Asymptotic statistics of cycles in Surrogate-Spatial Random Permutations

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15 November 2012

Outline

Spatial random permutations are motivated by a model from quantum mechanics and have recently studied by V. Betz and D. Ueltschi.

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We propose a natural approximation of spatial random permutations on the symmetric group S_n.

This approximation suggested here has a simpler structure and is thus more tractable.

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Furthermore, we obtain a few new results about the distribution of the cycle lengths.

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A cycle $(s_0\ s_1 \ldots s_{k-1})$ is a permutation which maps

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Two cycles $(s_0 \dots s_{k-1})$ and $(t_0 \dots t_{m-1})$ are called disjoint if the sets $\{s_0, \dots, s_{k-1}\}$ and $\{t_0, \dots, t_{m-1}\}$ are disjoint.

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Theorem (Shepp,Loyd (1966) $\vartheta = 1$, Watterson (1974) general ϑ)

$$(C_1, \dots, C_b) \xrightarrow{d} (Y_1, \dots, Y_b)$$

with Y_k independent Poisson distributed with $\mathbb{E}[Y_k] = \frac{\vartheta}{k}$

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Theorem (Goncharov (1942) $\vartheta = 1$, Watterson (1974) general ϑ)

$$\frac{T_n - \vartheta \log(n)}{\sqrt{\vartheta \log(n)}} \xrightarrow{d} N(0,1)$$

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Theorem (Vershik and Shmidt (1977) resp. Kingman (1977))

$$\left(\frac{\lambda_1}{n}, \frac{\lambda_2}{n}, \ldots\right) \xrightarrow{d} \mathcal{PD}(\vartheta), \qquad (n \to \infty)$$

with $\mathcal{PD}(\vartheta)$ the Poisson–Dirichlet distribution with parameter ϑ.

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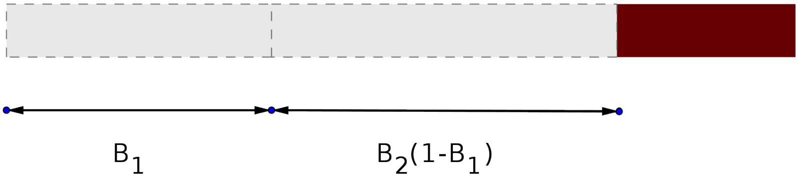

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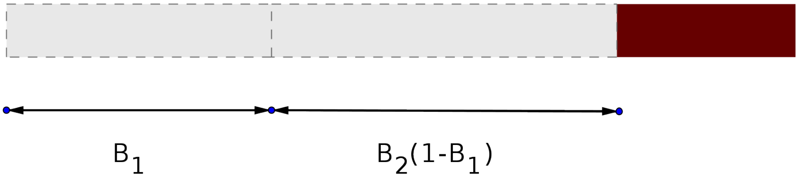

Ordering the sticks obtained by this process by size then has a Poisson–Dirichlet distribution.

Outline

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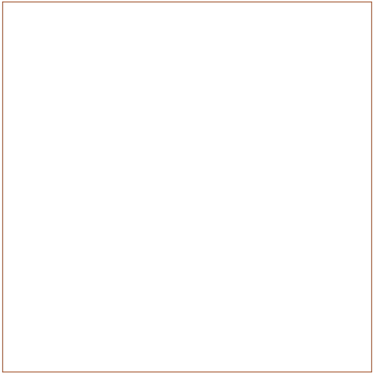
Let us first describe the idea of the model.



