Limit Theorems for a Periodically or Randomly Driven Semilinear Equation

Nikolaos Zygouras

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
Department of Mathematics
New York University
September 2004

S.R.S. Varadhan

Dedicated to my mother

Acknowledgements

I would like to express my gratitude to my advisor, professor S.R.S.Varadhan, for without him this work wouldn't have been possible. I would like to thank him for his interest in this work and his continuous support; he was always open to discussions and offered his invaluable ideas and encouragement.

I would also like to thank professors P. Deift, H.P. McKean, C. Newman, J. Shatah, E. Vanden-Eijnden, H.T. Yau and L.S. Young for their advice, help and the numerous things they taught me through courses and discussions.

I am also happy to have met Rongfeng, Paolo, Juyoung, Fanfu, Toufic, Matthias, Inés, Peter, Firas, Roberto and Tasos, whose friendship has been very important to me. Especially, I would like to thank Michail Loulakis not only for his frienship, but also for his guidance during my first years in New York.

I would also like to thank my family in Greece, for providing me with a feeling of security, that I could have any kind of support if the circumstances required it.

Finally, I owe a very special thanks to HsinPing. She has been a wonderful partner to me all these years, withstanding my uneven moods and being always there when I needed it most.

Abstract

We study the semilinear parabolic equation

$$u_t = u_{xx} - u^2 + \lambda(t)\,\delta_0(x), \qquad x \in \mathbb{R},\ t \in \mathbb{R},$$

driven by a source term $\lambda(t)\delta_0(x)$ at the origin. The intensity of the source is considered to be either a periodic function, or a stationary, ergodic process. In both cases the intensity is positive and bounded away from 0 and infinity, so that the dynamics is well defined. The solution of this equation describes the equilibrium state of a system, in which energy is supplied by the source, and is diffused and dissipated by the Laplacian and the nonlinearity, respectively. Our goal is to understand how the nonlinear dynamics transfer the perturbation from the origin to infinity. In particular, we study the asymptotics of the equilibrium state $u(\cdot, x)$, as x tends to infinity, and we prove that it is asymptotic to a steady state solution of the same equation, corresponding to an averaged constant intensity λ_*. We are also interested in studying the speed of the convergence, as well as the fluctuations around the limit.

Contents

List of Appendices

Chapter 1

Introduction

We study the semilinear parabolic equation

$$u_t = u_{xx} - u^2 + \lambda(t)\,\delta_0(x), \qquad x \in \mathbb{R},\, t \in \mathbb{R}, \tag{1.1}$$

driven by a source term $\lambda(t)\delta_0(x)$ at the origin. The intensity of the source is considered to be either a periodic function, or a stationary ergodic process. In both cases the intensity is positive and bounded away from 0 and infinity, so that the dynamics is well defined.

Our attitude differs from that of standard PDE's in two respects: first, we do not specify the initial data, and second, time is allowed to take both negative and postive values. This is because we want to study the equilibrium state of the process described by equation (1.1).

The system is expected to reach the equilibrium through the following mechanisms. The source provides energy to the system, which is then diffused by the diffusion term, and dissipated by the nonlinearity $-u^2$. After sufficient mixing time the system loses its memory, i.e. the effect of the initial condition disappears. More precisely, consider the equation (1.1) in a time interval $t > \tau$, for some $\tau \in \mathbb{R}$, and denote the solution of this initial value problem by $u^{(\tau)}(t,x)$. Then, as $\tau \to -\infty$, $u^{(\tau)}(t,x)$ converges to a function $u(t,x)$, which satisfies (1.1). Furthermore, it can be proved that, for any given $\lambda(\cdot)$, this equation has a unique solution, and this will also guarantee that $u(t,x)$ is a periodic function, or a stationary process, according to the choice of $\lambda(\cdot)$.

We want to analyse the asymptotic behaviour of $u(t,x)$ as $x \to \infty$ for arbitrary fixed $t \in \mathbb{R}$. The motivation of this question comes from work on several stochastically driven nonlinear PDEs, such as the Navier-Stokes, Ginzburg-Landau, etc. (see, for example, [EH], [EMS]). These are examples of infinite dimensional dynamical systems, which, when randomly perturbed, exhibit unique ergodicity. In other words, they posess a unique invariant measure.

After establishing unique ergodicity one would like to be able to characterize the invariant measure, but unfortunatelly there are almost no rigorous results in this direction. Apart from being an interesting question on its own, characterizing the invariant measure may also provide a better understanding of the nonlinear dynamics.

Assuming periodic boundary conditions, the solution of the models mentioned above can be written in terms of Fourier series as

$$u(t,x) = \sum_n u_n(t) e^{inx}.$$

Substituting this expansion into the PDEs, we find that the coefficients $u_n(t)$ satisfy an infinite system of ODEs of the form

$$\frac{du_n}{dt} = -\nu\, n^2 u_n + F_n(u_1, u_2, \dots), \quad n \in \mathbb{N}.$$

The first term on the right hand side is the Fourier representation of the Laplacian , while the second one is the Fourier representation of the nonlinear terms.

A random perturbation in the form of a finite dimensional white noise is then added to the right side of the above equations, so that, for appropriatelly chosen cut off N and $\sigma_n > 0$, the equations take the form

$$\frac{du_n}{dt} = -\nu\, n^2 u_n + F_n(u_1, u_2, \dots) + \sigma_n 1_{|n| \le N} \frac{d\beta_n(t)}{dt},$$

The noise is chosen to be finite dimensional in order to ensure that the perturbation does not overwhelm the natural dynamics of the nonlinear PDEs. An interesting question would then be to investigate the way in which the energy is transferred from low to high modes. In particular one would like to understand the asymptotic statistical properties, as n tends to infinity, of $u_n(\cdot)$, for the system in equilibrium.

The analogy between these problems and the one we are concerned with can be seen from the following correspondences: $n \leftrightarrow x$, $-\nu\, n^2 u_n \leftrightarrow u_{xx}$, $\{F_n\} \leftrightarrow -u^2$ and $\sigma_n 1_{|n| \le N} \frac{d\beta_n(t)}{dt} \leftrightarrow \lambda(t)\delta_0(x)$. In other words, we are interested to know how a perturbation at zero is transferred to infinity by the nonlinear dynamics. Because of the diffusive character of our equation, we expect the dynamics to have some sort of averaging property; however, the nonlinearity makes it difficult to see this explicitly. Our main focus will be to recover this averaging and show that as x tends to infinity the solution $u(t,x)$ is asymptotic to a time independent solution of (1.1), corresponding to a source with a constant intensity λ_*. We will also be interested in studying the speed of this convergence, as well as the fluctuations around the limit.

Finally let us mention that, although the motivation of our work was to give an example of a perturbed nonlinear equation for which the role of the nonlinearity can be studied in detail, the example we have chosen to study is related to an interacting particle system model. More specifically, the equation

$u_t = u_{xx} - u^2$ appears to be describing the continuous limit of the density of particles that perform independent random walks on the lattice $\mathbb{Z}^1$, and are annihilated upon meeting each other (see for example [S]). Our case has the extra feature that particles are born at the origin at a rate determined by $\lambda(\cdot)$ and our goal is to study the tails of the equilibrium density.

1.1 Description of the results

It will be convenient in our analysis to interpret the source term $\lambda(\cdot)\delta_0(x)$ as a boundary condition. If we assume that the solution $u(t,x)$ is symmetric around the origin, we can see that the corresponding boundary condition is $u_x(t,0) = -\frac{1}{2}\lambda(t)$, for $t \in \mathbb{R}$. So (1.1) is equivalent to the boundary value problem

$$u_t = u_{xx} - u^2, \qquad x > 0,\ t \in \mathbb{R}, \tag{1.2}$$

$$u_x(t,0) = -\frac{1}{2}\lambda(t)\ , \qquad t \in \mathbb{R}. \tag{1.3}$$

The general assumption we will put on $\lambda(\cdot)$ is that it is bounded away from 0 and infinity. In other words, we assume that there exist positive constants λ_1, λ_2 such that

$$0 < \lambda_1 \leq \lambda(t) \leq \lambda_2 < \infty \qquad \text{for every } t \in \mathbb{R}. \tag{1.4}$$

The fact that, for every $\lambda(\cdot)$ satisfying this condition, there is a unique solution to (1.2), (1.3) is proved in Chapter 2.

Let us now assume that $\lambda(\cdot)$ is equal to a constant λ_0. Then by uniqueness, the solution to (1.2), (1.3) will be independent of time, and will satisfy the ODE

$$u_{xx} - u^2 = 0, \qquad x > 0,$$
$$u_x(0) = -\frac{1}{\lambda_0}.$$

The only physically relevant solution to this equation is given by $u(x) = \frac{6}{(x+\alpha_0)^2}$, where the constant α_0 is equal to $\left(\frac{1}{24}\lambda_0\right)^{1/3}$. This is a steady state solution.

In the time dependent case the solution cannot be written explicitly. Nevertheless, by a standard comparison principle it can be seen that it satisfies the inequality

$$\frac{6}{(x+\alpha_1)^2} \leq u(t,x) \leq \frac{6}{(x+\alpha_2)^2}, \quad x > 0, \, t \in \mathbb{R},$$

where the left hand side corresponds to the solution in the case of the lowest particle input, λ_1, and the right hand side corresponds to the case of the highest particle input, λ_2. This means that we can still write the solution in the form

$$u(t,x) = \frac{6}{(x+\alpha(t,x))^2}. \tag{1.5}$$

So now, the study of the asymptotics of $u(t,x)$, as x goes to infinity, is reduced to the study of the asymptotics of $\alpha(t,x)$.

To guess what the behaviour should be, let us first consider the linear Dirichlet problem

$$u_t = u_{xx}, \qquad x > 0, \, t \in \mathbb{R}, \tag{1.6}$$
$$u(t,0) = u_0(t), \qquad t \in \mathbb{R}. \tag{1.7}$$

The solution to this problem can be written explicitly using functional integration. In particular if W_x denotes the Wiener measure on the Brownian paths

$\beta(s)$ starting from position $x > 0$, and $\tau_0 = \inf\{s > 0 \colon \beta(s) = 0\}$ denotes the first hitting time of 0, then

$$u(t,x) = E^{W_x} u_0(t - \tau_0) = \int u_0(t-\tau)\, W_x(\tau;0)\, d\tau,$$

where $W_x(s;0)$ is the density of the hitting time. Notice that this formula represents the solution to (1.6) as an average over the boundary values. If we assume that u_0 is periodic or stationary, then it is not difficult to see that as x goes to infinity, $u(t,x)$ converges to a constant. In other words the dynamics average the boundary data in such a way that for large x the system feels an averaged, constant perturbation.

The situation is similar in the nonlinear case, with the difference that the nonlinearity makes it impossible to see the averaging explicitly. However, we still expect the mechanism to be the same, so that for large x the solution should look like $6/(x+\alpha_*)^2$, where α_* is a constant corresponding to some averaged value of $\lambda(\cdot)$.

In Chapter 2 we study the case of a periodic intensity $\lambda(\cdot)$ and we prove that for every $t \in \mathbb{R}$, $\alpha(t,x)$ converges to a constant value α_*, exponentially fast.

The method we follow is to prove a Harnack type inequality for $u(t,x)$ along the lines $\{(t,x) \colon t \in \mathbb{R}\}$. To make this more clear let us consider equation (1.2) in the domain $x > x_0$, $t \in \mathbb{R}$, for some arbitrary $x_0 > 0$. If the situation was such that $\frac{\sup_t u(t,x_0)}{\inf_t u(t,x_0)} = 1$, then this would correspond to the time independent case and so $\alpha(t,x)$ would be a constant. So one might expect that, if the situation is such that $\frac{\sup_t u(t,x_0)}{\inf_t u(t,x_0)} = 1 + O(\delta^{x_0})$ for some $0 < \delta < 1$, then $\alpha(t,x)$ would also be, up to some exponentially small error, equal to a constant. We make this idea rigorous by the use of comparison principle, combined with probabilistic ideas from the theory of Markov processes. Let us mention that the periodicity is crucial, for it provides the uniformity, which is essential for the exponential convergence.

In Chapter 4 we study the case of a stationary, ergodic intensity $\lambda_\omega(\cdot)$. We assume that ω is an element of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and we will denote the expectation with respect to $\mathbb{P}$ by $\mathbb{E}$.

The solution $u_\omega(\cdot, x)$ of (1.1) will now be a random process indexed by the space variable x, and the uniqueness of solution implies that it is also stationary and ergodic. Clearly $\alpha_\omega(\cdot, x)$ is also a stationary and ergodic process.

We prove that for any fixed $t \in \mathbb{R}$, $\alpha_\omega(t, x)$ converges as x goes to infinity to a constant α_*, $\mathbb{P} - a.s.$. This constant is independent of the time t, as well as ω. The fact that this is a Law of Large Numbers (LLN) type of result will become more clear once we recover some form of the averaging. In order to achieve this we look at the equation describing the dynamics of $\alpha_\omega(t, x)$, which is deduced from equation (1.2):

$$\alpha_t = \alpha_{xx} - \frac{6\,\alpha_x}{x+\alpha} - \frac{3\alpha_x^2}{x+\alpha}, \qquad x > 0,\, t \in \mathbb{R}. \tag{1.8}$$

In Appendix B we prove the estimate $\int_0^\infty x\,\mathbb{E}\alpha_x^2\,dx < \infty$, which implies that we can consider the above dynamics as a perturbation of the linear dynamics

$$\tilde{\alpha}_t = \tilde{\alpha}_{xx} - \frac{6}{x}\,\tilde{\alpha}_x, \qquad x > 0,\, t \in \mathbb{R}.$$

If we denote by Q_x the measure corresponding to the process with generator $\mathcal{L} = \frac{\partial^2}{\partial x^2} - \frac{6}{x}\frac{\partial}{\partial x}$, then we can write the solution of the linear equation as an average in a similar fashion as we did for (1.6) in the form

$$\tilde{\alpha}_\omega(t, x) = E^{Q_x}\tilde{\alpha}_\omega(t - \tau_{x_0}, x_0) = \int \tilde{\alpha}_\omega(t - \tau; x_0)\, q_x(\tau, x_0)\, d\tau, \tag{1.9}$$

where $q_x(\cdot; x_0)$ denotes the density of the distribution of the first hitting time of level x_0 for the process Q_x. Since a LLN is easy to deduce for $\tilde{\alpha}$, we prove that the actual solution of (1.8) does not differ much from $\tilde{\alpha}$. In particular we manage to write $\alpha_\omega(t, x)$, up to some error, in the form $\int \xi_\omega(t - \tau) q_x(\tau; x_0)$, where $\xi_\omega(\cdot)$ is a stationary process, which, though, also depends in the values of α_ω. Much effort is devoted in showing that this error is of order $1/x$. The reason we do this is that we want to reduce the study of the fluctuations of $\alpha_\omega(t, x)$ around the limit, to the ones for $\int \xi_\omega(t - \tau) q_x(\tau; x_0)\, d\tau$, since if one assumes that the process $\xi_\omega(\cdot)$ decorrelates fast enough, then it is not difficult to see that the variance of $\int \xi_\omega(t-\tau) q_x(\tau; x_0)\, d\tau$ decays like $1/x^2$ (which implies that the correct scaling for the study of the fluctuations is x).

In section 4.2 we prove a Central Limit Theorem for $\alpha_\omega(t, x)$. In other words, we prove that as x goes to infinity $x(\alpha_\omega(t, x) - \mathbb{E}\alpha_\omega(t, x))$ converges in distribution to a Gaussian random variable. The convergence holds for any fixed $t \in \mathbb{R}$ and by stationarity the distribution of the limiting random variable is the same for every t. Besides the assumption on $\lambda(\cdot)$ that we imposed so far, we also ask that the process $\lambda(\cdot)$ decorrelates fast enough. In fact, we will assume that the values of $\lambda(\cdot)$ become independent, when considered at times that are separated enough. In particular, we will suppose that there exists a number L, such that

$$\mathbb{P}(A\, B) - \mathbb{P}(A)\, \mathbb{P}(B) = 0,$$
$$\text{for any } A \in \mathcal{F}_L^\infty,\ G \in \mathcal{F}_{-\infty}^0\,.$$

In the above expression, $\mathcal{F}_\sigma^\tau$ denotes the σ-algebra generated by $\{\lambda(s)\colon s \in (\sigma, \tau)\}$.

In order to prove the CLT we use the approximation mentioned above, and we prove a CLT for the average $\int_{\mathbb{R}_-} \xi_\omega(t-\tau)\, q_x(\tau; x_0)\, d\tau$. For this we investigate the decay in τ of the quantity $\mathbb{E}[\,(\xi_\omega(0) - \mathbb{E}[\xi_\omega(0)|\mathcal{F}^{\tau}_{-\tau}]\,)^2\,]$. In other words, we try to examine how well can the process $\xi_\omega(\cdot)$ be approximated by a stationary process with short range correlations.

In Appendix A we describe the properties of the process Q_x (which is a Bessel process) that are necessary to our approach. We also define two more related processes, the Bessel Bridge and the Entrance Law, which are used to obtain the representation for ξ_ω. In Appendix B we collect all the PDE estimates that are necessary to carry out the perturbation arguments.

