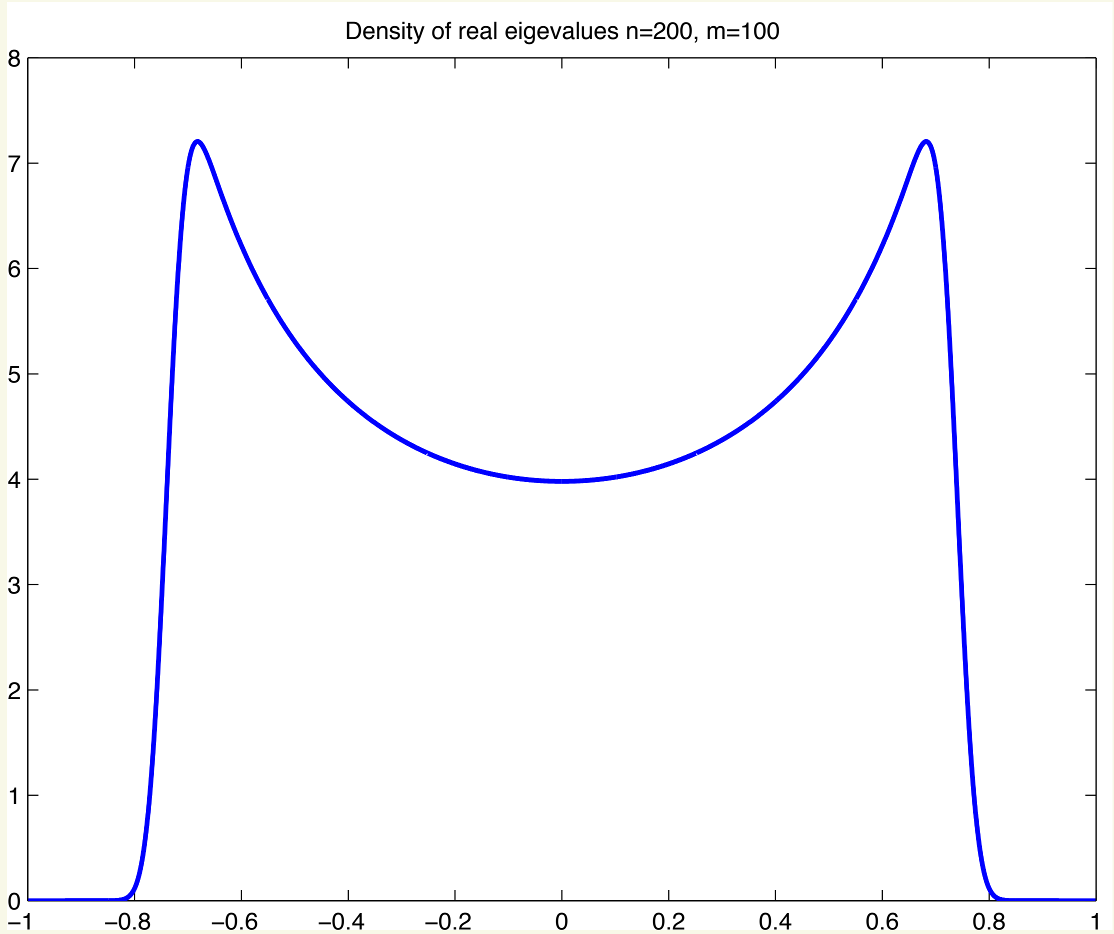
Cf.: $N_m^{(R)} \propto \sqrt{m}$ in the real Ginibre (Edelman,Kostlan & Shub, 1994);

$N_m^{(R)} \propto \log m$ for random real polynomials (Kac, 1948).

Strong non-orthogonality - density of real EVs,

Consider $m, l \to \infty$, $\frac{l}{m} \to \alpha > 0$. In this limit the distribution of the real EVs of **truncated Haar orthogonals** is described by the **'Artanh Law'**:

$$\rho_m^{(R)}(x) \simeq \sqrt{\frac{l}{2\pi}} \frac{1}{1-x^2} \Theta\left(\frac{1}{1+\alpha} - x^2\right).$$