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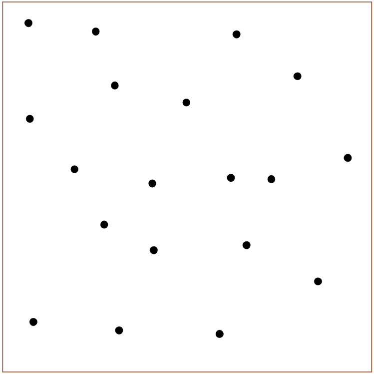

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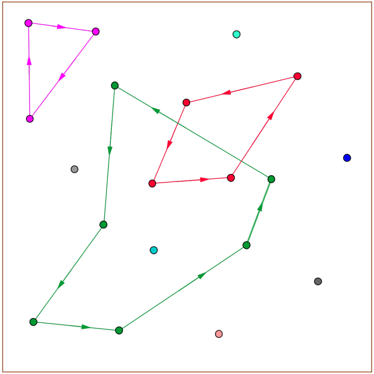

- A cube $\Lambda \subset \mathbb{R}^d$ with side length $L > 0$ $(d \geq 3)$
- n particles in the cube Λ
- A permutation σ of particles with the same state

We now define a measure on $\mathbb{P}[.]$ on $S_n \times \Lambda^n$ with

$$\mathbb{P}[\sigma, dx] = \frac{1}{Y_n n!} e^{-H(\sigma,x)} dx$$

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$$H(\sigma, x) = \sum_{k=1}^{n} \alpha_k C_k + \sum_{j=1}^{n} \|x_j - x_{\sigma(j)}\|$$

where $\alpha_k \in \mathbb{R}$, $x = (x_1, \ldots, x_n) \in \Lambda^n \subset \mathbb{R}^{nd}$ are the coordinates of the particles.

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In this setting large sets of particles of the same state correspond to long cycles in the cycle decomposition of $\sigma \in S_n$.

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This has the interpretation as the fraction of particles in cycles of length at least p.

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For the Ewens measure, we have $\nu = 1$.

It was shown by Betz and Ueltschi for
'$\alpha_k \to \alpha$' and $\alpha_k = \gamma \log(k)$ for $\gamma > 0$

$$\nu = \max\left\{0, 1 - \frac{\rho_c}{\rho}\right\}$$

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with

$$\rho_c := \sum_{k=1}^{\infty} e^{-\alpha_k} \int_{\mathbb{R}^d} e^{-k\|x\|}\, dx.$$

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In the case '$\alpha_k \to \alpha$' we have

$$\left(\frac{\lambda_1}{\nu n}, \frac{\lambda_2}{\nu n}, \dots \right) \xrightarrow{d} \mathcal{PD}(e^{-\alpha})$$

and in the case $\alpha_k = \gamma \log(k)$

$$\frac{\lambda_1}{\nu n} \xrightarrow{d} 1$$

Outline

Remember we had

$$\mathbb{P}[\sigma, dx] = \frac{1}{Y_n n!} e^{-H(\sigma,x)} dx$$

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"Periodizing" the boundary conditions gives and integrating out the x gives

$$\mathbb{P}\left[\sigma\right] := \frac{1}{Y_n n!} \prod_{k=1}^{n} \left(e^{-\alpha_k} \sum_{m \in \mathbb{Z}^d} e^{-k\|m/L\|} \right)^{C_k},$$

We now have for fixed k and using $\rho = \frac{n}{L^d}$

$$\frac{1}{L^d} \sum_{m \in \mathbb{Z}^d} e^{-k\|m/L\|} \approx \int_{\mathbb{R}^d} e^{-k\|x\|}\, dx$$

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and thus

$$\begin{aligned}
\mathbb{P}\left[\sigma\right] &= \frac{1}{Y_n n!} \prod_{k=1}^{n} \left(e^{-\alpha_k} \sum_{m \in \mathbb{Z}^d} e^{-k\|m/L\|} \right)^{C_k} \\
&\approx \frac{1}{Y_n n!} \prod_{k=1}^{n} \left(n \cdot \frac{e^{-\alpha_k}}{\rho} \int_{\mathbb{R}^d} e^{-k\|x\|}\, dx \right)^{C_k}.
\end{aligned}$$

We make the following Ansatz

Definition

Let $\Theta = (\theta_k)_{k\geq 1}$ and $\Upsilon = (\tau_k)_{k\geq 1}$ be given, with $\theta_k, \tau_k \geq 0$. We then define the *surrogate spatial* probability measure on permutations as

$$\mathbb{P}_n^{(sur)}[\sigma] := \frac{1}{H_n n!} \prod_{k=1}^{n} (n \cdot \tau_k + \theta_k)^{C_k},$$

with H_n some constant.