Finally, let us make a few of comments on the notation. By C we will denote a generic constant, the exact value of which is irrelevant, and does not depend on any of parameters of the quantities involved. The value of C might also change from line to line. $\|\cdot\|_{L^\infty}$ will stand for the supremum norm, when the supremum is considered over all the possible parameters, inclunding the randomness ω. Whenever we would like to consider the supremum taken over only a few of the parameters, we will indicate it by an argument next to L^∞. Last, let us mention that in several cases, when we need to apply the functional integration as in (1.9), for example, we will be forced, for technical reasons that we will mention, to choose the level x_0 large enough. Therefore, we will keep the convention that the level x_0 that appears in the rest of the paper, has been chosen so that all those technical criteria are satisfied.

Chapter 2

Existence Of The Dynamics

In this section we prove that there exists a unique bounded solution of the problem (1.1), that decays to zero as x tends to infinity. As we already mentioned, we want to think of the solution as the equilibrium state of the system starting from an arbitrary initial state $u_0(x)$. In order to do this, we consider the problem

$$\begin{aligned} u_t = u_{xx} - u^2 \quad &, \qquad x > 0,\, t > \tau, \\ u_x(t,0) = -1/2\ \lambda(t) \quad &, \qquad t > \tau, \\ u(\tau, x) = u_0(x) \quad &, \qquad x > 0. \end{aligned} \tag{2.1}$$

Since the system is driven by the source term $\lambda(t)\,\delta_0(x)$ we can assume, without loss of generality, that $u_0(x) \equiv 0$. It is easy to check that in this case the functions $\underline{u}(t,x) \equiv 0$ and $\overline{u}(t,x) = 6/(x+\alpha_2)^2$, with α_2 such that $12/\alpha_2^3 \geq \frac{1}{2}\lambda(t)$, $t > \tau$, are respectively sub- and super-solutions for the problem (2.1). Standard results (see [L], chpt 14) will then guarantee that there is a unique, $C^{1,2}((\tau,\infty)\times\mathbb{R}_+)$ solution of (2.1), which we denote by $u^{(\tau)}$. Moreover, we have that $0 \leq u^{(\tau)}(t,x) \leq 6/(x+\alpha_2)^2$, for $x > 0, t > \tau$. We will show that as $\tau \to -\infty$, $u^{(\tau)}$ converges to a $C^{1,2}(\mathbb{R} \times \mathbb{R}_+)$ function $u(t,x)$, which satisfies (1.1). Furthermore, this is the unique solution, and $0 \le u(t,x) \le 6/(x+\alpha_2)^2$. More precisely, we have

Theorem 1 *Consider equation (1.1), corresponding to a source term $\lambda(\cdot)$, that satisfies the bounds $0 < \lambda_1 \le \lambda(\cdot) \le \lambda_2 < \infty$, for some λ_1, λ_2. Then there exists a unique positive and bounded solution $u(t,x)$ of this equation, that decays to zero, uniformly in time, as x tends to infinity. Moreover, if $u^{(\tau)}(t,x)$ denotes the solution to problem (2.1), then $u^{(\tau)}(t,x)$ converges pointwise to $u(t,x)$, as $\tau \to -\infty$.*

Proof : Let us denote by $g(t,x;s,y)$ with $t > s$ and $x, y > 0$, the Green's function for the generator $-\frac{\partial}{\partial t} + \frac{\partial^2}{\partial x^2}$ with Neumann boundary conditions, i.e.

$$g(t,x;s,y) = \frac{1}{\sqrt{4\pi(t-s)}} \left(\exp\left(-\frac{(x-y)^2}{4(t-s)}\right) + \exp\left(-\frac{(x+y)^2}{4(t-s)}\right) \right),$$

and by $g_c(t,x;s,y) = e^{-c(t-s)} g(t,x;s,y)$, the Green's function for the generator $-\frac{\partial}{\partial t} + \frac{\partial^2}{\partial x^2} - c$, also with Neumann boundary conditions. We will consider c to be positive.

In order to guarantee that the integrals below are convergent, we add and subtract from the right hand side of (2.1) the quantity $c\,u$. We can now use the variation of constants formula to write for $t > \tau$,

$$\begin{aligned} u^{(\tau)}(t,x) &= \frac{1}{2} \int_\tau^t \lambda(s)\, g_c(t,x;s,0)\, ds \\ &\quad + \int_\tau^t \int_{\mathbb{R}_+} (-(u^{(\tau)})^2 + c u^{(\tau)})(s,y)\, g_c(t,x;s,y)\, dy\, ds. \end{aligned} \tag{2.2}$$

Let us note that the mapping $\tau \to u^{(\tau)}(t,x)$ is nonincreasing, for any arbitrary t, x. Indeed, consider the solutions $u^{(\tau_1)}, u^{(\tau_2)}$ for $\tau_1 > \tau_2$. In the domain $x > 0$, $t > \tau_1$, $u^{(\tau_1)}$ satisfies (2.1) with initial condition $u_0(x) \equiv 0$, while in the same domain $u^{(\tau_2)}$ satisfies the same problem, but with initial condition $u_0(x) = u^{(\tau_2)}(\tau_1, x) \geq 0$. Hence, by comparison, it follows that for any $t > \tau_1 > \tau_2$, and $x > 0$, $u^{(\tau_2)}(t,x) \geq u^{(\tau_1)}(t,x)$. From this it follows that, as $\tau \to -\infty$, $u^{(\tau)}(t,x)$ converges to a bounded function $u(t,x)$. Passing now to the limit, $\tau \to -\infty$, in (2.2) we see, by dominated convergence, that $u(t,x)$ also satisfies (2.2). Standard arguments now (see [L]) imply that $u(t,x)$ is a $C^{1,2}(\mathbb{R} \times \mathbb{R}_+)$ function and it satisfies the equation $u_t = u_{xx} - u^2$, for any $t \in \mathbb{R}$, $x \in \mathbb{R}_+$, as well as the boundary condition $u_x(t,0) = -1/2\,\lambda(t)$, for $t \in \mathbb{R}$.

Regarding the uniqueness, let us notice that we need to prove that (1.1) satisfies a maximum principle in terms of the data $\lambda(\cdot)$. In order to do this we will use functional integration to give a representation of the solution in terms of this data.

Let W_x denote the Wiener measure on continuous paths $\{\beta(t) : t \geq 0\}$ starting from position $x \in \mathbb{R}$, and E^{W_x} the expectation with respect to this measure. We will abuse the notation and use W_x as the Brownian motion speeded up by a factor of 2. This fits better to our setting, since the generator of this process is $\partial^2/\partial x^2$. For any $\sigma < \infty$, we can use the Feynman-Kac formula to write the solution to (1.1) as

$$u(t,x) = E^{W_x} \int_0^\sigma \lambda(t-s)\, \delta_0(\beta(s))\, e^{-\int_0^s u(t-r,\beta(r))\,dr} ds \\ + E^{W_x} \left[u(t-\sigma, \beta(\sigma))\, e^{-\int_0^\sigma u(t-r,\beta(r))\,dr} \right].$$

We can drop the exponential in the second term on the right hand side, to dominate it by $E^{W_x}[\, u(t-\sigma, \beta(\sigma))]$. Since $u(t,x)$ decays to zero, uniformly in time, as x tends to infinity, it is easy to see that, as $\sigma \to \infty$, the last term tends to zero. Thus, we obtain the expression,

$$u(t,x) = E^{W_x} \int_0^\infty \lambda(t-s)\, \delta_0(\beta(s))\, e^{-\int_0^s u(t-r,\beta(r))\, dr} ds.$$

Suppose, now, that $u(t,x)$ is also a solution to (1.1), corresponding to a source term $\tilde{\lambda}(\cdot)$. Subtracting the two equations we have that

$$(u-v)_t = (u-v)_{xx} - (u+v)(u-v) + (\lambda(t) - \tilde{\lambda}(t))\, \delta_0(x), \quad t \in \mathbb{R}, x \in \mathbb{R}.$$

A similar computation as before shows that the difference $u(t,x) - v(t,x)$ can be written as

$$u(t,x) - v(t,x) = E^{W_x} \int_0^\infty (\lambda(t-s) - \tilde{\lambda}(t-s))\, \delta_0(\beta(s))\, e^{-\int_0^s (u(t-r,\beta(r)) + v(t-r,\beta(r))\,)\, dr} ds.$$

From this expression, it is now clear that if $\lambda \geq \tilde{\lambda}$, then, pointwise, $u \geq v$. Finally, if $\lambda = \tilde{\lambda}$, then $u \equiv v$, which implies the uniqueness.

Remark 1: A direct consequence of the uniqueness is that any symmetry of the boundary data $\lambda(\cdot)$ transfers to the solution u. In particular, if $\lambda(\cdot)$ is periodic, then, for any $x > 0$, $u(\cdot, x)$ is also periodic, and if $\lambda(\cdot)$ is a stationary, ergodic process, then so is $u(\cdot, x)$.

Remark 2: From the construction of the solution it follows that $0 \leq u(t,x) \leq 6/(x+\alpha_2)^2$. The lower bound can be improved. If λ_1 is the lower bound of $\lambda(\cdot)$, then we can compare the solution u, with the one corresponding to the Neumann condition $-\frac{1}{2}\lambda_1$. The solution to the last problem is $6/(x+\alpha_1)^2$, with α_1 such that $12/\alpha_1^3 = \frac{1}{2}\,\lambda_1$. This comparison yields that $u(t,x) \geq 6/(x+\alpha_1)^2$.

Chapter 3

The Periodic Case

In this section we will study the case where the source term is a periodic funtion. Moreover, we will assume that there are positive constants λ_1, λ_2, such that $0 < \lambda_1 \leq \lambda(\cdot) \leq \lambda_2 < \infty$. We will prove that as $x \to \infty$ there is exponential convergence to a steady state solution. Specifically we will prove

Theorem 2 *Let $\lambda(t)$ be a periodic function and let λ_1, λ_2, be constants such that $0 < \lambda_1 \leq \lambda(\cdot) \leq \lambda_2 < \infty$. Then there exists a number $0 < \delta < 1$, a positive constant C and a positive constant α_*, such that if u is the solution of:*

$$u_t = u_{xx} - u^2 \qquad , \quad x > 0,\, t \in \mathbb{R}, \tag{3.1}$$

$$u_x(0,t) = -\frac{1}{2}\lambda(t) \qquad , \quad t \in \mathbb{R}, \tag{3.2}$$

then

$$\left| u(t,x) - \frac{6}{(x+\alpha_*)^2} \right| \leq C\delta^x,$$

for any $x > 0$.

As it will be seen, this Theorem reduces to

Theorem 3 *Let $\alpha(t,x)$ be defined from the solution $u(t,x)$ of (3.1), (3.2), by the relation $u(t,x) = 6/(x+\alpha(t,x))^2$, and assume the assumptions of Theorem 1 regarding $\lambda(t)$. Then there exist a constant α_*, a number $0 < \delta < 1$, a positive constant C and a postive function $f(x)$ of at most polynomial growth, such that for any $x > 0$*

$$|\alpha(t,x) - \alpha_*| < C\, f(x)\, \delta^x.$$

The method we are using to prove Theorem 2 is a combination of the parabolic comparison principle along with probabilistic techniques. In the following discussion we show how the proof of Theorem 2 reduces to the proof of Theorem 3, and we also point out the key point of its proof.

To begin with, let us denote by $\overline{u}(x) = \sup_{-\infty<t<+\infty} u(t,x)$, and by $\underline{u}(x) = \inf_{-\infty<t<+\infty} u(x,t)$. Notice that in the periodic case the above supremum and infimum need only to be taken over one period interval and let us assume that the period is 1. The function $\alpha(t,x)$ is given in terms of $u(t,x)$ by the formula:

$$\alpha(x,t) = \sqrt{\frac{6}{u(x,t)}} - x.$$

Denote, also, by

$$\overline{\alpha}(x) \equiv \sqrt{\frac{6}{\overline{u}(x)}} - x \text{ and } \underline{\alpha}(x) \equiv \sqrt{\frac{6}{\underline{u}(x)}} - x.$$

It is clear that $u(t,x) \le \overline{u}(x) = 6/(x+\overline{\alpha}(x))^2$ for any x, t and this bound is optimal, in the sense that, since the supremum is taken over a finite interval, there will be a t, such that equality holds.

On the other hand, suppose that we fix an arbitrary $x_0 > 0$, and let w solve the boundary problem

$$w_t = w_{xx} - w^2, \qquad x \geq x_0,\ t \in \mathbb{R},$$
$$w(t, x_0) = \overline{u}(x_0), \qquad t \in \mathbb{R}.$$

By the comparison principle it is clear that $u(t,x) \leq w(t,x)$, for any $x \geq x_0, t \in \mathbb{R}$. Since the equation for w corresponds to the time homogeneous case, it can be solved explicitly and yields that $w(t,x) = 6/(x + \overline{\alpha}(x_0))^2$, $x \geq x_0, t \in \mathbb{R}$.
If we consider the two previous bounds for u evaluated at $x = x_0 + 1$, and recall that the first is optimal we get that $6/(x_0+1+\overline{\alpha}(x_0+1))^2 \leq 6/(x_0+1+\overline{\alpha}(x_0))^2$, and consenquently $\overline{\alpha}(x_0) \leq \overline{\alpha}(x_0+1)$. In the same way we can get that $\underline{\alpha}(x_0+1) \leq \underline{\alpha}(x_0)$.
Moreover, by the definition, $\overline{\alpha}(x_0) \leq \alpha(t, x_0) \leq \underline{\alpha}(x_0)$ and , thus, we have proved the following monotonicity property:

$$\overline{\alpha}(x_0) \leq \overline{\alpha}(x_0+1) \leq \alpha(t, x_0+1) \leq \underline{\alpha}(x_0+1) \leq \underline{\alpha}(x_0).$$

Since, x_0 is arbitrary, Theorem 2 boils down to proving that $\underline{\alpha}(x) - \overline{\alpha}(x)$ decays exponentially fast, i.e. it boils down to proving Theorem 3. Finally, since

$$\underline{\alpha}(x) - \overline{\alpha}(x) = \frac{\sqrt{6}}{\sqrt{\overline{u}(x)\underline{u}(x)}(\sqrt{\underline{u}(x)} + \sqrt{\overline{u}(x)})}(\overline{u}(x) - \underline{u}(x)),$$

and $\overline{u}(x), \underline{u}(x)$ have inverse power lower bounds, Theorem 3 will be established once we prove that the difference $\overline{u}(x) - \underline{u}(x)$ decays exponentially fast.

Let us denote by $\rho(x) \equiv \overline{u}(x) - \underline{u}(x)$. In order to get the exponential decay we are are going to show that there is a constant $0 < \delta < 1$, such that, for any $x > 0, \rho(x+1) < \delta\rho(x)$. The proof of this fact will be the subject of the rest of this section. The method we follow to get this bound is probabilistic.

The first lemma shows that $\rho(x)$ is a nonincreasing in x.

Lemma 1 *If ρ is defined as above, and $x_0 > 0$ is arbitrary, then for any $x \geq x_0$, $\rho(x) \leq \rho(x_0)$.*

Proof: For arbitrary $t_1, t_2 \in \mathbb{R}$ define the function

$$v(t,x) \equiv u(t_1 + t, x) - u(t_2 + t, x).$$

Then, using the fact that $u(t_i{+}t, x)$ satisfies (3.1), $v(t,x)$ will satisfy the equation

$$\frac{\partial v(t,x)}{\partial t} = \frac{\partial^2 v(t,x)}{\partial x^2} - (u(t_1+t,x) + u(t_2+t,x))v(t,x).$$

Solve this equation in the region $x > x_0, t \in \mathbb{R}$ (for arbitrary x_0) with Dirichlet boundary condition $v(\cdot, x_0)$, on $x = x_0$. Notice that, since $v(x_0, t) \leq \overline{u}(x_0) - \underline{u}(x_0) = \rho(x_0)$ and

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} + (u(x, t+t_1) + u(x, t+t_2))\right)\rho(x_0) \geq 0,$$

then, by comparison, $\rho(x_0)$ is an upper solution for the above problem i.e. $v(x,t) \leq \rho(x_0)$. Since t_1, t_2 are arbitrary, it follows that $\rho(x) \leq \rho(x_0)$ for $x \geq x_0$.

In what follows, we develop the probabilistic framework in which we are going to work. After this framework is set, we work towards the proof of the contraction property for ρ, through a series of lemmas.

As in the previous sections, let W_x denote the Wiener measure on continuous paths $\{\beta(t) : t \geq 0\}$ starting from position $x \in \mathbb{R}$, and speeded by a factor of 2, and E^{W_x} the expectation with respect to this measure. Let also $\tau_y = \inf\{t > 0 : \beta(t) = y\}$, be the hitting time of a level $y > 0$.

