Annihilating and coalescing particle systems as extended Pfaffian point processes

Roger Tribe, Kwan Yip and Oleg Zaboronski

Department of Mathematics, University of Warwick

May 6 2013

Outline

1. Spin variables
   - Definition
   - From spin variables to correlation functions
2. Interacting particle systems and Pfaffians
   - Pfaffians
   - Annihilating Brownian motions on **R**
   - Pfaffian Point Processes
   - Extended Pfaffian Point Process
   - Applications
3. Conclusions

Spin variables

$N(dx)$ simple point measure on $\mathbf{R}$

Spin variable $s(x) = (-1)^{N(0,x)}$

$x < y$ implies $s(x)s(y) = (-1)^{N(x,y)}$

Spin correlations $\mathbb{E}\left[s(x_1)\ldots s(x_{2n})\right]$

Spin variables → correlation functions

$$\frac{d}{dx}(-1)^{N(x,y)} = (-2)(-1)^{N(x,y)}N(dx)$$

$$-\frac{1}{2}\lim_{y\downarrow x}\frac{d}{dx}s(x)s(y) = N(dx)$$

$$\begin{aligned}&\rho(x_1, x_2, \ldots, x_n)\\ = \quad &\left(-\frac{1}{2}\right)^n \lim_{x_2\downarrow x_1}\ldots\lim_{x_{2n}\downarrow x_{2n-1}}\\ &\quad\frac{d}{dx_1}\frac{d}{dx_3}\ldots\frac{d}{dx_{2n-1}}E\left[s(x_1)\ldots s(x_{2n})\right]\end{aligned}$$

Pfaffians

Recall, for anti-symmetric real $2n \times 2n$ matrix A,
$\det(A) = (\mathrm{Pf}(A))^2$.

$$\mathrm{Pf}\begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix} = af - be + cd.$$

Annihilating Brownian motions on $\mathbf{R}$

$$N_t(A) = \text{Number of particles in } A \text{ at time } t$$

$$\mathbb{E}[s(x_1)\dots s(x_{2n})] = \mathrm{Pf}\,(\mathbb{E}[s(x_i)s(x_j)] : i < j)$$

Proof: Both sides solve

$$\partial_t u^{(2n)} = \frac{1}{2}\Delta u^{(2n)} \quad \text{on } V_{2n} = \{x_1 < x_2 < \dots x_{2n}\}$$

with boundary conditions

$$u^{(2n)}|_{x_i = x_{i+1}} = u^{(2n-2)}(t, x_1, \dots, x_{i-1}, x_{i+2}, \dots, x_{2n})$$

Pfaffian Point Processes

.

Corollary: N_t is a Pfaffian Point process.

.

That is, $\rho_t(x_1, x_2, \ldots, x_n) = \mathrm{Pf}(K(x_i, x_j) : i < j)$

Special case: maximal entrance law

$$K_t(z) = \begin{pmatrix} K_t^{11}(z) & K_t^{12}(z) \\ K_t^{21}(z) & K_t^{22}(z) \end{pmatrix} = \begin{pmatrix} -\frac{F''(zt^{-1/2})}{t^{-1}} & -\frac{F'(zt^{-1/2})}{t^{-1/2}} \\ \frac{F'(zt^{-1/2})}{t^{-1/2}} & \mathrm{sgn}(z)F(|z|t^{-1/2}) \end{pmatrix}$$

and F is the Gaussian error function given by

$$F(z) = \frac{1}{2\pi^{1/2}} \int_z^\infty e^{-x^2/4} dx$$

Extended Pfaffian Point process

Let

$$\begin{aligned} & \rho((t_1,x_1),\ldots,(t_n,x_n))dx_1\ldots dx_n \\ = \quad & P(\text{particles at times } t_i \text{ at positions } dx_i) \end{aligned}$$

Then

$$\rho((t_1,x_1),\ldots,(t_n,x_n)) = \mathrm{Pf}(K((t_i,x_i),(t_j,x_j) : i<j))$$

Under maximal entrance law: for $t > s$ and $i,j \in \{1,2\}$

$$K^{ij}((t,x);(s,y)) = G_{t-s}K^{ij}_s(y-x) - 2I_{\{i=1,j=2\}}g_{t-s}(y-x);$$

Proof: double induction over space and time points.

Applications

- **Coalescing case**: The same structure as ABM's. The kernel is rescaled by 2.
  Proof. Use empty interval formula

$$I\left(N_t([y_1,y_2]) = N_t([y_3,y_4]) = \ldots = N_t([y_{2m-1},y_{2m}]) = 0\right)$$

  in place of product spin formula.
- **Negative dependence**:

$$\rho_t(x_1,x_2,\ldots,x_n) = \frac{A_n}{t^{n/2}}\left|\Delta\left(\frac{\mathbf{x}}{\sqrt{t}}\right)\right|\left(1+O(t^{-1/2})\right)$$

Conclusions

- Coalescing (annihilating) Brownian motions on **R** can be characterized as an extended Pfaffian point process
- The one dimensional law of C(A)BM's at $t = 1$ coincides with the law of real eigenvalues for the real Ginibre ensemble in the limit $N \to \infty$ (Borodin, Sinclair; Forrester, Nagao)
- The Pfaffian structure allows for a detailed study of the structure of correlations in C(A)BM's including negative dependencies between particles
- **Further research**: asymptotics of multi-time correlations, the distribution of inter-particle spacings (Janossi densities), models with immigration
- **Reference**: Roger Tribe, Oleg Zaboronski, K. Yip, *One dimensional annihilating and coalescing particle systems as extended Pfaffian point processes*, ECP vol.**17** (2012)