Are the error-terms θ_k important?

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If one computes the τ_k arising from spatial permutations, one gets

'$\alpha_k \to \alpha$'	$\alpha_k = \gamma \log(k)$
$\tau_k = \frac{1}{\rho} e^{-\alpha} k^{-d/2}$	$\tau_k = \frac{1}{\rho} k^{-d/2-\gamma}$

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$\tau_k = \frac{1}{\rho} e^{-\alpha} k^{-d/2}$	$\tau_k = \frac{1}{\rho} k^{-d/2-\gamma}$
$\left(\frac{\lambda_1}{\nu n}, \frac{\lambda_2}{\nu n}, \dots \right) \xrightarrow{d} \mathcal{PD}(e^{-\alpha})$	$\frac{\lambda_1}{\nu n} \xrightarrow{d} 1$

Strategy to analyse large$-n$ asymptotics of surrogate-spatial permutations:

- Use generating functions to give ‘nice’ expressions
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- Use generating functions to give ‘nice’ expressions
- Apply Cauchy’s integral formula

We illustrate this with the normalisation constant H_n in the surrogate-spatial measure.

We work here only with class functions on S_n and it is well known that the conjugacy class of S_n can be parametrised with partitions.

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Lemma

For any class function $u : S_n \to \mathbb{C}$, there is the identity

$$\frac{1}{n!}\sum_{\sigma\in S_n} u(\sigma) = \sum_{\lambda\in\mathcal{P}_n} \frac{1}{z_\lambda} u(\mathcal{C}_\lambda)$$

where $\mathcal{C}_\lambda$ is the conjugacy class corresponding to partition λ and $z_\lambda := \prod_{k=1}^{n} k^{C_k} C_k!$.

The main tool in this talk to write down generating functions

Lemma (Polya)

Let $(a_m)_{m\in\mathbb{N}}$ be a sequence of complex numbers. Then

$$\sum_{\lambda\in\mathcal{P}} \frac{1}{z_\lambda} \left(\prod_{m=1}^{\ell(\lambda)} a_{\lambda_m} \right) t^{|\lambda|} = \exp\left(\sum_{m=1}^{\infty} \frac{1}{m} a_m t^m \right)$$

with z_λ as above.
If one of the sums is absolutely convergent then so is the other one.

It follows form the definition of $\mathbb{P}_n^{(sur)}[\cdot]$ that

$$H_n = \sum_{\lambda \in \mathcal{P}_n} \frac{1}{z_\lambda} \prod_{k=1}^{\infty} (n \cdot \tau_k + \theta_k)^{C_k}$$

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Unfortunately we can not directly compute

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We thus introduce for $v \in \mathbb{N}$

$$h_n(v) = \sum_{\lambda \in \mathcal{P}_n} \frac{1}{z_\lambda} \prod_{k=1}^{\infty} (v \cdot \tau_k + \theta_k)^{C_k}$$

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with

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$$H_n = [t^n]\left[\exp\left(g_\Theta(t) + n \cdot p_\Upsilon(t)\right)\right]$$

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$$\underbrace{a(R) := Rp'_\Upsilon(R) > 1}_{\text{sub-critical}} \quad \text{and} \quad \underbrace{a(R) < 1}_{\text{super-critical}} \quad \text{and} \quad \underbrace{a(R) = 1}_{\text{critical}}$$

Sub-critical case

If we apply Cauchy's integral formula, we get

$$H_n = \frac{1}{2\pi i} \int_{\gamma} \frac{\exp\left(g_\Theta(z) + n \cdot p_\Upsilon(z)\right)}{z^{n+1}} dz$$

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We choose $\gamma(\varphi) = re^{i\varphi}$ with $\varphi \in [-\pi, \pi]$.