Since the solution is smooth we can use the Feynman-Kac formula to write it implicitly in the following way:

$$u(t,x) = E^{W_x}\left[u(t-\tau_y, y)\exp\left(-\int_0^{\tau_y} u(t-r,\beta(r))dr\right)\right]. \tag{3.3}$$

This formula is valid for any $y > 0$, but we will be choosing y to be less than x. This formula is implicit, since we don't know a priori the values appearing in the right hand side of (3.3). Nevertheless, this expression provides the main tool to establish the exponential decay. In order to make more clear the way to proceed, let us rewrite the formula in the following way:

$$\begin{aligned}
& E^{W_x}\left[u(t-\tau_y, y)\exp\left(-\int_0^{\tau_y} u(t-r,\beta(r))dr\right)\right] \\
= \;& E^{W_x}\left[E^{W_x}\left[u(t-\tau_y, y)\exp\left(-\int_0^{\tau_y} u(t-r,\beta(r))dr\right)|\tau_y = s\right]\right] \\
= \;& \int_0^{\infty} u(t-s,y)g(s;t)ds \\
= \;& \int_{-\infty}^{t} u(s,y)g(t-s;t)ds,
\end{aligned} \tag{3.4}$$

where the measure [1] $g(s;t)\,ds$, that appears in the above formula is equal to

$$E^{W_x}\left[\exp\left(-\int_0^{\tau_y} u(t-r,\beta(r))dr\right)|\tau_y = s\right] W_x(\tau_y \in ds), \tag{3.5}$$

and

$$W_x(\tau_y \in ds) = \frac{|x-y|}{\sqrt{4\pi s^3}}\exp(-\frac{|x-y|^2}{4s})ds,$$

is the hitting time density for Brownian motion. Let us finally write $g_t(s) \equiv g(t-s;t)$, and then write the representation of u as $u(t,x) = \int_{-\infty}^{t} u(s,y)g_t(s)ds$. Having this representation, the proof of the main estimate will follow the lines of proof of the exponential convergence to equilibrium of an irreducible Markov

[1] we have made explicit the dependence on t, but suppressed the one on u, x and y

chain on a compact state space (see [V]). In our case there is some extra complexity, because of the fact that the density $g_t(s)$ depends on u. Also, the total mass, $\int_{-\infty}^{t} g_t(s)ds = E^{W_x}\left[\exp\left(-\int_0^{\tau_y} u(t-s,\beta(s))ds\right)\right]$, is not constant in time. Let us denote the difference in masses between different times by $m(t_1,t_2) \equiv \int_{-\infty}^{t_1} g_{t_1}(s)ds - \int_{-\infty}^{t_2} g_{t_2}(s)ds$. For the rest of the section the densities $g_t(s)$ and the mass difference $m(t_1,t_2)$ will correspond to $y = x-1$.

Lemma 2 *For arbitrary* $0 \le t_1, t_2 \le 1, x > 0$ *we have that*

$$u(t_1,x) - u(t_2,x) \le (\overline{u}(x-1) - \underline{u}(x-1)) \int_{I_+} g_{t_1t_2}(s)ds + \overline{u}(x-1)\, m(t_1,t_2),$$

where

$$g_{t_1t_2}(s) \equiv g_{t_1}(s) - g_{t_2}(s),$$

$$I_+ \equiv \{s \in (-\infty, t_1) : g_{t_1t_2}(s) \ge 0\}.$$

Proof: Without loss of generality suppose that $t_1 < t_2$. By the representation (3.4) we have that

$$\begin{aligned}
u(t_1,x) - u(t_2,x) &= \int_{-\infty}^{t_1} u(s,x-1)g_{t_1}(s)ds - \int_{-\infty}^{t_2} u(s,x-1)g_{t_2}(s)ds \\
&= \int_{-\infty}^{t_1} u(s,x-1)g_{t_1t_2}(s)ds - \int_{t_1}^{t_2} u(s,x-1)g_{t_2}(s)ds \\
&= \int_{I_+} u(s,x-1)g_{t_1t_2}(s)ds + \int_{(-\infty,t_1)\setminus I_+} u(s,x-1)g_{t_1t_2}(s)ds \\
&\quad - \int_{t_1}^{t_2} u(s,x-1)g_{t_2}(s)ds \\
&\le \overline{u}(x-1)\int_{I_+} g_{t_1t_2}(s)ds + \underline{u}(x-1)\int_{(-\infty,t_1)\setminus I_+} g_{t_1t_2}(s)ds \\
&\quad - \int_{t_1}^{t_2} u(s,x-1)g_{t_2}(s)ds.
\end{aligned}$$

Let us denote by $I_- \equiv (-\infty, t_1) \setminus I_+$, and notice that

$$\begin{aligned}\int_{(-\infty,t_1)\setminus I_+} g_{t_1t_2}(s)ds &= \int_{-\infty}^{t_1} g_{t_1}(s)ds - \int_{-\infty}^{t_2} g_{t_2}(s)ds \\ &\quad + \int_{t_1}^{t_2} g_{t_2}(s)ds - \int_{I_+} g_{t_1t_2}(s)ds.\end{aligned}$$

Substitution gives us

$$\begin{aligned}u(t_1,x) - u(t_2,x) &\leq (\overline{u}(x-1) - \underline{u}(x-1)) \int_{I_+} g_{t_1t_2}(s)ds \\ &\quad + \underline{u}(x-1) \int_{t_1}^{t_2} g_{t_2}(s)ds - \int_{t_1}^{t_2} u(s,x-1) g_{t_2}(s)ds \\ &+ \underline{u}(x-1)\, m(t_1,t_2),\end{aligned}$$

and the result now follows by noticing that $u(s,x-1) \geq \underline{u}(x-1)$

Lemma 3 *There exists a* $0 < \delta < 1$ *such that:*

$$sup_{0 \leq t_1,t_2 \leq 1} \int_{I_+} g_{t_1t_2}(s)ds \leq \delta.$$

Proof: First, by the definition of g_t and since the solution u is positive, we see that

$$\int_{-\infty}^{t} g_t(s)ds \leq 1.$$

Also recall that

$$\int_{I_+} g_{t_1t_2}(s)ds = \int_{-\infty}^{t_1} g_{t_1}(s)ds - \int_{I_-} g_{t_1}(s)ds - \int_{I_+} g_{t_2}(s)ds.$$

This can be bounded above by

$$\begin{aligned}& 1 - \int_{I_-} g_{t_1}(s)ds - \int_{I_+} g_{t_2}(s)ds \qquad (3.6)\\ = \;& 1 - \int_{I_-} E^{W_x}\left[e^{-\int_0^{\tau_{x-1}} u(t_1-r,\beta(r))\,dr} \mid \tau_{x-1} = t_1 - s\right] W_x(t_1-s;x-1)\,ds \\ & - \int_{I_+} E^{W_x}\left[e^{-\int_0^{\tau_{x-1}} u(t_2-r,\beta(r))\,dr} \mid \tau_{x-1} = t_2 - s\right] W_x(t_2-s;x-1)\,ds.\end{aligned}$$

where $W_x(s; x-1)$ is the density of the measure $W_x(\tau_{x-1} \in ds)$. So, it suffices to bound from below the sum of the two integrals by a positive number.

By the comparison principle, we know that there is a positive constant α_2, such that $u(t,x) \le 6/(x+\alpha_2)^2$, for any $x > 0$ and $t \in \mathbb{R}$ (see Chapter 2). In particular, since the Brownian motion in (3.6) lies on the right of the level $x-1$, we have that $u(t_1 - r, \beta(r)) \le 6/(x-1+\alpha_2)^2$, and, thus, the first integral is bounded below by

$$\int_{I_-} e^{-\frac{6}{(x-1+\alpha_2)^2}(t_1-s)} W_x(t_1-s; x-1)ds$$
$$\ge \int_{I_-\cap(-2,-1)} e^{-\frac{6}{(x-1+\alpha_2)^2}(t_1-s)} W_x(t_1-s; x-1)ds.$$

and similarly for the second one. It can be checked that for any $c > 0$, $e^{-cs}W_x(s; x-1)$ is decreasing for $s \ge 1$, and since $t_1, t_2 \in [0,1]$, we see that the last integrand is bounded below by $e^{-18/(x-1+\alpha_2)^2}W_x(3; x-1) \ge \varepsilon$, for some positive ε. In the same way we can bound the second integral in (3.6) from below, and hence estimate their sum by

$$\varepsilon\,|I_- \cap (-2,-1)| + \varepsilon\,|I_+ \cap (-2,-1)| \ge \varepsilon.$$

Thus, we see that we can choose δ to be $1-\varepsilon$.

Remark: Let us point out that from the proof of this lemma it follows that the number δ that appears here, as well as in Theorems 2, 3, does not depend on $\lambda(\cdot)$. In other words the exponential decay rate depends only on the dynamics of the system.

Lemma 4 *For the mass difference* $m(t_1, t_2)$ *between arbitrary times* $0 \le t_1, t_2 \le 1$, *the following bound holds:*

$$m(t_1,t_2) \le (x+\alpha_1)^2 \ln(\frac{x+\alpha_1}{x-1+\alpha_1})\left(\overline{u}(x-1) - \underline{u}(x-1)\right),$$

where α_1 is the constant that appears in Remark 2 of Chapter 2.

Proof: By (3.5) we have that $m(t_1, t_2)$ is equal to

$$E^{W_x}[\exp(-\int_0^{\tau_{x-1}} u(t_1 - r, \beta(r))\, dr)] - E^{W_x}[\exp(-\int_0^{\tau_{x-1}} u(t_2 - r, \beta(r))\, dr\,)]$$

$$\leq E^{W_x}[\exp(-\int_0^{\tau_{x-1}} \underline{u}(\beta(r))\, dr)] - E^{W_x}[\exp(-\int_0^{\tau_{x-1}} \overline{u}(\beta(r))\, dr)]$$

Denote by $\underline{f}(x; x-1)$ and $\overline{f}(x; x-1)$, respectively, the functions that appear in the last difference. We can replace $x-1$ with an arbitrary x_0 and note that $\underline{f}(x; x_0)$ and $\overline{f}(x; x_0)$ solve, respectively, the equations:

$$\underline{f}_{xx} - \underline{u}\, \underline{f} = 0 \quad , x > x_0,$$

and

$$\overline{f}_{xx} - \overline{u}\, \overline{f} = 0 \quad , x > x_0,$$

with boundary condition on x_0: $\underline{f}(x_0; x_0) = \overline{f}(x_0; x_0) = 1$. Subtract the equations to get that

$$(\underline{f} - \overline{f})_{xx} + \overline{u}\, \overline{f} - \underline{u}\, \underline{f} = 0,$$

or

$$(\underline{f} - \overline{f})_{xx} - \underline{u}\, (\underline{f} - \overline{f}) + (\overline{u} - \underline{u})\, \overline{f} = 0.$$

Since the difference on the boundary is 0, we get by Feynman-Kac formula that

$$(\underline{f} - \overline{f})(x; x_0) = E^{W_x}\left[\int_0^{\tau_{x_0}} \big((\overline{u} - \underline{u})\overline{f}\big)\, (\beta(s))\ \exp(-\int_0^s \underline{u}(\beta(r))dr)ds\right].$$

By Lemma 1, this is

$$\leq \rho(x_0) E^{W_x}\left[\int_0^{\tau_{x_0}} \overline{f}(\beta(s); x_0) \exp(-\int_0^s \underline{u}(\beta(r))dr)ds\right]$$

$$< \rho(x_0) E^{W_x}\left[\int_0^{\tau_{x_0}} \overline{f}(\beta(s); x_0) ds\right].$$

In order to bound the last quantity let us first get a bound for $\overline{f}$. Again we can use the comparison principle to bound $\overline{u}$ below as in Remark 2 of Chapter 2 , and get that

$$\overline{f}(x;x_0) \le E^{W_x}\left[\exp(-\int_0^{\tau_{x_0}} \frac{6}{(\beta(r)+\alpha_1)^2}dr)\right].$$

The right hand side of the above relation, which we denote by $h(x;x_0)$, solves the equation: $h_{xx} - \frac{6}{(x+\alpha_1)^2}\ h = 0$, $x > x_0$ with boundary condition $h(x_0;x_0) = 1$. This equation can be solved and gives us $h(x;x_0) = ((x_0+\alpha_1)/(x+\alpha_1))^2$. Thus,

$$\begin{aligned} E^{W_x}\left[\int_0^{\tau_{x_0}} \overline{f}(\beta(s);x_0)ds\right] &\le E^{W_x}\left[\int_0^{\tau_{x_0}} (\frac{x_0+\alpha_1}{\beta(s)+\alpha_1})^2 ds\right] \\ &= (x_0+\alpha_1)^2 \ln(\frac{x+\alpha_1}{x_0+\alpha_1}), \end{aligned}$$

where the last equality follows by solving the equation

$$\hbar_{xx} + ((x_0+\alpha_1)/(x+\alpha_1))^2 = 0,$$

for $x \ge x_0$ and $\hbar(x_0) = 0$. Setting now $x_0 = x-1$ the result follows.

Proposition 1 *For x large enough, there is a $0 < \delta < 1$ such*

$$\rho(x) \le \delta\ \rho(x-1).$$

Proof: The proof follows immediately by substitution of the estimates in Lemma 3 and 4 into the estimate in Lemma 2, and noticing that $\overline{u}(x-1)\cdot(x+\alpha_1)^2\ln(((x+\alpha_2)/(x-1+\alpha_2))^2)$ goes to zero as x goes to infinity.

Chapter 4

The Stationary Case

In this section we consider the case that the boundary data is a stationary process $\{\lambda_\omega(t)\colon t \in \mathbb{R}\}$. This means that there is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, such that λ_ω is a mapping from this space to the space of paths on $\mathbb{R}$, and that the time shift θ_τ acting on the space of paths as $\theta_\tau \lambda_\omega(t) = \lambda_{\theta_\tau \omega}(t) = \lambda_\omega(-\tau + t)$ preserves the distribution of λ_ω.

The solution of equation (1.1) will now be a random process $u_\omega(\cdot, x)$, indexed by the space variable x. By the uniqueness of the solution, $u_\omega(\cdot, x)$ will be stationary and ergodic under the time shift θ_τ. Clearly, $\alpha_\omega(\cdot, x)$ also inherits the same properties.

We wish to study the asymptotics of $u_\omega(\cdot, x)$, or $\alpha_\omega(\cdot, x)$, as $x \to \infty$. In particular, we will prove the existence of an *a.s.* limit α^* (LLN) for $\alpha_\omega(\cdot, x)$, which by an easy Taylor expansion transfers to an almost sure asymptotic behaviour for the original function $u_\omega(\cdot . x)$. These results are summarised in Theorems 4 and 5. The proof of Theorem 4 is essentially an application of the ergodic theorem tailored for the nonlinear process $\alpha_\omega(\cdot, x)$. Though it provides no information about the limit α^*.

A different method provides an expression for the limit (see Proposition 4). This latter method is developed in order to study the fluctuations around the limit point α^*, which, after the appropriate scaling, are expected to be Gaussian. In particular, this method reduces the study of the fluctuations of $\alpha_\omega(\cdot, x)$ to those of another quantity that is easier to analyse.

In the second section we analyse the quantity that we mentioned above and we prove that the Central Limit Theorem (CLT) is valid for $\alpha_\omega(t,x)$. In other words we prove that, as x tends to infinity, $x(\alpha_\omega(t,x) - \mathbb{E}\alpha_\omega(t,x))$ converges in distribution to a Gaussian random variable, independent of $t \in \mathbb{R}$.

4.1 Law of Large Numbers & Approximation

To begin with let us write the equation that $\alpha_\omega(t,x)$ satisfies. Recalling the relation $u_\omega(t,x) = 6/(x + \alpha_\omega(t,x))^2$, the equation is easily derived from the one for u and is

$$\alpha_t = \alpha_{xx} - \frac{6\,\alpha_x}{x+\alpha} - \frac{3\alpha_x^2}{x+\alpha}, \qquad x > 0,\, t \in \mathbb{R}. \tag{4.1}$$

The exact boundary condition that $\alpha(t,x)$ satisfies on $x = 0$ will be irrelevant for our purposes, but whenever it is necessary we will consider it to be Dirichlet. By this we mean that we will consider $\alpha(t,x)$ to be equal to some function $\alpha(t,0)$, on $x = 0$, which will also not be given explicitly, but on the other hand will allow us to obtain represenations like the ones in the previous chapter.

Let us go one step further by adding and subtracting the term $\frac{6}{x}\,\alpha_x$ in the last equation to bring it into the form

$$\alpha_t = \alpha_{xx} - \frac{6}{x}\,\alpha_x + \left(-\frac{3\,\alpha_x^2}{\alpha(x+\alpha)} + \frac{6\,\alpha_x}{x(x+\alpha)} \right) \alpha, \qquad x > 0,\, t \in \mathbb{R}. \tag{4.2}$$

The reason we do this is that we want to think of the dynamics as a perturbation of the linear dynamics $\alpha_t = \alpha_{xx} - \frac{6}{x}\,\alpha_x$, since in this case the representation $E^{Q_x}\alpha_\omega(t-\tau_0, 0)$ for the solution, the Bessel process Q_x is defined in Section A.1), in combination with the ergodic theorem provides the LLN. Furthermore in this case, assuming that the correlations of the process $\alpha_\omega(\cdot, 0)$ decay fast enough, Gaussian fluctuations are not difficult to obtain.

In the sequel we will try to obtain an expression for $\alpha_\omega(t,x)$ in a similar manner as in Chapter 2. In this section when using functional integration, we will be dealing with several time dependent diffusions and it will be convenient to consider them going forward in time instead of backwards. In order to achieve this we define $a_\omega(t,x)$ to be equal to $\alpha_\omega(-t,x)$. It is easy to see that the new function satisfies the equation

$$a_t + a_{xx} - \frac{6}{x}\,a_x + \left(-\frac{3\,a_x^2}{a(x+a)} + \frac{6\,a_x}{x(x+a)} \right) a = 0, \qquad x > 0,\, t \in \mathbb{R}. \tag{4.3}$$

In the rest of the section we will study this equation. Let us point out that, besides notational covenience, nothing really changes, since the results for $a_\omega(t,x)$ directly transfer to $\alpha_\omega(t,x)$. Moreover the estimates of Appendix B, which are the basis of our analysis, read exactly the same for $a_\omega(t,x)$ as for $\alpha_\omega(t,x)$.