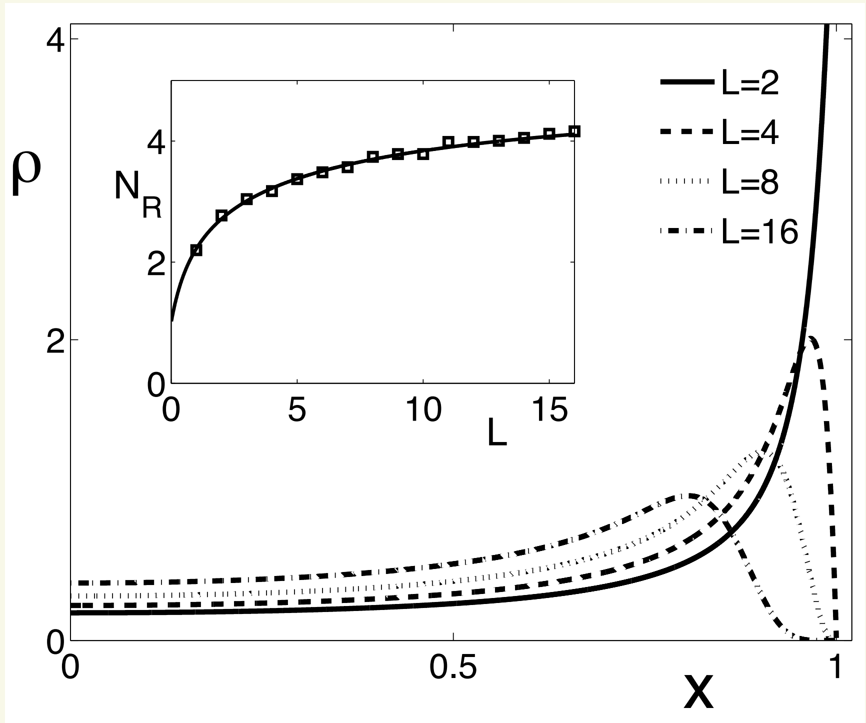


Weak non-orthogonality - density of real EVs

Consider $m \to \infty$, l is finite. In this limit

$$\rho_m^{(R)}(x) \simeq \sqrt{\frac{l}{2\pi}}\frac{1}{1-x^2}, \quad x \in (-1+\varepsilon, 1-\varepsilon).$$

Have two accumulation points ± 1. The expected no. of real EVs away from these two points is finite.



$$\rho := \rho_{32}^{(R)}(x)/N_{32}^{(R)}$$

Weak non-orthogonality - density of real EVs near the acc. pnts

Profile of EV distribution near the accumulation point $x = 1$:

On rescaling $x = 1 - \frac{u}{m}$, we have $\lim_{m\to\infty} \frac{1}{m}\rho_m^{(R)}\left(1 - \frac{u}{m}\right) = p(u)$ where

$$p(u) = \frac{u^{\frac{l}{2}-1}e^{-u}}{2\Gamma(\frac{l}{2})} \frac{\int_u^\infty t^{\frac{l}{2}-1}e^{-t}\,dt}{\Gamma(\frac{l}{2})} + \frac{1}{2u}\frac{1}{B(\frac{l}{2},\frac{1}{2})}\frac{\int_0^{2u} t^{l-1}e^{-t}\,dt}{\Gamma(l)}.$$

The 1st term determines behaviour **for small u; have** $p(u) \simeq \frac{u^{\frac{l-2}{2}}}{2\Gamma(\frac{l}{2})}$.

The 2nd term determines behaviour **for large u; have** $p(u) \simeq \frac{1}{2uB(l/2,1/2))}$, **heavy tail** leading to the $\log m$ growth of the number of real eigenvalues.

Compare with Kac polynomials: same density (and corr fncs) away from the accumulation points [Forrester 2010]. Different profile near ± 1 [Aldous-Fyodorov 2004].

Strong non-orthogonality - density of complex EVs

Consider $m, l \to \infty$, $\frac{l}{m} = \alpha > 0$. Away from the real line :

$$\rho_m^{(C)}(z) \simeq \frac{l}{\pi} \frac{1}{(1-|z|^2)^2} \Theta\left(\frac{1}{1+\alpha} - |z|^2\right), \quad y \neq 0.$$

Same limiting form as for **truncated unitaries** (Życzkowski & Sommers 2000). However, finite size corrections near the real line differ due to $\rho_m^{(C)}(x+i0) = 0$.

Close to the real line ($y \propto \frac{1}{\sqrt{m}}$) the density of complex EVs of **truncated orthogonals** is described by the **scaling law**

$$\rho_m^{(C)}(z) \simeq \rho_m^{(R)}(x)^2 h(y\rho_m^{(R)}(x)),\ h(y) = 4\pi|y|\mathrm{e}^{4\pi y^2}\,\mathrm{erfc}(\sqrt{4\pi}|y|)$$

where $\rho_m^{(R)}(x) = \sqrt{\frac{l}{2\pi}\frac{1}{1-x^2}}$ is the density of real EVs (note factorisation in curvilinear coordinates) . Same form as for the real Ginibre except $\rho_m^{(R)}$ is not constant now. **Universality?**

Conclusions

A non-Gaussian emsemble of real asymmetric matrices – truncations of random orthogonal matrices – is solved exactly:

- jpdf of eigenvalues obtained;
- EV densities and corr fncs obtained in closed form for finite matrix dimensions, and asymptotically in various regimes;
- real Ginibre correlations recovered in the regime of strong non-orthogonality;
- scaling laws obtained for expected number of real eigenvalues and corresponding densities.

THANK YOU!