Sub-critical case

If we apply Cauchy's integral formula, we get

$$H_n = \frac{1}{2\pi i} \int_\gamma \frac{\exp\left(g_\Theta(z) + n \cdot p_\Upsilon(z)\right)}{z^{n+1}} dz \approx \int \Big(f(x)\Big)^n dx$$

We choose $\gamma(\varphi) = re^{i\varphi}$ with $\varphi \in [-\pi, \pi]$. It's clear that $\mathrm{Re}(p_\Upsilon(re^{i\varphi}))$ has a maximum for $\varphi = 0$.

We expand

$$p_\Upsilon(re^{i\varphi}) = p_\Upsilon(r) + i\varphi a(r) - \frac{\varphi^2}{2} b(r) + O(\varphi^3),$$

with $a(r) = rp'_\Upsilon(r)$ and $b(r) = rp'_\Upsilon(r) + r^2 p''_\Upsilon(r)$.

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$$\frac{\exp\left(n \cdot p_\Upsilon(r)\right)}{2\pi r^n \sqrt{n}} \int_{-\kappa_n\sqrt{n}}^{\kappa_n\sqrt{n}} e^{i\sqrt{n}\varphi(a(r)-1)} e^{-b(r)\frac{x^2}{2}} (1 + o(1)) dx$$

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We now choose r to be the solution of $a(r) = 1$ (if it's possible).

Theorem

Assume that $g_\Theta(t)$ and $p_\Upsilon(t)$ have radius of convergence $R > 0$. Suppose that

$$a(R) > 1 \text{ with } a(R) := \sup_{0<r<R} a(r) \leq \infty$$

and let r_1 be the unique solution of $a(r_1) = 1$. Suppose further that $p_\Upsilon(t) \neq f(t^k)$ with $k > 1$ and f holomorphic.
We then have as $n \to \infty$

$$H_n \sim \frac{\exp\big(g_\Theta(r_1) + np_\Upsilon(r_1)\big)}{r_1^{n-d}\sqrt{2\pi nb(r_1)}}$$

Super-critical case

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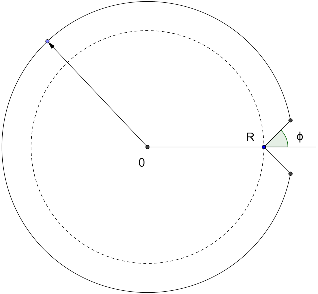
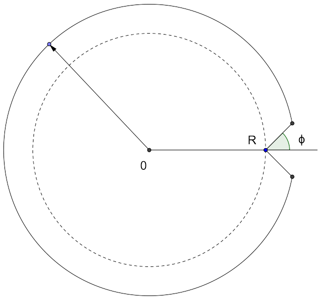
If $g_\Theta(t)$ is diverging at R, we can hope that H_n is determined by the behaviour of p_Υ and g_Θ near R.



$$g_\Theta(t) = -\vartheta \log(1 - t/R) + O(1 - t/R),$$
$$p_\Upsilon(t) = p_\Upsilon(R) + a(R)(t/R - 1) + O((1 - t/R)^2).$$



$$g_\Theta(t) = -\vartheta \log(1 - t/R) + O(1 - t/R),$$
$$p_\Upsilon(t) = p_\Upsilon(R) + a(R)(t/R - 1) + O((1 - t/R)^2).$$

Theorem

For $a(R) < 1$ and under the above assumptions we have

$$H_n \sim \frac{n^{\vartheta-1} \exp(n p_\Upsilon(R))}{R^n} \frac{(1 - a(R))^{\vartheta-1}}{\Gamma(\vartheta)}$$

Let us define the quantity r_* as

$$r_* := \begin{cases} r_1, & a(R) \geq 1, \\ R, & a(R) \leq 1, \end{cases} \qquad (1)$$

where r_1 is the (unique) solution of the equation

$$a(r_1) = 1$$

Cycle counts

Theorem

We have

$$\mathbb{E}_n^{(sur)}\left[(C_k)^m\right] \sim \left(n \cdot \frac{\tau_k r_*^k}{k}\right)^m$$

and $\frac{C_k}{n}$ *converges in law to the constant* $\frac{\tau_k r_*^k}{k}$.