Let us introduce the notation:

$$\begin{aligned} c_\omega(t,x) &\equiv -\frac{3\,a_x^2}{a(x+a)} + \frac{6\,a_x}{x(x+a)}, \\ \zeta_\omega(\sigma,\tau) &\equiv \int_\sigma^\tau c_\omega(s, x(s))\,ds, \end{aligned} \tag{4.4}$$

As before we will use functional integration to linearize the equation. Because of stationarity it will be enough and more convenient to consider the solution evaluated at $t = 0$. Let us now choose and fix $x_0 > 0$ appropriatelly large, and use Feynman-Kac formula to write $a_\omega(0, x)$ as

$$a_\omega(0,x) = E^{Q_x}\left[a_\omega(\tau_{x_0}, x_0) e^{\zeta_\omega(0,\tau_{x_0})}\right]. \tag{4.5}$$

The fact that the quantity in the left hand side is well defined, and thus the representation is valid, is justified by the following proposition

Proposition 2 *For any $x_0 > 0$ large enough and any $y \geq x_0$, the following bound holds*

$$\sup_\omega \sup_{x>y} E^{Q_x}\left[e^{\zeta_\omega(\tau_y,\tau_{x_0})}\right] < \infty. \tag{4.6}$$

Proof: By writing out the form of $\zeta_\omega(\tau_y, \tau_{x_0})$ as this is given in (4.4) we have that

$$\begin{aligned}
E^{Q_x}\left[e^{\zeta_\omega(\tau_y,\tau_{x_0})}\right] &= E^{Q_x}\left[\exp\left(\int_{\tau_y}^{\tau_{x_0}} \left(-\frac{3\,a_x^2}{a\,(x+a)} + \frac{6\,a_x}{x\,(x+a)}\right) ds\right)\right] \\
&= E^{Q_x}\left[\exp\left(\int_{\tau_y}^{\tau_{x_0}} \left(-\frac{3}{a\,(x+a)}(a_x - \frac{a}{x})^2 + \frac{3\,a}{x^2\,(x+a)}\right) ds\right)\right] \\
&\leq E^{Q_x}\left[\exp\left(\int_{\tau_y}^{\tau_{x_0}} \frac{3\,a}{x^2\,(x+a)}\, ds\right)\right] \\
&\leq E^{Q_x}\left[\exp\left(\int_0^{\tau_{x_0}} \frac{3\,a}{x^2\,(x+a)}\, ds\right)\right],
\end{aligned}$$

where, as usual, the integrand is evaluated over the path $(s, x(s))$. By Khasminskii's lemma (see Lemma (13) the last expectation will be uniformly bounded, as long as we have the uniform estimate

$$\sup_{x>x_0} E^{Q_x}\left[\int_0^{\tau_{x_0}} \frac{3\,a(s,x(s))}{x^2(s)\,(x(s)+a(s,x(s)))}\, ds\right] < 1.$$

But by Proposition 18, since a is uniformly bounded away from zero and infinity, this is true as long as we choose and fix x_0 large enough.

We are now ready to give the proof of the LLN.

Theorem 4 *Let $\lambda_\omega(t)$ be a stationary, ergodic process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, satisfying the same bounds as in Theorem 2 . Let $\alpha_\omega(t,x)$ be defined from the solution $u_\omega(t,x)$ of (3.1), (3.2) by the relation $u(t,x) = 6/(x+\alpha_\omega(t,x))^2$. Then there exists a constant α_* such that for each $t \in \mathbb{R}$*

$$\alpha_\omega(t,x) \longrightarrow \alpha_*, \qquad \mathbb{P}\ a.s.\ as\ x \to \infty.$$

An immediate corollary of this Theorem is

Theorem 5 *Let $\lambda_\omega(t)$ be a stationary, ergodic process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, satisfying the same bounds as in Theorem 2 . Then if u satisfies (3.1), (3.2) there exists a positive constant α_* such that for each $t \in \mathbb{R}$*

$$\left|(x+\alpha_*)^3(u(x,t) - \frac{6}{(x+\alpha_*)^2})\right| \longrightarrow 0, \qquad \mathbb{P}\ a.s.,\ as\ x \to \infty.$$

Proof of Theorem 4 : It is enough to prove the result for $a_\omega(0,x)$. As we mentioned the strategy we follow is to consider $a_\omega(0,x)$ as a perurbation of $E^{Q_x} a_\omega(\tau_{x_0}, x_0)$, where x_0 is arbitrary, but such that Proposition 2 is valid. Thus, we write the quantity $a_\omega(0,x) - \mathbb{E}\left[a_\omega(\cdot,x)\right]$ as :

$$\begin{aligned}\big(a_\omega(0,x) - E^{Q_x} a_\omega(\tau_{x_0}, x_0)\big) \;&+\; \big(E^{Q_x} a_\omega(\tau_{x_0}, x_0) - \mathbb{E} a_\omega(\cdot, x_0)\big) \\ &+\; \left(\mathbb{E} a_\omega(\cdot, x_0) - \mathbb{E} a_\omega(\cdot, x)\right). \end{aligned} \tag{4.7}$$

Now the steps towards the LLN are the following:

Step 1: The perturbation $e^{\zeta_\omega(0,\tau_{x_0})}$ turns out to be weak enough to guarantee that

$$\lim_{x_0\to\infty}\sup_{x>x_0}\left(a_\omega(0,x)-E^{Q_x}a_\omega(\tau_{x_0},x_0)\right)=0.$$

Step 2 Using the ergodic theorem we can prove that, as $x\to\infty$, the second term in the above expression converges to 0, $\mathbb{P}$ $a.s.$.

Step 3 Using an energy estimate we can prove that, as $x\to\infty$, $\mathbb{E}a_\omega(\cdot,x)$ converges to a constant.

The LLN follows from these steps, if in the expression (4.7) we first take the limit $x\to\infty$ and then the limit $x_0\to\infty$.

Proof of Step 1: Using the representation of α_ω as in (4.5), we have that

$$\begin{aligned}
|a_\omega(0,x)-E^{Q_x}a_\omega(\tau_{x_0},x_0)| &\leq E^{Q_x}\left[a_\omega(\tau_{x_0},x_0)\left|e^{\zeta_\omega(0,\tau_{x_0})}-1\right|\right]\\
&\leq \|a_\omega\|_{L^\infty}E^{Q_x}\left[\left(e^{\zeta_\omega(0,\tau_{x_0})}+1\right)|\zeta_\omega(0,\tau_{x_0})|\right]\\
&\leq \|a_\omega\|_{L^\infty}E^{Q_x}\left[\left(e^{\zeta_\omega(0,\tau_{x_0})}+1\right)^2\right]^{1/2}\cdot\\
&\qquad\cdot E^{Q_x}\left[\zeta_\omega^2(0,\tau_{x_0})\right]^{1/2},
\end{aligned}$$

by Taylor's expansion and the Cauchy-Schwarz inequality. The first expectation is uniformly bounded by Proposition 2. Regarding the second term, we expand $\zeta_\omega(0,\tau_{x_0})$ as in (4.4) and then similar calculations as in Lemmas 5 and 6 will show that $\lim_{x_0\to\infty}\sup_{x>x_0}\left(E^{Q_x}[|\zeta_\omega(0,\tau_{x_0})|^2]\right)^{1/2}=0$.

Proof of Step 2: In this step we prove essentially an ergodic theorem for the weighted average with weight equal the density $q_x(\tau;x_0)$. More precisely, we have

$$\begin{aligned}
E^{Q_x}a_\omega(\tau_{x_0},x_0) &= \int_{\mathbb{R}_+}a_\omega(\tau,x_0)q_x(\tau;x_0)\,d\tau\\
&= \int_{\mathbb{R}_+}\frac{d}{d\tau}\left(\int_0^\tau a_\omega(r,x_0)dr\right)\frac{1}{x^2}q_1(\frac{\tau}{x^2};\frac{x_0}{x})\,d\tau
\end{aligned}$$

Integrating by parts, taking into account that $a_\omega(r,x)$ is bounded and that $q_x(\tau;x_0) = O(\tau^{-9/2})$, as $\tau \to \infty$ (see Appendix A), and making a change of variables we get that the last quantity is equal to :

$$-\frac{1}{x^4}\int_{\mathbb{R}_+}\int_0^{\tau} a_\omega(r,x_0)\,dr\cdot q_1'(\frac{\tau}{x^2};\frac{x_0}{x})\,d\tau \;=-\frac{1}{x^2}\int_{\mathbb{R}_+}\int_0^{x^2\tau} a_\omega(r,x_0)\,dr\cdot q_1'(\tau;\frac{x_0}{x})\,d\tau$$

$$=-\int_{\mathbb{R}_+}\frac{1}{x^2\tau}\int_0^{x^2\tau} a_\omega(r,x_0)\,dr\cdot\tau\, q_1'(\tau;\frac{x_0}{x})\,d\tau.$$

By the ergodic theorem the inner average converges $\mathbb{P}-a.s.$ to $\mathbb{E}a_\omega(\cdot,x_0)$, as $x\to\infty$, for every $\tau>0$. The result now follows by the dominated convergence theorem and the fact that $-\int_0^\infty \tau\, q_1'(\tau;0)\,d\tau = 1$.

Proof of Step 3: Taking expectations with respect to ω in equation (4.2) and using the stationarity of a_ω we manage to eliminate the time dependence. Thus, we see that $\mathbb{E}a_\omega(\cdot,x)$ satisfies the equation

$$\frac{\partial^2\,\mathbb{E}a_\omega(\cdot,x)}{\partial x^2} - \frac{6}{x}\frac{\partial\,\mathbb{E}a_\omega(\cdot,x)}{\partial x} + \mathcal{E}(f) = 0, \qquad x > x_0,$$

for any arbitrary $x_0>0$. On x_0 we can assign the natural boundary value $\mathbb{E}a_\omega(\cdot,x_0)$. $\mathcal{E}(f)$ stands for $\mathbb{E}f$, with $f(t,x) = -\frac{3a_x^2}{x+a}+\frac{6aa_x}{x(x+a)}$. Proposition 18 in Section A.1 allows us to write the solution of the last equation as

$$\mathbb{E}a_\omega(\cdot,x) = \int_{x_0}^{\infty}\mathcal{E}(f)(y)\cdot y^{-6}\left((x\wedge y)^7 - x_0^7\right)\,dy + \mathbb{E}\alpha(\cdot,x_0).$$

We will show that $\{\mathbb{E}a_\omega(\cdot,x)\}_{x>x_0}$ is Cauchy. Indeed, for any $x'\geq x>x_0$ we have that

$$|\mathbb{E}a_\omega(\cdot,x)-\mathbb{E}a_\omega(\cdot,x')| \;\leq\; \int_x^{x'}|\mathcal{E}(f)(y)|\cdot y^{-6}\left|(x'\wedge y)^7-(x\wedge y)^7\right|\,dy$$

$$
\begin{aligned}
&\le \int_x^{x'} \mathbb{E}\left[\frac{3\,a_x^2}{y+a} + \frac{6\,|\,a\,a_x|}{y\,(y+a)}\right] \cdot y^{-6}\left((x'\wedge y)^7 + (x\wedge y)^7\right)\,dy \\
&\le \int_x^{x'} \left(\frac{3\,\mathbb{E}\,a_x^2}{y} + \frac{6\,\mathbb{E}\,|\,a\,a_x|}{y^2}\right) \cdot y^{-6}\left((x'\wedge y)^7 + (x\wedge y)^7\right)\,dy \\
&\le C\int_x^{x'} \left(\mathbb{E}\,a_x^2 + \frac{\mathbb{E}\,|\,a\,a_x|}{y}\right)\,dy \\
&\le C\int_x^{x'} \left(\mathbb{E}\,a_x^2 + \frac{1}{y^2}\right)\,dy.
\end{aligned}
$$

Finally, Proposition 27 guarantees that the right hand side goes to 0, as $x \to \infty$.

This finishes the proof of Theorem 4.

Theorem 4 does not give any information about the limit α^*. Moreover, the proof of it does not indicate a way to study the fluctuations around this limit. In order to achieve this a more detailed analysis of the perturbation is required. This will be the subject of the rest of the section. What we will try to do is to write a_ω as a sum of two terms, one of which is neglible in the limit, after rescaling, and the other is an average over a stationary process. Thus, the question of fluctuation reduces to the question of the decay of the correlations for that process.

In order to proceed we need to modify the representation (4.5). First, let us condition on the hitting time τ_{x_0} to write (4.5) as

$$
\int_{\mathbb{R}_+} a_\omega(\tau, x_0)\, E^{Q_x^{\tau,x_0}}[e^{\zeta_\omega(0,\tau)}]\, q_x(\tau; x_0)d\tau.
$$

The Bessel Bridge Q_x^{τ,x_0} is defined in Section A.2, as the Bessel process starting at time 0 from position x and conditioned to hit level x_0 for the first time at time τ. If the process starts from a time σ different than 0, then we will indicate it in the measure as $Q_{\sigma,x}^{\tau,x_0}$.

We would like to pin down the Bessel Bridge at a fixed point and let the point (more precisely the time) that the process starts vary. We do this as follows: First, notice that the uniqueness of the solution, implies that $c_{\theta_{-\tau}\omega}(-\tau+s,\cdot) = c_\omega(s,\cdot)$, for every s, once we have fixed an arbitrary τ. Recall that $c_\omega(s,\cdot)$ is defined in (4.4). The same holds for a_ω. Using this observation and shifting the path $x(\cdot)$, τ units down, we see that

$$E^{Q_x^{\tau,x_0}}[\,e^{\zeta_\omega(0,\tau)}] = E^{Q^{0,x_0}_{-\tau,x}}[\,e^{\zeta_{\theta_{-\tau}\omega}(-\tau,0)}]. \tag{4.8}$$

Finally, we obtain the following expression for $a_\omega(0,x)$

$$\begin{aligned} a_\omega(0,x) &= \int_{\mathbb{R}_+} a_{\theta_{-\tau}\omega}(0,x_0) E^{Q^{0,x_0}_{-\tau,x}}[\,e^{\zeta_{\theta_{-\tau}\omega}(-\tau,0)}]\, q_x(\tau;x_0)\, d\tau \\ &= \int_{\mathbb{R}_-} a_{\theta_{\tau}\omega}(0,x_0) E^{Q^{0,x_0}_{\tau,x}}[\,e^{\zeta_{\theta_{\tau}\omega}(\tau,0)}]\, q_x(-\tau;x_0)\, d\tau, \end{aligned} \tag{4.9}$$

where in the first line we also used the fact that $a_\omega(\tau,x_0) = a_{\theta_{-\tau}\omega}(0,x_0)$. The second line follows from the first by the simple change of variables $\tau := -\tau$, and we will prefer it since it is notationally more convenient.

Before proceeding we would like to explain the idea by giving a heuristic argument.

Let us use the scaling property (A.4) in Section A.1 , and make the change of variables $\tau := x^2\tau$ in (4.9) so that

$$a_\omega(0,x) = \int_{\mathbb{R}_-} a_{\theta_{x^2\tau}\omega}(0,x_0) E^{Q^{0,x_0}_{x^2\tau,x}}[\,e^{\zeta_{\theta_{x^2\tau}\omega}(x^2\tau,0)}]\, q_1(-\tau;\frac{x_0}{x})\, d\tau. \tag{4.10}$$

Disregard for the moment the shift $\theta_{x^2\tau}$, which will be taken care by the stationarity. In Section A.3 we see that, as $x\to\infty$, $Q^{0,x_0}_{x^2\tau,x} \Rightarrow Q^{0,x_0}_\infty$, for any $\tau < 0$, where the process Q^{0,x_0}_∞ (defined also in section A.3) represents a conditioned Bessel process starting from infinity . Moreover, by the energy estimate in Proposition 28 in Appendix B we expect that for any $\tau < 0$, $\zeta_\omega(x^2\tau, 0) \sim \zeta_\omega(-\infty, 0)$, as $x \to \infty$. So we can expect that asymptotically, as $x \to \infty$, (4.10) behaves like

$$\int_{\mathbb{R}_-} a_{\theta_{x^2\tau}\omega}(0, x_0) E^{Q_\infty^{0,x_0}}[\, e^{\zeta_{\theta_{x^2\tau}\omega}(-\infty,0)}]\, q_1(-\tau, \frac{x_0}{x})\, d\tau,$$

and by reversing the scaling and a change of variables, the last expression is found to be equal to

$$\int_{\mathbb{R}_-} a_{\theta_\tau\omega(0,x_0)} E^{Q_\infty^{0,x_0}}[\, e^{\zeta_{\theta_\tau\omega}(-\infty,0)}] q_x(-\tau; x_0)\, d\tau. \tag{4.11}$$

In this way we eliminate the dependence on x of the nonlinear term, and represent $a_\omega(0, x)$ as an average over the stationary process $a_{\theta_\tau\omega}(0, x_0) E^{Q_\infty^{0,x_0}}[\, e^{\zeta_{\theta_\tau\omega}(-\infty,0)}]$. Moreover, as is also seen by Proposition 5 and Proposition 6, the study of the fluctuation of $a_\omega(0, x)$ can be reduced to the study of the decay of correlations of that process.

