The real Ginibre evolution

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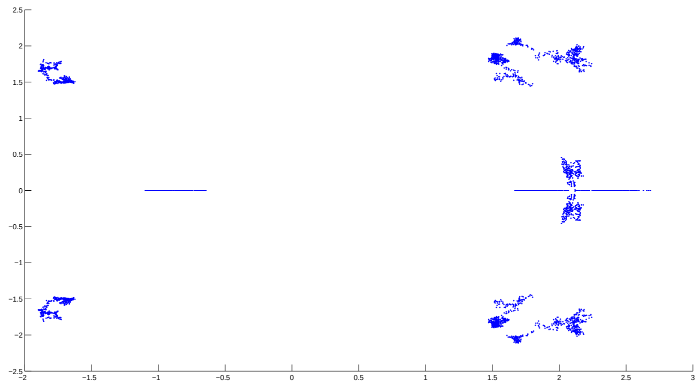
Plan

- Review of the model
- Counting real eigenvalues: spin variables
- Spin correlation functions for the real Ginibre evolution
- Open problems
- References

Ginibre evolutions



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- $[M_t]_{ij} = B_{ij}(t)$, independent BM's.
- The law of M_1 is called the real Ginibre random matrix ensemble

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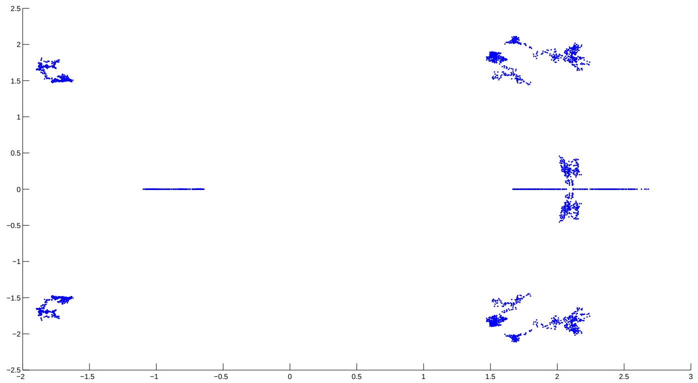

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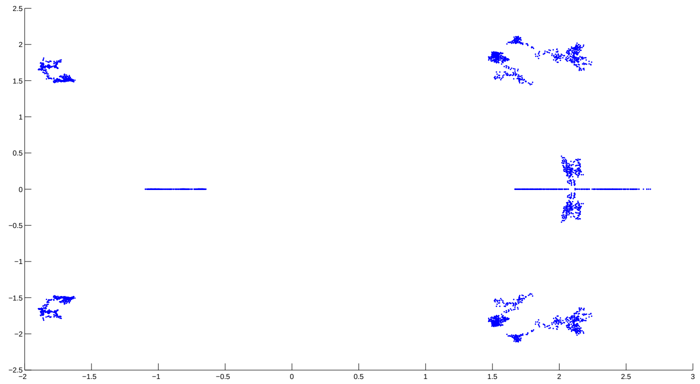

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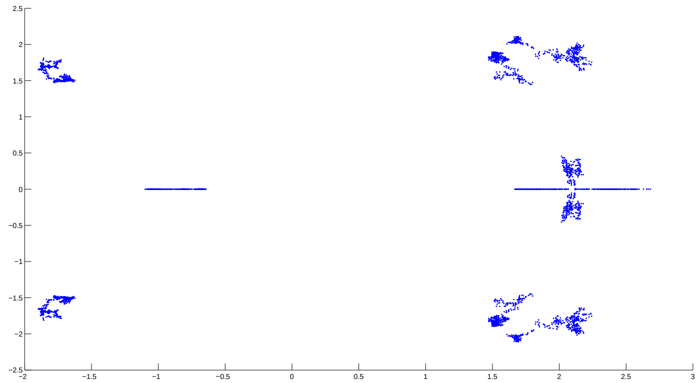

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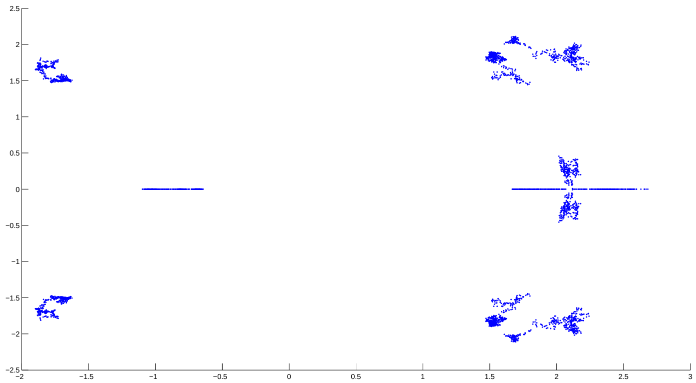

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- Is there an interpretation of the evolution of real eigenvalues in terms of an annihilation process?