Using the previous result, we obtain for each $p \in \mathbb{N}$

$$\nu_p = \liminf_{n\to\infty}\left(1 - \frac{1}{n}\mathbb{E}\left[\sum_{k=1}^{p} kC_k\right]\right) = 1 - \sum_{k=1}^{p} \tau_k r_*^k$$

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$$\nu = \lim_{p\to\infty} \left(1 - \sum_{k=1}^{p} \tau_k r_1^m\right) = 1 - a(r_*)$$
$$= \begin{cases} 0 & \text{if } a(R) \geq 1 \\ 1 - a(R) & \text{if } a(R) < 1 \end{cases}$$

Total number of cycles

Theorem

(a) *Let* $a(R) > 1$. *Then,*

$$\frac{T_n - np_\Upsilon(r_1)}{\sqrt{n\left[p_\Upsilon(r_1) - 1/b(r_1)\right]}} \xrightarrow{d} \mathcal{N}(0,1), \qquad n \to \infty, \tag{2}$$

(b) *If* $a(R) < 1$ *then,*

$$\frac{T_n - np_\Upsilon(R)}{\sqrt{np_\Upsilon(R)}} \xrightarrow{d} \mathcal{N}(0,1), \qquad n \to \infty. \tag{3}$$

First comparison of models

If we take $\tau_k = \frac{k^{-\alpha}}{\rho}$ with $\alpha > 1$, we get the polylogarithm

$$p_\Upsilon(t) = \frac{\mathrm{Li}_{\alpha+1}(t)}{\rho} = \sum_{k=1}^{\infty} \frac{t^k}{\rho k^{\alpha+1}} \text{ and } a(t) = \frac{\mathrm{Li}_\alpha(t)}{\rho}.$$

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We have in this case $R = 1$

$$a(1) = \frac{\mathrm{Li}_\alpha(1)}{\rho} < \infty \quad \text{and} \quad \nu = \max\left\{0, 1 - \frac{\mathrm{Li}_\alpha(1)}{\rho}\right\}$$

The study of the long cycles requires further assumptions on the derivative.

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$$\left(\frac{\partial}{\partial t}\right)^d p_\Upsilon(t) = \left(\left(\frac{\partial}{\partial t}\right)^d Li_\alpha(t)\right)(1 + o(1))$$

$$\left(\frac{\partial}{\partial t}\right)^d g_\Theta(t) = \left(\left(\frac{\partial}{\partial t}\right)^d \vartheta \log\left(\frac{1}{1-t}\right)\right)(1 + o(1))$$

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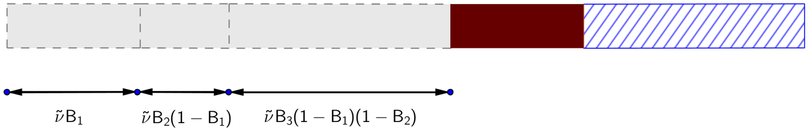

Theorem

Suppose that $\theta_k \to \vartheta > 0$ *and* $\nu > 0$. *Then*

$$\left(\frac{\lambda_1}{\nu n}, \frac{\lambda_2}{\nu n}, \dots\right) \xrightarrow{d} \mathcal{PD}(\vartheta)$$

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Theorem

Suppose that $\theta_k \equiv 0$ *and* $\nu > 0$. *Then*

$$\frac{\lambda_1}{\nu n} \xrightarrow{d} 1 \tag{4}$$

Thank you