In order to carry out this program rigorously let us first add and subtract in (4.9) the quantity $\int_{\mathbb{R}_-} a_{\theta_\tau\omega}(0, x_0) E^{Q_\infty^{0,x_0}}[\, e^{\zeta_{\theta_\tau\omega}(-\infty,0)}]\, q_x(-\tau; x_0)\, d\tau$, to write it as

$$\begin{aligned} a_\omega(0, x) &= \int_{\mathbb{R}_-} a_{\theta_\tau\omega}(0, x_0) E^{Q_\infty^{0,x_0}}[\, e^{\zeta_{\theta_\tau\omega}(-\infty,0)}]\, q_x(-\tau; x_0)\, d\tau \\ &+ \int_{\mathbb{R}_-} a_{\theta_\tau\omega}(0, x_0) \Big(E^{Q_{\tau,x}^{0,x_0}}[\, e^{\zeta_{\theta_\tau\omega}(\tau,0)}] - E^{Q_\infty^{0,x_0}}[e^{\zeta_{\theta_\tau\omega}(-\infty,0)}] \Big)\, q_x(-\tau; x_0)\, d\tau. \end{aligned}$$

The main focus of this section will be to prove that

Proposition 3 *The quantity*

$$x\, \mathbb{E} \left| \int_{\mathbb{R}_-} a_{\theta_\tau\omega}(0, x_0) \Big(E^{Q_{\tau,x}^{0,x_0}}[e^{\zeta_{\theta_\tau\omega}(\tau,0)}] - E^{Q_\infty^{0,x_0}}[e^{\zeta_{\theta_\tau\omega}(-\infty,0)}] \Big)\, q_x(-\tau; x_0) d\tau \right|,$$

converges to 0*, as* x *tends to infinity.*

Once Proposition 3 is established, it is easy to deduce the expression for the limit of $\alpha_\omega(0,x)$, as described in the next proposition.

Proposition 4 *Let α_ω be the solution of (4.1). For any x_0 large enough, so that the representation (4.5) is valid, we have that*

$$\alpha_\omega(0,x) \longrightarrow \mathbb{E}\left[\alpha_\omega(0,x_0) E^{Q_\infty^{0,x_0}}[\, e^{\zeta_\omega(-\infty,0)}]\right], \quad \mathbb{P}-a.s., \, as \; x \to \infty$$

Proof: By Proposition 3 it follows that the limit of $a_\omega(0,x)$ is the same as the limit of $\int_{\mathbb{R}_-} a_{\theta_\tau\omega}(0,x_0) E^{Q_\infty^{0,x_0}}[\, e^{\zeta_{\theta_\tau\omega}(-\infty,0)}]\, q_x(-\tau;x_0)\, d\tau$. The last integral, though, is an average over the stationary process $a_{\theta_\tau\omega}(0,x_0) E^{Q_\infty^{0,x_0}}[\, e^{\zeta_{\theta_\tau\omega}(-\infty,0)}]$, and the ergodic theorem, as this is applied in Step 2 in the proof of the LLN, can be now used to show the $a.s.$ convergence of this integral to
$\mathbb{E}\left[\alpha_\omega(0,x_0) E^{Q_\infty^{0,x_0}}[\, e^{\zeta_\omega(-\infty,0)}]\right]$. Since $a_\omega(0,x)$ is equal to $\alpha_\omega(0,x)$, this completes the proof.

Also, from Porposition 3 it follows that the study of the fluctuations of $\alpha_\omega(0,x)$ can be reduced to the study of the fluctuations of

$$\int_{\mathbb{R}_-} a_{\theta_\tau\omega}(0,x_0) E^{Q_\infty^{0,x_0}}[\, e^{\zeta_{\theta_\tau\omega}(-\infty,0)}] q_x(-\tau;x_0)\, d\tau.$$

The last quantity has the advantage that is given as an explicit average over the stationary process $\xi_\omega(\tau) \equiv a_{\theta_\tau\omega}(0,x_0) E^{Q_\infty^{0,x_0}}[\, e^{\zeta_{\theta_\tau\omega}(-\infty,0)}]$, and thus reduces the question to the decay of correlations of the latter. This is summarised in the next two propositions.

Proposition 5 *Let $\xi_\omega(\tau) \equiv a_{\theta_\tau\omega}(0,x_0) E^{Q_\infty^{0,x_0}}[\, e^{\zeta_{\theta_\tau\omega}(-\infty,0)}]$, and assume that as $x \to \infty$, $x \int_{\mathbb{R}_-} (\xi_\omega(\tau) - \mathbb{E}[\xi_\omega(\tau)])\, q_x(-\tau;x_0)\, d\tau$ converges in distribution to a random variable $\mathcal{N}$. Then $x(a_\omega(0,x) - \mathbb{E}[a_\omega(0,x)])$ converges in distribution to the same random variable $\mathcal{N}$.*

The next proposition explains why we insist on the scaling of order x.

Proposition 6 *Denote by $\mathcal{R}_\xi(\tau)$ the correlation function of the process $\xi_\omega(\tau) \equiv a_{\theta_\tau \omega}(0, x_0) E^{Q_\infty^{0,x_0}}[\, e^{\zeta_{\theta_\tau \omega}(-\infty,0)}]$, i.e.*

$$\mathcal{R}_\xi(\tau) = \mathbb{E}\,[\, \xi_\omega(\tau)\, \xi_\omega(0)] - \mathbb{E}\,[\, \xi_\omega(\tau)]\, \mathbb{E}\,[\, \xi_\omega(0)],$$

and assume that $\int_{\mathbb{R}} |\mathcal{R}_\xi(\tau)|\, d\tau < \infty$. Then the variance of $\int_{\mathbb{R}_-} \xi_\omega(\tau)\, q_x(-\tau; x_0)\, d\tau$ has the property that

$$x^2\, \mathbb{E}\left[\left(\int_{\mathbb{R}_-} (\xi_\omega(\tau) - \mathbb{E}\,[\, \xi_\omega(\tau)])\, q_x(-\tau; x_0)\, d\tau \right)^2 \right] \to \int_{\mathbb{R}} \mathcal{R}_\xi(\tau)\, d\tau \int_{\mathbb{R}_-} q_1^2(-\tau; 0)\, d\tau$$

Proof: By writing the square as a double integral, using the scaling property of the density $q_x(\tau; x_0)$, and also making a simple change of variables we see that

$$x^2\, \mathbb{E}\left[\left(\int_{\mathbb{R}_-} (\xi_\omega(\tau) - \mathbb{E}\,[\, \xi_\omega(\tau)])\, q_x(-\tau; x_0)\, d\tau \right)^2 \right] =$$
$$2 \int_{\mathbb{R}_-} \mathcal{R}_\xi(\tau') \int_{\mathbb{R}_-} q_1(-\tau; \frac{x_0}{x})\, q_1(-\tau - \frac{\tau'}{x^2}; \frac{x_0}{x})\, d\tau\, d\tau'.$$

The rest follows by dominated convergence.

Before giving the proof of Proposition 3 we would like to make a few remarks that will help to make the method more clear.

By stationarity we can disregard the shift θ_τ, and also take into account the boundedness of a_ω, so the expression to be controled is

$$x\, \mathbb{E} \int_{\mathbb{R}_-} \left| E^{Q_{\tau,x}^{0,x_0}}[e^{\zeta_\omega(\tau,0)}] - E^{Q_\infty^{0,x_0}}[e^{\zeta_\omega(-\infty,0)}] \right| q_x(-\tau; x_0) d\tau. \tag{4.12}$$

In order to estimate it we could neglect, due to the energy estimate of Proposition 28 in Appendix B , the contribution of the Q_∞^{0,x_0} paths in the time interval $(-\infty, \tau_x)$, where $\tau_x = \inf\{s < 0 : x(s) = x\}$, and then try to control the difference

$$E^{Q^{0,x_0}_{\tau,x}}[e^{\zeta_\omega(\tau,0)}] - E^{Q^{0,x_0}_{\infty}}[e^{\zeta_\omega(\tau_x,0)}]$$

Though, by the definition of Q^{0,x_0}_{∞}, see (A.11), this difference is equal to

$$E^{Q^{0,x_0}_{\tau,x}}[e^{\zeta_\omega(\tau,0)}] - \int_{\mathbb{R}_-} q_x(-\tau; x_0) E^{Q^{0,x_0}_{\tau,x}}[e^{\zeta_\omega(\tau,0)}]\, d\tau,$$

and this, clearly, cannot be made small. In other words, the fact that the marginals of $Q^{0,x_0}_{\tau,x}$ and Q^{0,x_0}_{∞} at the level x are singular, implies that we should consider a level εx, $0 < \varepsilon < 1$, before which the contributions of the paths of both the processes can be neglected and also the difference

$$E^{Q^{0,x_0}_{\tau,x}}[e^{\zeta_\omega(\tau_{\varepsilon x},0)}] - E^{Q^{0,x_0}_{\infty}}[e^{\zeta_\omega(\tau_{\varepsilon x},0)}] = \\ \int (q^{0,x_0}_{\tau,x}(\tau_1; \varepsilon x) - q^{0,x_0}_{\infty}(\tau_1; \varepsilon x)) E^{Q^{0,x_0}_{\tau_1,x_1}}[e^{\zeta_\omega(\tau_1,0)}] d\tau_1,$$

could be made small. Notice that the smaller ε is, the smaller this difference can be made, but on the other hand the contribution of the paths in the interval $(-\infty, \tau_{\varepsilon x})$ becomes more significant. A calculation shows that trying to balance this competition by considering only one level, εx, is not enough to control the difference when this is multiplied by x as in (4.12). To achieve this, a more detailed analysis, considering a sequence of levels, needs to be made.

Proof of Proposition 3: Let us define the sequence $x_i = 2^i x_0$, $i = 1, \cdots, N$, with N chosen to be the largest integer such that $2^N x_0 < \varepsilon x$, for some $0 < \varepsilon < 1$ fixed. Also define the sequence of stopping times $\tau_{x_i} = \inf\{s \in \mathbb{R} \colon x(s) = x_i\}$.

Now, we can write the expectations as

$$E^{Q^{0,x_0}_{\tau,x}}[e^{\zeta_\omega(\tau,0)}] = \qquad E^{Q^{0,x_0}_{\tau,x}}[e^{\zeta_\omega(\tau,0)} - e^{\zeta_\omega(\tau_{x_N},0)}]$$

$$
\begin{aligned}
&+ \quad E^{Q^{0,x_0}_{\tau,x}}\big[e^{\zeta\omega(\tau_{x_N},0)} - e^{\zeta\omega(\tau_{x_{N-1}},0)}\big] \\
&\vdots \\
&+ \quad E^{Q^{0,x_0}_{\tau,x}}\big[e^{\zeta\omega(\tau_{x_2},0)} - e^{\zeta\omega(\tau_{x_1},0)}\big] \\
&+ \quad E^{Q^{0,x_0}_{\tau,x}}\big[e^{\zeta\omega(\tau_{x_1},0)}\big],
\end{aligned}
$$

and similarly,

$$
\begin{aligned}
E^{Q^{0,x_0}_{\infty}}\big[e^{\zeta\omega(-\infty,0)}\big] = \quad & E^{Q^{0,x_0}_{\infty}}\big[e^{\zeta\omega(-\infty,0)} - e^{\zeta\omega(\tau_{x_N},0)}\big] \\
+ \quad & E^{Q^{0,x_0}_{\infty}}\big[e^{\zeta\omega(\tau_{x_N},0)} - e^{\zeta\omega(\tau_{x_{N-1}},0)}\big] \\
\vdots \quad & \\
+ \quad & E^{Q^{0,x_0}_{\infty}}\big[e^{\zeta\omega(\tau_{x_2},0)} - e^{\zeta\omega(\tau_{x_1},0)}\big] \\
+ \quad & E^{Q^{0,x_0}_{\infty}}\big[e^{\zeta\omega(\tau_{x_1},0)}\big].
\end{aligned}
$$

Thus, (4.12) can be estimated by

$$
\sum_{i=1}^{N} \int_{\mathbb{R}_-} \mathbb{E}\,|\, E^{Q^{0,x_0}_{\tau,x}}\big[e^{\zeta\omega(\tau_{x_i},0)} - e^{\zeta\omega(\tau_{x_{i-1}},0)}\big] \tag{4.13}
$$

$$
\begin{aligned}
& -E^{Q^{0,x_0}_{\infty}}\big[e^{\zeta\omega(\tau_{x_i},0)} - e^{\zeta\omega(\tau_{x_{i-1}},0)}\big]\,|\, q_x(-\tau;x_0)\ d\tau \\
+ \quad & \int_{\mathbb{R}_-} \mathbb{E}\left|E^{Q^{0,x_0}_{\tau,x}}\big[e^{\zeta\omega(\tau,0)} - e^{\zeta\omega(\tau_{x_N},0)}\big]\right| q_x(-\tau;x_0)\ d\tau \\
+ \quad & \int_{\mathbb{R}_-} \mathbb{E}\left|E^{Q^{0,x_0}_{\infty}}\big[e^{\zeta\omega(-\infty,0)} - e^{\zeta\omega(\tau_{x_N},0)}\big]\right| q_x(-\tau;x_0)d\tau.
\end{aligned} \tag{4.14}
$$

Each term in the first row in (4.13) can be now written as

$$
\begin{aligned}
\int_{\mathbb{R}_-} d\tau\, q_x(-\tau;x_0)\,\mathbb{E}\,|\int_{\mathbb{R}_-} \big(q^{0,x_0}_{\tau,x}(\tau_i;x_i) - q^{0,x_0}_{\infty}(\tau_i;x_i)\big)\,\cdot \\
\cdot E^{Q^{0,x_0}_{\tau_i,x_i}}\left[e^{\zeta\omega(\tau_{x_i},0)} - e^{\zeta\omega(\tau_{x_{i-1}},0)}\right]\ d\tau_i\,|.
\end{aligned}
$$

Taking into account that $q^{0,x_0}_{\tau,x}(\tau_i;x_i) = \frac{q_x(\tau_i-\tau;x_i)\ q_{x_i}(-\tau_i;x_0)}{q_x(-\tau;x_0)}$, and $q^{0,x_0}_{\infty}(\tau_i;x_i) = q_{x_i}(-\tau_i;x_0)$ and also passing the absolute value inside the second integral, we get that this is less than or equal to

$$
\begin{aligned}
& \int_{\mathbb{R}_-} d\tau\, q_x(-\tau;x_0) \int_{\mathbb{R}_-} \left| \frac{q_x(\tau_i-\tau;x_i)\; q_{x_i}(-\tau_i;x_0)}{q_x(-\tau;x_0)} - q_{x_i}(-\tau_i,x_0) \right| \cdot \\
& \qquad \cdot \mathbb{E}\, E^{Q^{0,x_0}_{\tau_i,x_i}} \left[\left| e^{\zeta\omega(\tau_{x_i},0)} - e^{\zeta\omega(\tau_{x_{i-1}},0)} \right| \right] d\tau_i \\
= & \int_{\mathbb{R}_-} d\tau\, q_x(-\tau;x_0) \int_{\mathbb{R}_-} \left| \frac{q_x(\tau_i-\tau;x_i)}{q_x(-\tau;x_0)} - 1 \right| q_{x_i}(-\tau_i,x_0) \cdot \\
& \qquad \cdot \mathbb{E}\, E^{Q^{0,x_0}_{\tau_i,x_i}} \left[\left| e^{\zeta\omega(\tau_{x_i},0)} - e^{\zeta\omega(\tau_{x_{i-1}},0)} \right| \right] d\tau_i .
\end{aligned}
$$

Using Fubini's Theorem we can write this as

$$
\begin{aligned}
& \int_{\mathbb{R}_-} d\tau_i\, q_{x_i}(-\tau_i;x_0) \quad \cdot \quad \mathbb{E}E^{Q^{0,x_0}_{\tau_i,x_i}} \left[\left| e^{\zeta\omega(\tau_{x_i},0)} - e^{\zeta\omega(\tau_{x_{i-1}},0)} \right| \right] \cdot \\
& \qquad \cdot \int_{\mathbb{R}_-} \left| q_x(\tau_i-\tau;x_i) - q_x(-\tau;x_0)) \right| d\tau .
\end{aligned}
$$

Proposition 17 in Section A.1 provides the estimate

$$
\int |q_x(\tau_i-\tau;x_i) - q_x(-\tau;x_0))| d\tau \le c_1 (\frac{x_i}{x})^2 + c_2 \frac{|\tau_i|}{x^2},
$$

for some positive constants c_1, c_2. So what we need to estimate is

$$
\begin{aligned}
& c_1 \left(\frac{x_i}{x}\right)^2 \int_{\mathbb{R}_-} q_{x_i}(-\tau_i;x_0) \cdot \mathbb{E}E^{Q^{0,x_0}_{\tau_i,x_i}} \left[\left| e^{\zeta\omega(\tau_{x_i},0)} - e^{\zeta\omega(\tau_{x_{i-1}},0)} \right| \right] d\tau_i \\
+ \quad & c_2 \frac{1}{x^2} \int_{\mathbb{R}_-} q_{x_i}(-\tau_i;x_0)\; |\tau_i| \;\cdot \mathbb{E}E^{Q^{0,x_0}_{\tau_i,x_i}} \left[\left| e^{\zeta\omega(\tau_{x_i},0)} - e^{\zeta\omega(\tau_{x_{i-1}},0)} \right| \right] d\tau_i \\
\equiv \quad & c_1 (\frac{x_i}{x})^2 \cdot I_i + c_2 \frac{1}{x^2} \cdot II_i
\end{aligned}
\tag{4.15}
$$

We will estimate the quantities I_i, II_i separately in what follows, but for the moment we will assume Proposition 8 below, which combines these estimates to provide an estimate for the first line of (4.13). For the two remainder terms we will use Proposition 9.