ABM's and random matrices

Theorem 1 *(Borodin-Sinclair, 2009) For any $N = 1, 2, \ldots$, the law of real eigenvalues for the real Ginibre(N) random matrix ensemble is a Pfaffian point process.*

Corollary 2

$$K_t^{ABM}(x,y) = \frac{1}{\sqrt{2t}} K_{rr}^{Ginibre}\left(\frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}}\right),$$

where $K_{rr}^{Ginibre}$ is the $N \to \infty$ limit of the kernel of the Pfaffian point process characterising the law of real eigenvalues in the real Ginibre(N) ensemble.

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- $S_t(x) = sign\left(\det\left(M_t - x\right)\right)$

Tools.

1. **Berezin integral:** Let $A \in R^{N \times N}$

$$\det(A) = \int_{\mathbf{R}^{0|2N}} d\alpha d\beta e^{\alpha^T A \beta}$$

2. **Integral representation for spin variables**

$$sign\left(\det(A)\right) = \lim_{\epsilon \downarrow 0} \int_{\mathbf{R}^{N|2N}} \frac{dx d\alpha d\beta}{\pi^{N/2}} e^{-x^T (A^T A + \epsilon^2 I) x + \alpha^T A \beta}$$

3. **Fyodorov formula:**

$$\int_{\mathbf{R}^{Nm}} \prod_{k=1}^{m} dV_k F\left(\{V_i \cdot V_j\}_{1 \le i,j \le m}\right) = \int_{Sym_m^{(+)}} dQ det(Q)^{\frac{N-m-1}{2}} F(Q)$$

4. **Laplace method for $N \to \infty$ limit**

Matrix integral representation for the product expectation of spins

Theorem 3

$$\lim_{N\to\infty} \tilde{\rho}_1(x_1, x_2, \ldots, x_K) = C_K \, |\Delta(\mathbf{x})| \int_{U(K)} \mu_H(dU) e^{-\frac{1}{2} Tr \left(H - H^R\right)^2},$$

where C_K is a positive constant, $H = UXU^\dagger$ is a Hermitian matrix with eigenvalues $x_1, x_2, \ldots, x_K$, μ_H is Haar measure on the unitary group $U(K)$, $H^R = JH^T J$ is a symplectic involution of matrix H, J is a canonical symplectic matrix, $\Delta(\mathbf{x}) = \prod_{i>j}(x_i - x_j)$ is the Van-der-Monde determinant.

To restore Borodin-Sinclair result: use HCIZ theorem, then de Brujn formula.

The two-time correlation functions of spins for Ginibre evolutions.

Theorem 4

$$\lim_{N\to\infty} \mathbb{E}_N \left(S_{t+\tau}(x) S_t(0)\right) = 0$$

$$\lim_{N\to\infty} \mathbb{E}_N \left(S_{t+\frac{T}{N}}(x) S_t(0)\right) = erfc\left(\sqrt{\frac{x^2}{t} + \frac{T}{2t}}\right)$$

Observation. ABM's $\neq$ Ginibre(∞) evolutions as processes. An interesting instance of a solvable long-range system?

Open problems

- Multi-time correlation functions for Ginibre evolutions
- Dynamic characterization of the real Ginibre evolutions in terms of SDE's and/or an interacting particle system (similar to Dyson BM's)

References

1. *Multi-Scaling of the n-Point Density Function for Coalescing Brownian Motions* , CMP Vol. **268**, No. **3**, December 2006;

2. *Pfaffian formulae for one dimensional coalescing and annihilating systems*, arXiv Math.PR: 1009.4565; EJP, vol. **16**, Article 76 (2011)

3. *One dimensional annihilating and coalescing particle systems as extended Pfaffian point processes*, ECP, vol. **17**, Article 40 (2012)

4. *The Ginibre evolution in the large-N limit*, arXiv:1212.6949 $[$math.PR$]$ (2012), rejected by EJP