Summarising we have that

$$
\begin{aligned}
& x\,\mathbb{E}\int_{\mathbb{R}_-}\left|E^{Q^{0,x_0}_{\tau,x}}[e^{\zeta_\omega(\tau,0)}]-E^{Q^{0,x_0}_\infty}[e^{\zeta_\omega(-\infty,0)}]\right| q_x(-\tau;x_0)d\tau \\
\leq\ & C\varepsilon\left(\int_{x_0}^\infty z\,\mathbb{E}a_x^2\ dz\right)^{1/2}+\frac{Cx}{x_N}\left(\int_{x_N}^\infty z\,\mathbb{E}a_x^2\,dz\right)^{1/2}
\end{aligned}
$$

Let now x go to infinity. Since x/x_N tends to a constant (it depends on ε), the second term vanishes, which implies that

$$
\begin{aligned}
\limsup_{x\to\infty}\ & x\,\mathbb{E}\int_{\mathbb{R}_-}\left|E^{Q^{0,x_0}_{\tau,x}}[e^{\zeta_\omega(\tau,0)}]-E^{Q^{0,x_0}_\infty}[e^{\zeta_\omega(-\infty,0)}]\right| q_x(-\tau;x_0)d\tau \\
& \leq C\varepsilon\left(\int_{x_0}^\infty z\,\mathbb{E}a_x^2\ dz\right)^{1/2}.
\end{aligned}
$$

Since ε was arbitrary we see, by letting $\varepsilon\to 0$, that the left hand side is equal to 0, and this finishes the proof.

Estimate on I_i.

First, notice that $\zeta_\omega(\tau_{x_i},0)-\zeta_\omega(\tau_{x_{i-1}},0)=\zeta_\omega(\tau_{x_i},\tau_{x_{i-1}})$. Using Taylor's expansion we can estimate I_i by

$$
\int_{\mathbb{R}_-} q_{x_i}(-\tau_i;x_0)\mathbb{E}\cdot E^{Q^{0,x_0}_{\tau_i,x_i}}\left[|e^{\zeta_\omega(\tau_{x_i},0)}+e^{\zeta_\omega(\tau_{x_{i-1}},0)}|\cdot|\zeta_\omega(\tau_{x_i},\tau_{x_{i-1}})|\right]\ d\tau_i.
$$

Using Cauchy-Scwharz on $E^{Q^{0,x_0}_{\tau_i,x_i}}$ we can bound this by

$$
\begin{aligned}
\int_{\mathbb{R}_-} & q_{x_i}(-\tau_i;x_0)\mathbb{E}\cdot E^{Q^{0,x_0}_{\tau_i,x_i}}\left[|e^{\zeta_\omega(\tau_{x_i},0)}+e^{\zeta_\omega(\tau_{x_{i-1}},0)}|^2\right]^{1/2}\cdot \\
& \cdot E^{Q^{0,x_0}_{\tau_i,x_i}}\left[|\zeta_\omega(\tau_{x_i},\tau_{x_{i-1}})|^2\right]^{1/2}\ d\tau_i,
\end{aligned}
$$

and using Cauchy-Schwarz again we can bound it by

$$
\begin{aligned}
& \left(\int_{\mathbb{R}_-} q_{x_i}(-\tau_i;x_0)\cdot\mathbb{E}E^{Q^{0,x_0}_{\tau_i,x_i}}\left[|e^{\zeta_\omega(\tau_{x_i},0)}+e^{\zeta_\omega(\tau_{x_{i-1}},0)}|^2\right]\ d\tau_i\right)^{1/2}\cdot \\
& \cdot\left(\int_{\mathbb{R}_-} q_{x_i}(-\tau_i;x_0)\cdot\mathbb{E}E^{Q^{0,x_0}_{\tau_i,x_i}}\left[|\zeta_\omega(\tau_{x_i},\tau_{x_{i-1}})|^2\right]\ d\tau_i\right)^{1/2}.
\end{aligned}
$$

If we use the stationarity, and shift the path $x(\cdot)$ forward, in other words opposite to what we did to obtain (4.8), we see that the last quantity is equal to

$$\left(\mathbb{E}E^{Q_{x_i}}\left[|e^{\zeta_\omega(\tau_{x_i},\tau_{x_0})}+e^{\zeta_\omega(\tau_{x_{i-1}},\tau_{x_0})}|^2\right]\right)^{1/2}\cdot\left(\mathbb{E}E^{Q_{x_i}}\left[|\zeta_\omega(\tau_{x_i},\tau_{x_{i-1}})|^2\right]\right)^{1/2}. \quad (4.16)$$

The first term can be bounded uniformly as in the Proposition 2. In order to control the second term in (4.16) we will use the energy estimate of Proposition 28 of Appendix B. In particular, writing out what $\zeta_\omega(\tau_{x_i},\tau_{x_{i-1}})$ is, we have that

$$\begin{aligned}\left(\mathbb{E}E^{Q_{x_i}}\left[|\zeta_\omega(\tau_{x_i},\tau_{x_{i-1}})|^2\right]\right)^{1/2} &\le \left(\mathbb{E}E^{Q_{x_i}}\left[\left|\int_{\tau_{x_i}}^{\tau_{x_{i-1}}}\frac{3\,\alpha_x^2}{\alpha\,(x+\alpha)}\,ds\right|^2\right]\right)^{1/2}\\ &+ \left(\mathbb{E}E^{Q_{x_i}}\left[\left|\int_{\tau_{x_i}}^{\tau_{x_{i-1}}}\frac{6\,\alpha_x}{x(x+\alpha)}\,ds\right|^2\right]\right)^{1/2}\end{aligned}$$

We will estimate each of the above terms seperately in the following lemmas:

Lemma 5 *There is a positive constant C such that*

$$\left(\mathbb{E}E^{Q_{x_i}}\left[\left|\int_{\tau_{x_i}}^{\tau_{x_{i-1}}}\frac{3\,\alpha_x^2}{\alpha\,(x+\alpha)}\,ds\right|^2\right]\right)^{1/2}\le\frac{C}{x_{i-1}}\left(\int_{x_{i-1}}^{\infty}z\,\mathbb{E}a_x^2\,dz\right)^{1/2}.$$

Proof: By the positivity of a it will be enough to estimate

$$\left(\mathbb{E}E^{Q_{x_i}}\left[\left|\int_{\tau_{x_i}}^{\tau_{x_{i-1}}}\frac{\alpha_x^2(s,x(s))}{x(s)}\,ds\right|^2\right]\right)^{1/2}.$$

Now, as in the proof of Khasminskii's lemma, we will write the square of the integral as a double integral i.e.

$$\begin{aligned}&2\,\mathbb{E}E^{Q_{x_i}}\int_0^{\tau_{x_{i-1}}}\int_{s_1}^{\tau_{x_{i-1}}}\frac{a_x^2(s_1,x(s_1))}{x(s_1)}\,\frac{a_x^2(s_2,x(s_2))}{x(s_2)}\,ds_1\,ds_2\\ =\;&2\,\mathbb{E}E^{Q_{x_i}}\int_0^{\tau_{x_{i-1}}}ds_1\frac{a_x^2(s_1,x(s_1))}{x(s_1)}E^{Q_{x_i}}\left[\int_{s_1}^{\tau_{x_{i-1}}}\frac{a_x^2(s_2,x(s_2))}{x(s_2)}\,ds_2\,|\,\mathcal{F}_{s_1}\right]\\ =\;&2\,\mathbb{E}E^{Q_{x_i}}\int_0^{\tau_{x_{i-1}}}ds_1\frac{a_x^2(s_1,x(s_1))}{x(s_1)}\,E^{Q_{x(s_1)}}\int_0^{\tau_{x_{i-1}}}\frac{a_x^2(s_1+s_2,x(s_2))}{x(s_2)}\,ds_2.\end{aligned}$$

By Proposition 25 this is less or equal to

$$C\,\mathbb{E}E^{Q_{x_i}}\int_0^{\tau_{x_{i-1}}} ds_1\frac{a_x^2(s_1,x(s_1))}{x(s_1)}\,E^{Q_{x(s_1)}}\int_0^{\tau_{x_{i-1}}}\frac{ds_2}{(x(s_2))^3},$$

and by Proposition 18 this is equal to

$$\begin{aligned}
&C\,\mathbb{E}E^{Q_{x_i}}\int_0^{\tau_{x_{i-1}}} ds_1\,\frac{a_x^2(s_1,x(s_1))}{x(s_1)}\int_{x_{i-1}}^{\infty}\frac{dy}{y^3}\,y^{-6}\left((x(s_1)\wedge y)^7-x_{i-1}^7\right)\\
\leq\;& \frac{C}{x_{i-1}}\,\mathbb{E}E^{Q_{x_i}}\int_0^{\tau_{x_{i-1}}}\frac{a_x^2(s_1,x(s_1))}{x(s_1)}\,ds_1\\
=\;& \frac{C}{x_{i-1}}\int_{x_{i-1}}^{\infty}\frac{\mathbb{E}a_x^2(\cdot,z)}{z}\,z^{-6}\left((x_i\wedge z)^7-x_{i-1}^7\right)\,dz\;\leq\;\frac{C}{x_{i-1}^2}\int_{x_{i-1}}^{\infty} z\,\mathbb{E}a_x^2\,dz.
\end{aligned}$$

This completes the proof.

Lemma 6 *There is a positive constant* C *such that*

$$\left(\mathbb{E}E^{Q_{x_i}}\left[\left|\int_{\tau_{x_i}}^{\tau_{x_{i-1}}}\frac{6\,a_x}{x\,(x+a)}\,ds\right|^2\right]\right)^{1/2}\leq\frac{C}{x_{i-1}}\left(\int_{x_{i-1}}^{\infty} z\;\mathbb{E}a_x^2\;dz\right)^{1/2}.$$

Proof: Once again we will write the square as a double integral

$$2\;\mathbb{E}E^{Q_{x_i}}\int_0^{\tau_{x_{i-1}}}\int_{s_1}^{\tau_{x_{i-1}}}\frac{6a_x(s_1,x(s_1))}{x(s_1)(x(s_1)+a(s_1,x(s_1)))}\cdot\frac{6a_x(s_2,x(s_2))}{x(s_2)(x(s_2)+a(s_2,x(s_2)))}\;ds_1\;ds_2.$$

We can now pass the $\mathbb{E}$ inside the integral, use the positivity of a to bound the denominators below, the Cauchy-Schwarz inequality and the same conditioning trick as in the previous lemma, to obtain the bound

$$C\left(\sup_{x\geq x_{i-1}}\;E^{Q_x}\int_0^{\tau_{x_{i-1}}}\frac{\mathbb{E}[a_x^2(\cdot,x(s))]^{1/2}}{(x(s))^2}\;ds\right)^2.$$

In terms of the Green's function this reduces to

$$\begin{aligned}
&\left(\sup_{x\geq x_{i-1}}\int_{x_{i-1}}^{\infty}\frac{\mathbb{E}[a_x^2(\cdot,z)]^{1/2}}{z^2}\cdot z^{-6}\left((x\wedge z)^7-x_{i-1}^7\right)\;dz\right)^2\\
<\;&\left(\int_{x_{i-1}}^{\infty}\frac{\mathbb{E}[a_x^2(\cdot,z)]^{1/2}}{z}\;dz\right)^2\\
\leq\;&\frac{1}{x_{i-1}^2}\int_{x_{i-1}}^{\infty} z\;\mathbb{E}a_x^2\;dz,
\end{aligned}$$

where in the last inequality we used the Cauchy-Schwarz inequality . The result now follows.

Proposition 7 *There is a constant* C *such that*

$$I_i \leq C \frac{1}{x_{i-1}} \left(\int_{x_{i-1}}^{\infty} z \; \mathbb{E}\alpha_x^2 \; dz \right)^{1/2} .$$

Proof: This follows immediatelly from Lemmas (5) and (6).

Estimate on II_i.

We first use Taylor's expansion in the difference $|e^{\zeta_\omega(\tau_{x_i},0)} - e^{\zeta_\omega(\tau_{x_{i-1}})}|$, and then Cauchy-Schwarz on $E^{Q^{0,x_0}_{\tau_i,x_i}}$ to get that

$$\begin{aligned} II_i &\leq \mathbb{E} \int_{\mathbb{R}_-} q_{x_i}(-\tau_i; x_0) \; |\tau_i| \cdot E^{Q^{0,x_0}_{\tau_i,x_i}} \left[|e^{\zeta_\omega(\tau_{x_i},0)} + e^{\zeta_\omega(\tau_{x_{i-1}},0)}||\zeta_\omega(\tau_{x_i}, \tau_{x_{i-1}})| \right] \; d\tau_i \\ &\leq \mathbb{E} \int_{\mathbb{R}_-} q_{x_i}(-\tau_i; x_0) \; |\tau_i| \cdot E^{Q^{0,x_0}_{\tau_i,x_i}} \left[|e^{\zeta_\omega(\tau_{x_i},0)} + e^{\zeta_\omega(\tau_{x_{i-1}},0)}|^2 \right]^{1/2} \cdot \\ &\qquad \cdot E^{Q^{0,x_0}_{\tau_i,x_i}} [|\zeta_\omega(\tau_{x_i}, \tau_{x_{i-1}})|^2]^{1/2} \; d\tau_i \; . \end{aligned}$$

Furthermore, using Holder's inequality with exponents $(p, q, r) = (3, 6, 2)$ we can bound this by

$$\left(\int_{\mathbb{R}_-} q_{x_i}(-\tau_i; x_0) \; \tau_i^3 \; d\tau_i \right)^{\frac{1}{3}} \left(\int_{\mathbb{R}_-} q_{x_i}(-\tau_i; x_0) \; \mathbb{E}E^{Q^{0,x_0}_{\tau_i,x_i}} [|\zeta_\omega(\tau_{x_i}, \tau_{x_{i-1}})|^2] d\tau_i \right)^{\frac{1}{2}}$$
$$\cdot \left(\int_{\mathbb{R}_-} q_{x_i}(-\tau_i; x_0) \; \cdot \mathbb{E}E^{Q^{0,x_0}_{\tau_i,x_i}} \left[|e^{\zeta_\omega(\tau_{x_i},0)} + e^{\zeta_\omega(\tau_{x_{i-1}},0)}|^6 \right] \; d\tau_i \right)^{\frac{1}{6}}$$

The second and the third term can be estimated as for I_i, and yield a bound of $\frac{C}{x_{i-1}} \left(\int_{x_{i-1}}^{\infty} z\mathbb{E}\, \alpha_x^2 \; dz \right)^{1/2}$. Regarding the second term, the scaling shows that it is equal to

$$\left(\int_{\mathbb{R}_+} \frac{1}{x_i^2} \cdot q_1(\frac{\tau_i}{x_i^2}; \frac{x_0}{x_i}) \tau_i^3 \; d\tau_i \right)^{1/3} = x_i^2 \left(\int_{\mathbb{R}_+} q_1(\tau_i; \frac{x_0}{x_i}) \tau_i^3 \; d\tau_i \right)^{1/3}$$

The tail decay of $q_1(\tau_i; x_0/x_i)$ guarantees that the integral is uniformly bounded. Hence, putting all the bounds together we get that

$$II_i \leq x_i^2 \frac{C}{x_{i-1}} \left(\int_{x_{i-1}}^{\infty} z\mathbb{E}\, \alpha_x^2 \, dz \right)^{1/2} .$$

The following Proposition brings together the estimates on I_i and II_i to give a bound of the first line in (4.13).

Proposition 8 *The contribution of the summation in (4.13) is bounded by*

$$\frac{C\varepsilon}{x} \left(\int_{x_0}^{\infty} z\mathbb{E}\alpha_x^2 \, dz \right)^{1/2} .$$

Proof : Using the estimates on I_i and II_i in (4.15), and also the fact that, for any $i \geq 1$, $\int_{x_{i-1}}^{\infty} z\, \mathbb{E}a_x^2 \, dz \leq \int_{x_0}^{\infty} z\, \mathbb{E}a_x^2 \, dz$, , we can bound the sum in (4.13) by
$\frac{C}{x^2} \left(\int_{x_0}^{\infty} z\mathbb{E}\alpha_x^2 \, dz \right)^{1/2} \sum_{i=1}^{N} x_i$. The last sum is equal to $\sum_{i=1}^{N} 2^i x_0 = (2^{N+1} - 2)x_0$. But N is chosen as the largest integer such that $2^N x_0 < \varepsilon x$, and this yields the result.

The next proposition provides the estimate on the remainder terms in (4.13). To prove it we need the two following lemmas.

Lemma 7 *For any $x > x_0$,*

$$\mathbb{E}E^{Q_{\infty}^{0,x_0}}[e^{\zeta_\omega(-\infty,\tau_x)}] \leq \sup_{y \geq x} \mathbb{E}E^{Q_y}[e^{\zeta_\omega(0,\tau_x)}]$$

Proof: By Fatou's lemma

$$\begin{aligned}\mathbb{E}E^{Q_{\infty}^{0,x_0}}[e^{\zeta_\omega(-\infty,\tau_x)}] &\le \liminf_{y\to\infty}\ \mathbb{E}E^{Q_{\infty}^{0,x_0}}[e^{\zeta_\omega(\tau_y,\tau_x)}] \\ &= \liminf_{y\to\infty}\ \mathbb{E}\int_{\mathbb{R}_-} q_y(-\tau;x_0)E^{Q_{\tau,y}^{0,x_0}}[e^{\zeta_\omega(\tau,\tau_x)}]\,d\tau.\end{aligned}$$

As before, we shift the Bessel paths forward so that $E^{Q_{\tau,y}^{0,x_0}}[e^{\zeta_\omega(\tau,\tau_x)}] = E^{Q_y^{-\tau,x_0}}[e^{\zeta_{\theta_{-\tau}\omega}(\tau,\tau_x)}]$. This fact, combined with stationarity, shows that the last quantity is equal to

$$\liminf_{y\to\infty}\mathbb{E}E^{Q_y}[e^{\zeta_\omega(0,\tau_x)}],$$

and the result follows.

Lemma 8 *For any $x > x_0$, $\mathbb{E}E^{Q_{\infty}^{0,x_0}}|\zeta_\omega(-\infty,\tau_x)|^2 \le \sup_{y>x}\mathbb{E}E^{Q_y}|\zeta_\omega(0,\tau_x)|^2$*

Proof: The proof is the same as for the previous lemma.

Proposition 9 *The sum of the last two terms in (4.13) is bounded by*

$$\frac{C}{x_N}\left(\int_{x_N}^{\infty} z\,\mathbb{E}a_x^2\,dz\right)^{1/2}.$$

Proof: The proof follows the same lines as in *Estimate I_i*, in combination with Lemmas 7 and 8. Let us point out that the lemmas are only used to control the second term of the remainder terms in (4.13).

4.2 Central Limit Theorem

In this section we study the fluctuations of the average $\int_{\mathbb{R}_-}\xi_\omega(\tau)\,q_x(-\tau;x_0)\,d\tau$, as x goes to infinity, where ξ_ω is defined in Proposition 6. We prove that $x\int_{\mathbb{R}_-}(\xi_\omega(\tau)-\mathbb{E}\xi_\omega(\tau))\;q_x(-\tau;x_0)\,d\tau$, converges in distribution to a Gaussian random variable. By Proposition 5, this implies that $x\left(\alpha_\omega(\cdot,x)-\mathbb{E}\alpha_\omega(\cdot,x)\right)$, also converges to the same Gaussian random variable; that is, the Central Limit Theorem is valid.

We prove this result under the assumption that the values of the process $\lambda_\omega(\cdot)$ become independent, when considered at times that are far apart. More specifically we suppose that there is a number L, such that

$$\mathbb{P}(A\,B)-\mathbb{P}(A)\,\mathbb{P}(B)=0,$$
$$\text{for any } A\in\mathcal{F}_L^{\infty},\ G\in\mathcal{F}_{-\infty}^{0}\,. \qquad (A)$$

where $\mathcal{F}_\sigma^\tau$ denotes the σ-algebra generated by $\{\lambda_\omega(s)\colon s\in(\sigma,\tau)\}$.

In order to prove the CLT, we write $\xi_\omega(\tau)$ in the form

$$\xi_\omega(\tau)=\sum_{k\geq 1}\left(\mathbb{E}[\,\xi_\omega(\tau)|\mathcal{F}_{\tau-2^k}^{\tau+2^k}\,]-\mathbb{E}[\,\xi_\omega(\tau)|\mathcal{F}_{\tau-2^{k-1}}^{\tau+2^{k-1}}\,]\right)+\mathbb{E}[\,\xi_\omega(\tau)|\mathcal{F}_{\tau-1}^{\tau+1}\,].$$

Notice that, for any $l>0$, the CLT is valid for the first l terms of the above series. This is because

$$\mathbb{E}[\,\xi_\omega(\tau)|\mathcal{F}_{\tau-2^l}^{\tau+2^l}\,]=\sum_{k=1}^{l}\left(\mathbb{E}[\,\xi_\omega(\tau)|\mathcal{F}_{\tau-2^k}^{\tau+2^k}\,]-\mathbb{E}[\,\xi_\omega(\tau)|\mathcal{F}_{\tau-2^{k-1}}^{\tau+2^{k-1}}\,]\right)+\mathbb{E}[\,\xi_\omega(\tau)|\mathcal{F}_{\tau-1}^{\tau+1}\,],$$

and the left hand side of the above formula is a stationary process with short range correlations. One can refer to [HH], chapter 5, for the validity of the CLT for such processes.

Let us denote by $\xi_{\omega,\kappa}(\tau)\equiv\mathbb{E}[\,\xi_\omega(\tau)|\mathcal{F}_{\tau-2^k}^{\tau+2^k}\,]-\mathbb{E}[\,\xi_\omega(\tau)|\mathcal{F}_{\tau-2^{k-1}}^{\tau+2^{k-1}}\,]$. In order to show that the CLT is valid for the original process $\xi_\omega(\cdot)$, we need to check that

$$\lim_{l\to\infty}\lim_{x\to\infty}x\,\mathbb{E}\left[\Big(\sum_{k\geq l}\int_{\mathbb{R}_-}\xi_{\omega,\kappa}(\tau)\,q_x(-\tau;x_0)\,d\tau\Big)^2\right]^{\frac{1}{2}}=0 \qquad (4.17)$$

The next proposition gives a sufficient condition for the validity of (4.17).

Proposition 10 *(4.17) is valid if the integral* $\int \tau^{\delta} \mathbb{E}\left[\left(\xi_\omega(0) - \mathbb{E}[\xi_\omega(0)|\mathcal{F}^{\tau}_{-\tau}]\right)^2\right] d\tau$ *is finite, for some positive number* δ.

Proof: Using the triangle inequality, we can bound the expectation in relation (4.17) by

$$
\begin{aligned}
&\sum_{k\geq l} \mathbb{E}\left[\left(\int_{\mathbb{R}_-} \xi_{\omega,\kappa}(\tau)\, q_x(-\tau;x_0)\, d\tau\right)^2\right]^{\frac{1}{2}} \\
=\ & \sum_{k\geq l} \left(\int\int \mathbb{E}[\xi_{\omega,\kappa}(\tau)\, \xi_{\omega,\kappa}(\tau')\,]\, q_x(-\tau;x_0)\, q_x(-\tau';x_0)\, d\tau\, d\tau'\right)^{\frac{1}{2}} \\
=\ & \sum_{k\geq l} \left(2\int \mathbb{E}[\xi_{\omega,\kappa}(\tau)\, \xi_{\omega,\kappa}(0)\,] \int q_x(-\tau-\tau';x_0)\, q_x(-\tau';x_0)\, d\tau\, d\tau'\right)^{\frac{1}{2}} \\
=\ & \frac{1}{x}\sum_{k\geq l} \left(2\int \mathbb{E}[\xi_{\omega,\kappa}(\tau)\, \xi_{\omega,\kappa}(0)\,] \int q_1(-\frac{\tau}{x^2}-\tau';\frac{x_0}{x})\, q_1(-\tau';\frac{x_0}{x})\, d\tau\, d\tau'\right)^{\frac{1}{2}}.
\end{aligned}
$$

By assumption (A) we see that the last quantity is equal to

$$
\sum_{k\geq l} \left(2\int_0^{2^{k+1}+L} \mathbb{E}[\xi_{\omega,\kappa}(-\tau)\, \xi_{\omega,\kappa}(0)\,] \int q_1(\frac{\tau}{x^2}+\tau';\frac{x_0}{x})\, q_1(\tau';\frac{x_0}{x})\, d\tau\, d\tau'\right)^{\frac{1}{2}},
$$

and by the stationarity of $\xi_{\omega,\kappa}(\cdot)$, and the Cauchy-Schwarz inequality we can bound the last quantity by

$$
\begin{aligned}
&\sum_{k\geq l} \left(2\int_0^{2^{k+1}+L} \mathbb{E}[\,(\xi_{\omega,\kappa}(0))^2\,] \int q_1(\frac{\tau}{x^2}+\tau';\frac{x_0}{x})\, q_1(\tau';\frac{x_0}{x})\, d\tau\, d\tau'\right)^{\frac{1}{2}} \\
\leq\ & \sum_{k\geq l}\Bigg(2\int_0^{2^{k+1}+L} d\tau \mathbb{E}[\,(\xi_{\omega,\kappa}(0))^2\,]\,\cdot \\
&\qquad \cdot\left(\int q_1^2(\frac{\tau}{x^2}+\tau';\frac{x_0}{x})\, d\tau'\right)^{\frac{1}{2}} \left(\int q_1^2(\tau';\frac{x_0}{x})\, d\tau'\right)^{\frac{1}{2}}\Bigg)^{\frac{1}{2}} \\
\leq\ & C\sum_{k\geq l} 2^{\frac{k}{2}}\,\mathbb{E}[\,(\xi_{\omega,\kappa}(0))^2\,]^{\frac{1}{2}} = C\sum_{k\geq l} 2^k\, \frac{\mathbb{E}[\,(\xi_{\omega,\kappa}(0))^2\,]^{\frac{1}{2}}}{2^{\frac{k}{2}}}
\end{aligned}
$$

$$= \; C \sum_{k \geq l} 2^k \, \frac{1}{2^{\frac{k}{2}}} \, \mathbb{E}\left[\left(\mathbb{E}[\xi_\omega(0) | \mathcal{F}^{2^k}_{-2^k}] - \mathbb{E}[\xi_\omega(0) | \mathcal{F}^{2^{k-1}}_{-2^{k-1}}] \right)^2 \right]^{\frac{1}{2}}$$

$$\leq \; C \sum_{k \geq l} 2^k \, \frac{1}{2^{\frac{k}{2}}} \, \mathbb{E}\left[\left(\xi_\omega(0) - \mathbb{E}[\xi_\omega(0) | \mathcal{F}^{2^{k-1}}_{-2^{k-1}}] \right)^2 \right]^{\frac{1}{2}}$$

$$= \; C \sum_{k \geq l-1} 2^k \, \frac{1}{2^{\frac{k}{2}}} \, \mathbb{E}\left[\left(\xi_\omega(0) - \mathbb{E}[\xi_\omega(0) | \mathcal{F}^{2^k}_{-2^k}] \right)^2 \right]^{\frac{1}{2}} .$$

Since the quotient in the last summand forms a decreasing sequence, the last series converges if and only if the series $\sum_k \frac{1}{\sqrt{k}} \mathbb{E}\left[\left(\xi_\omega(0) - \mathbb{E}[\xi_\omega(0) | \mathcal{F}^{k}_{-k}] \right)^2 \right]^{\frac{1}{2}}$ converges, and, for the same reason, the series converges if the corresponding integral converges. Using the Cauchy-Schwarz inequality in the last integral, we see that the convergence and, thus (4.17), are implied by the convergence of the integral $\int \tau^\delta \mathbb{E}\left[\left(\xi_\omega(0) - \mathbb{E}[\xi_\omega(0) | \mathcal{F}^{\tau}_{-\tau}] \right)^2 \right] d\tau$ for some positive δ, and this completes the proof of the proposition.

In the first part of this section we investigate how well we can approximate the solution of equation (1.1), by a solution of the same equation, but with data $\tilde{\lambda}(\cdot)$, that agree with $\lambda_\omega(\cdot)$ on some interval $(-\tau, \tau)$ and otherwise is arbitrary, but within the bounds λ_1, λ_2. We use some heat kernel estimates to control the error of this approximation. This approximation is used in the second part, to prove the validity of the condition in Propositon 10.

To begin with, let us denote by $u^{(\tau)}$ the solution of the problem

$$u^{(\tau)}_t = u^{(\tau)}_{xx} - (u^{(\tau)})^2 + \left(\lambda_\omega(t) \, 1_{|t| < \tau} + \tilde{\lambda}(t) \, 1_{|t| > \tau} \right) \delta_0(x), \quad t \in \mathbb{R}, x \in \mathbb{R}. \quad (4.18)$$

τ is considered to be positive. Notice that $u^{(\tau)}$ is measurable with respect to the σ-algebra generated by $\{\lambda_\omega(s) \colon |s| < \tau\}$, and this will suffice to transfer the mixing assumptions on $\lambda_\omega(\cdot)$ to $u(\cdot, x)$, for finite x.

To this end, subtract equation (4.18) from (1.1) to get the equation

$$(u-u^{(\tau)})_t = (u-u^{(\tau)})_{xx} - (u+u^{(\tau)})\,(u-u^{(\tau)}) + (\lambda(t)-\tilde{\lambda}(t))\,1_{|t|>\tau}\,\delta_0(x).$$

Using the Feynman-Kac formula we have that, for any $|t|<\tau$,

$$\begin{aligned} u(t,x)-u^{(\tau)}(t,x) = E^{W_x}\int_{t+\tau}^{\infty} (\lambda(t-s)-\tilde{\lambda}(t-s))\,\delta_0(\beta(s))\,\cdot \\ \cdot\exp\{-\int_0^s \left(u(t-r,x(r))+u^{(\tau)}(t-r,x(r))\right)\,dr\}\,ds. \end{aligned}$$

By Remark 2 of Chapter 2 we have that $u(t,x)+u^{(\tau)}(t,x)\geq 12/(|x|+\alpha_1)^2$, and this yields the bound

$$\left|u(t,x)-u^{(\tau)}(t,x)\right| \leq \|\lambda(\cdot)-\tilde{\lambda}(\cdot)\|_{L^\infty}\int_{t+\tau}^{\infty} E^{W_x}\left[\delta_0(\beta(s))\,e^{-\int_0^s \frac{12}{(|\beta(r)|+\alpha_1)^2}\,dr}\right]\,ds; \tag{4.19}$$

moreover,

$$\begin{aligned} \sup_{-\frac{\tau}{2}\leq t\leq\frac{\tau}{2}} \left|u(t,x)-u^{(\tau)}(t,x)\right| \leq \\ \leq \|\lambda(\cdot)-\tilde{\lambda}(\cdot)\|_{L^\infty}\int_{\frac{\tau}{2}}^{\infty} E^{W_x}\left[\delta_0(\beta(s))\,e^{-\int_0^s \frac{12}{(|\beta(r)|+\alpha_1)^2}\,dr}\right]\,ds. \end{aligned} \tag{4.20}$$

To proceed further we need to estimate the decay in s of the above integrand. To do so we will investigate the existence of moments of the heat kernel

$$p(s,x)\equiv E^{W_x}\left[\delta_0(\beta(s))\exp\{-\int_0^s V(\beta(r))\,dr\}\right],$$

of the process with generator $\frac{\partial^2}{\partial x^2}-V$. $V(x)$ equals $\frac{b}{(|x|+a)^2}$, with a,b being positive constants, which in our case correspond to the values α_1 and 12 respectively.

0^{th} **Momement.**

Let us consider the equation

$$\frac{\partial^2 p^{(0)}}{\partial x^2} - Vp^{(0)} = -\delta_0(x), \quad x\in\mathbb{R}. \tag{4.21}$$

If the above equation admits a positive, bounded solution that decays at infinity, then this would be equal to the 0^{th}-moment, $\int p(s,x)\,ds$. It is easy to check that the function $p^{(0)}(x) \equiv \frac{C_b^{(0)}}{(|x|+a)^{k_b}}$, satisfies these requirements, for $k_b = \frac{-1+\sqrt{1+4b}}{2}$ and $C_b^{(0)} = \frac{a^{k_b+1}}{2k_b}$. In the case that $b = 12$, it turns out that $k_b = 3$.

1^{st} **Moment.**

Let us consider the equation

$$\frac{\partial^2 p^{(1)}}{\partial x^2} - Vp^{(1)} + p^{(0)} = -c\,\delta_0(x), \quad x \in \mathbb{R}, \tag{4.22}$$

where c is a positive constant, that will be appropriatelly chosen. We will show that if this equation admits a positive, bounded solution, that decays at infinity, then this solution provides an upper bound for the first moment $\int s\,p(s,x)\,ds$. Indeed, such a solution will satisfy

$$\begin{aligned}
p^{(1)}(x) &= E^{W_x} \int_0^\infty p^{(0)}(\beta(s))\, e^{-\int_0^s V(\beta(r))\,dr}\, ds \\
&\quad + c\, E^{W_x} \int_0^\infty \delta_0(x(s))\, e^{-\int_0^s V(\beta(r))\,dr}\, ds \\
&\geq E^{W_x} \int_0^\infty p^{(0)}(\beta(s))\, e^{-\int_0^s V(\beta(r))\,dr}\, ds \\
&= E^{W_x} \int_0^\infty ds \int_0^\infty dr\, E^{W_{\beta(s)}} \left[\delta_0(\beta(r))\, e^{-\int_0^r V(\beta(r'))\,dr'}\right] e^{-\int_0^s V(\beta(r'))\,dr'} \\
&= E^{W_x} \int_0^\infty ds \int_0^\infty dr\, E^{W_x} \left[\delta_0(\beta(r+s))\, e^{-\int_s^{r+s} V(\beta(r'))\,dr'} | \mathcal{F}_s\right] e^{-\int_0^s V(\beta(r'))\,dr'} \\
&= \int_0^\infty ds \int_0^\infty dr\, E^{W_x} \left[\delta_0(\beta(r+s))\, e^{-\int_0^{r+s} V(\beta(r'))\,dr'}\right] \\
&= \int_0^\infty ds \int_s^\infty dr\, E^{W_x} \left[\delta_0(\beta(r))\, e^{-\int_0^r V(\beta(r'))\,dr'}\right] \\
&= \int_0^\infty r\, E^{W_x} \left[\delta_0(\beta(r))\, e^{-\int_0^r V(\beta(r'))\,dr'}\right] dr \\
&= \int_0^\infty r\, p(r,x)\, dr,
\end{aligned}$$

where in the third line we used the form of the 0^{th} moment, as this was previously obtained, in the fourth line the Markov property and in the last Fubini's theorem. We can now check that $p^{(1)}(x) \equiv \frac{C_b^{(1)}}{(|x|+a)^{k_b-2}}$, with $C_b^{(1)} = \frac{-C_b^{(0)}}{(k_b-2)(k_b-1)-b}$ is a positive, bounded solution of (4.22), as long as $k_b > 2$ and $(k_b-2)(k_b-1)-b < 0$. Both conditions are satisfied when $b = 12$.

Remark: The motivation to look at this equation comes from the fact that the first moment, if it exists, will be the limit, as $\mu \to 0$, of $\int e^{-\mu s} s\, p(s,x)\, ds$. The last integral is the derivative with respect to μ of the Laplace transform of $p(s,x)$, and it can be easily checked that it satisfies the equation

$$\frac{\partial^2 p^{(1)}}{\partial x^2} - (\mu + V)p^{(1)} + p^{(0)} = 0.$$

The Dirac function on the right hand side of (4.22) appears because of the singularity of the second derivative of $\frac{C_b^{(1)}}{(|x|+a)^{k_b-2}}$ at 0, and the constant c is chosen so that $\frac{\partial p^{(1)}}{\partial x}|_{x=0^+} = -c/2$.

Higher Moments

To establish existence of higher moments we look at the equation

$$\frac{\partial^2 p^{(n)}}{\partial x^2} - Vp^{(n)} + np^{(n-1)} = -c_n\, \delta_0(x), \quad x \in \mathbb{R}, \tag{4.23}$$

where as before c_n is an appropriatelly chosen constant. The existence of a positive, bounded solution for this equation will guarantee that the n^{th} moment of $p(s,x)$ exists. The proof is by induction and follows exactly the same lines as in the treatment of the first moment, and so we will skip it.

The function $p^{(n)}(x) = \frac{C_b^{(n)}}{(|x|+a)^{k_b-2n}}$, with $C_b^{(n)} = \frac{-C_b^{(n-1)}}{(k_b-2n)(k_b-2n+1)-b}$, satisfies the above requirements as long as $k_b > 2n$ and $(k_b-2n)(k_b-2n+1)-b < 0$. The constant c_n is chosen so that $\frac{\partial p^{(n)}}{\partial x}|_{x=0} = -c_n/2$. In the case that $b = 12$, $k_b = 3$ the above conditions are satisfied for $n < 3/2$.

In the proof of the main estimate of Proposition 10 we will need the fact that the left hand side of (4.20) is integrable with respect to τ , and for this the existence of the first moment is enough.

Proposition 11 *Let $u^{(\tau)}(t,x)$ be defined by (4.18), and $\alpha^{(\tau)}(t,x)$ be defined by the formula $\alpha^{(\tau)}(t,x) = \sqrt{\frac{6}{u^{(\tau)}(t,x)}} - x$. Then for any x, the following estimates hold true:*

$$(i) \qquad \int_{\mathbb{R}_+} \sup_{\omega} \sup_{-\frac{\tau}{2}\le t\le \frac{\tau}{2}} |u_\omega(t,x) - u^{(\tau)}(t,x)|\, d\tau < \infty,$$

$$(ii) \qquad \int_{\mathbb{R}_+} \sup_{\omega} \sup_{-\frac{\tau}{2}\le t\le \frac{\tau}{2}} |\alpha_\omega(t,x) - \alpha^{(\tau)}(t,x)|\, d\tau < \infty.$$

Proof : (i) follows from (4.20) (which is uniform in ω) combined with the existence of the first moment of the heat kernel, and (ii) follows from (i) by noting that

$$\alpha_\omega(t,x) - \alpha^{(\tau)}(t,x) = \frac{\sqrt{6}\ (u^{(\tau)}(t,x) - u_\omega(t,x))}{\sqrt{u_\omega(t,x)\, u^{(\tau)}(t,x)}(\sqrt{u_\omega(t,x)} + \sqrt{u^{(\tau)}(t,x)})},$$

and that, for any fixed x, $u(t,x)$ and $u^{(\tau)}(t,x)$ are uniformly away from zero.

We will now concentrate on proving the condition of Proposition 10. It is not clear at first sight how we could estimate the decay of $\mathbb{E}[(\xi_\omega - \mathbb{E}[\xi_\omega|\mathcal{F}^\tau_{-\tau}])^2]$ via equation. (4.1). On the other hand we can use functional integration to obtain L^∞ estimates for the difference between two solutions of equation (4.1), corresponding to two different Dirichlet boundary data $a(\cdot, x_0)$ and $b(\cdot, x_0)$, as in Proposition 26, and then we could use Proposition 11 to measure the dependence of ξ_ω on the σ-algebra $\mathcal{F}^\tau_{-\tau}$. The connection between these two quantities is established in Proposition 12, but in order to state it we need the following construction:

Let us consider on $\{(t,0)\colon t \in \mathbb{R}\}$ a stochastic process ω, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let us denote the distribution of this process with respect to $\mathbb{P}$ by $\mu(d\omega)$. In our case ω will correspond to the process $\lambda_\omega(\cdot)$ and the measure $\mu(d\omega)$ will be such that, for every measurable function F, $\int F(\omega)\mu(d\omega) = \mathbb{E}[F(\lambda_\omega(\cdot))]$ (this is the reason why we use the same notation for the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ as before). Finally we denote by $\mathcal{F}_\sigma^\tau$ the σ-algebra generated by $\{\omega(s)\colon s \in (\sigma, \tau)\}$.

Let us now decompose ω into two parts $(\omega_0^\tau, \omega_1^\tau)$, where $\omega_0^\tau = \{\omega(s)\colon s \in (-2\tau, 2\tau)\}$ and $\omega_1^\tau = \{\omega(s)\colon s \in (-2\tau, 2\tau)^c\}$, and consider the stochastic process $(\omega_0^\tau, \omega_1^\tau, \tilde{\omega}_1^\tau)$, the distribution $\mu^{\otimes 2}$ of which is defined by

$$\mu^{\otimes 2}(d\omega_0^\tau, d\omega_1^\tau, d\tilde{\omega}_1^\tau) = \mu(d\omega_0^\tau)\, \mu(d\omega_1^\tau |\, \omega_0^\tau)\, \mu(d\tilde{\omega}_1^\tau |\, \omega_0^\tau).$$

In the above formula $\mu(d\omega_0^\tau)$ is the marginal distribution of ω_0^τ with respect to μ, and $\mu(d\omega_1^\tau |\, \omega_0^\tau)$ and $\mu(d\tilde{\omega}_1^\tau |\, \omega_0^\tau)$ are two copies of the conditional distribution of ω_1^τ given ω_0^τ.

Notice that, conditionally on ω_0^τ, the processes ω_1^τ and $\tilde{\omega}_1^\tau$ are independent. Moreover, the marginal distributions of $(\omega_0^\tau, \omega_1^\tau)$ and $(\omega_0^\tau, \tilde{\omega}_1^\tau)$ are the same and identical to μ.

The importance of this construction in our case lies in the following Proposition.

Proposition 12 *Let $\mathbb{E}$ denote the expectation with respect to $\mu(d\omega)$ and $\mathbb{E}^{\otimes 2}$ the expectation with respect to $\mu^{\otimes 2}$. Suppose that $\xi(\omega_0^\tau, \omega_1^\tau)$ is a measurable function with respect to $\mathcal{F}$. Then*

$$\mathbb{E}\left[\left(\xi - \mathbb{E}[\,\xi|\mathcal{F}_{-\tau}^\tau]\right)^2\right] = \frac{1}{2}\,\mathbb{E}^{\otimes 2}\left[(\xi(\omega_0^\tau, \omega_1^\tau) - \xi(\omega_0^\tau, \tilde{\omega}_1^\tau))^2\right].$$

Proof : The proof is given by a straightforward computation. More specifically, by expanding the right hand side and using the preceding comments we have that

$$\begin{aligned}
\mathbb{E}^{\otimes 2}\left[(\xi(\omega_0^\tau,\omega_1^\tau)-\xi(\omega_0^\tau,\tilde{\omega}_1^\tau))^2\right] &= 2(\int \mu(d\omega_0^\tau)\int \xi^2(\omega_0^\tau,\omega_1^\tau)\mu(d\omega_1^\tau|d\omega_0^\tau) \\
&\quad - \int \mu(d\omega_0^\tau)\int\int \xi(\omega_0^\tau,\omega_1^\tau)\,\xi(\omega_0^\tau,\tilde{\omega}_1^\tau)\mu(d\omega_1^\tau|d\omega_0^\tau)\mu(d\tilde{\omega}_1^\tau|d\omega_0^\tau)) \\
&= 2\,(\int \mu(d\omega_0^\tau)\int \xi^2(\omega_0^\tau,\omega_1^\tau)\mu(d\omega_1^\tau|\omega_0^\tau) \\
&\quad - \int \mu(d\omega_0^\tau)\,(\int \xi(\omega_0^\tau,\omega_1^\tau)\mu(d\omega_1^\tau|\omega_0^\tau))^2\,) \\
&= 2\int \mu(d\omega_0^\tau)\int(\xi(\omega_0^\tau,\omega_1^\tau)-(\int \xi(\omega_0^\tau,\omega_1^\tau)\mu(d\omega_1^\tau|\omega_0^\tau)\,)^2 \\
&= 2\,\mathbb{E}\,[(\xi(\omega_0^\tau,\omega_1^\tau)-\mathbb{E}[\xi(\omega_0^\tau,\omega_1^\tau)|\mathcal{F}_{-\tau}^\tau])^2],
\end{aligned}$$

and this completes the proof.

We are now heading towards the proof of the integrability of $\tau^\delta\,\mathbb{E}[(\xi_\omega(0)-\xi_\omega^\tau(0))^2]$, where we denote by

$$\xi_\omega^\tau(s) \equiv \mathbb{E}\left[\xi_\omega(s)|\mathcal{F}_{-\tau+s}^{\tau+s}\right]. \tag{4.24}$$

In order to achieve this, it will be necessary to obtain first an estimate of the form

$$\mathbb{E}\,[\,|a_\omega(\cdot,x)-\mathbb{E}a_\omega(\cdot,x)|] \le \frac{C}{\sqrt{x}}, \tag{4.25}$$

for some $C>0$. In fact, the L^∞ estimates of Proposition 26 are marginally insufficient to provide the required integrability, but on the other hand they are enough to establish (4.25). Once this estimate is established we can combine it with the L^∞ estimates to obtain the integrability.

Proposition 13 *Let ξ_ω^τ denote the random variable $\xi_\omega^\tau(0)$ as this is defined in (4.24), and suppose that there is a positive number δ, such that*

$$\int_{\mathbb{R}_+} \mathbb{E}[(\xi_\omega - \xi_\omega^\tau)^2\,]\,\frac{d\tau}{\tau^{(1-\delta)/2}} < \infty. \tag{4.26}$$

Then there is a positive constant C such that

$$\mathbb{E}[|a_\omega(\cdot,x) - \mathbb{E}a_\omega(\cdot,x)|] \leq \frac{C}{\sqrt{x}}.$$

Proof : By Proposition 3 we see that it suffices to show that

$$\mathbb{E}[(\int \tilde{\xi_\omega}(\tau)\, q_x(-\tau;x_0)\,d\tau)^2]^{\frac{1}{2}} \leq \frac{C}{\sqrt{x}},$$

where $\tilde{\xi_\omega}(\tau)$ denotes the centered random variable $\xi_\omega(\tau) - \mathbb{E}\xi_\omega(\tau)$. Following the same steps as in Proposition 10 we have that

$$\begin{aligned}
&\mathbb{E}[(\int \tilde{\xi_\omega}(\tau)\, q_x(-\tau;x_0)\,d\tau)^2]^{\frac{1}{2}} \\
&\leq \sum_k \left(\int \mathbb{E}[\,\tilde{\xi}_{\omega,\kappa}(\tau)\tilde{\xi}_{\omega,\kappa}(0)\,] \int q_x(-\tau';x_0)q_x(-\tau-\tau';x_0)\,d\tau'\,d\tau\right)^{\frac{1}{2}} \\
&\leq \sum_k \left(\int d\tau' q_x(-\tau';x_0)\cdot(\int \mathbb{E}[\tilde{\xi}_{\omega,\kappa}(\tau)\tilde{\xi}_{\omega,\kappa}(0)]^2 d\tau)^{\frac{1}{2}}\,(\int q_x^2(-\tau-\tau';x_0)\,d\tau)^{\frac{1}{2}}\right)^{\frac{1}{2}},
\end{aligned}$$

and by assumption (A) and the Cauchy-Schwarz inequality, this is bounded by

$$\begin{aligned}
&\sum_k \left(\int d\tau' q_x(-\tau';x_0)\cdot(\,(2^{k+1}+L)\mathbb{E}[\,(\tilde{\xi}_{\omega,\kappa}(0))^2]^2\,)^{\frac{1}{2}}\,(\int q_x^2(-\tau-\tau';x_0)\,d\tau)^{\frac{1}{2}}\right)^{\frac{1}{2}} \\
\leq\;& C\sum_k 2^{\frac{k}{4}}\,\mathbb{E}[\,(\tilde{\xi}_{\omega,\kappa}(0))^2]^{\frac{1}{2}}\left(\int d\tau' q_x(-\tau';x_0)\cdot(\int q_x^2(-\tau-\tau';x_0)\,d\tau)^{\frac{1}{2}}\right)^{\frac{1}{2}} \\
=\;& \frac{C}{\sqrt{x}}\sum_k 2^{\frac{k}{4}}\,\mathbb{E}[\,(\tilde{\xi}_{\omega,\kappa}(0))^2]^{\frac{1}{2}}\left(\int d\tau' q_1(-\tau';\frac{x_0}{x})\cdot(\int q_1^2(-\tau-\tau';\frac{x_0}{x})\,d\tau)^{\frac{1}{2}}\right)^{\frac{1}{2}}.
\end{aligned}$$

The result will be now implied as long as the last series converges. As in Proposition 10, we write the series in the form $\sum_k 2^k\,\frac{1}{2^{3k/4}}\,\mathbb{E}[\,(\tilde{\xi}_{\omega,\kappa}(0))^2]^{\frac{1}{2}}$, and we see that the convergence of this series is equivalent to the convergence of the integral $\int \mathbb{E}[(\xi_\omega - \xi_\omega^\tau)^2]^{\frac{1}{2}} \frac{d\tau}{\tau^{3/4}}$, and by the Cauchy-Schwarz inequality this will be implied by the convergence of the integral $\int \mathbb{E}[(\xi_\omega - \xi_\omega^\tau)^2] \frac{d\tau}{\tau^{(1-\delta)/2}}$, for some positive number δ, and the result follows.

In the next lemma we use Proposition 12 and simple manipulations on ξ_ω to bring the difference $\mathbb{E}[(\xi_\omega - \xi_\omega^\tau)^2]$ into a more neat form.

Lemma 9 *Let $a(\cdot, x) \equiv a_{(\omega_0^\tau, \omega_1^\tau)}(\cdot, x)$ and $b(\cdot, x) \equiv b_{(\omega_0^\tau, \omega_1^\tau)}(\cdot, x)$ denote two solutions of equation 4.3 corresponding to (possibly different) boundary data $\lambda_\omega(\cdot)$, which we denote by $\omega = (\omega_0^\tau, \omega_1^\tau)$, in accordance with the construction of Proposition 12. Then there is a positive constant C, such that for $r > 0$ (that will be chosen to be small),*

$$\begin{aligned} \mathbb{E}[(\xi_\omega - \xi_\omega^\tau)^2] \;&\le\; C\,\mathbb{E}^{\otimes 2} E^{Q_\infty^{0,x_0}} \left[\left| \int_{-\infty}^0 \frac{\|a-b\|_{L^\infty\left(B_{rx(s)}(s,x(s))\right)}}{(x(s))^3}\, ds \right|^2 \right] \\ &+ C\,\mathbb{E}^{\otimes 2}\left[\,|\, a(0,x_0) - b(0,x_0)\,|^2\,\right] \end{aligned}$$

The set $B_r(t,x)$ is defined to be the set $\{(s,y) \in \mathbb{R}^2 \colon s \in (t-r^2, t), y \in (x - r, x + r)\}$.

Proof : By Proposition 12 we have that $\mathbb{E}[(\xi_\omega - \xi_\omega^\tau)^2] = \frac{1}{2}\mathbb{E}^{\otimes 2}[(\xi(\omega_0^\tau, \omega_1^\tau) - \xi(\omega_0^\tau, \tilde{\omega}_1^\tau))^2]$. Recall that $\xi_\omega = a(0, x_0) E^{Q_\infty^{0,x_0}}[\, e^{\zeta_\omega(-\infty,0)}]$. Then we have the bound

$$\begin{aligned} \mathbb{E}[(\xi_\omega - \xi_\omega^\tau)^2] \;&\le\; \mathbb{E}^{\otimes 2}\left[\left(|\, a(0,x_0) - b(0,x_0)\,|\; E^{Q_\infty^{0,x_0}}\left[e^{\zeta(\omega_0^\tau, \omega_1^\tau)} \right] \right)^2 \right] \\ &+ \mathbb{E}^{\otimes 2}\left[\left(b(0,x_0)\, E^{Q_\infty^{0,x_0}}\left[|e^{\zeta(\omega_0^\tau,\omega_1^\tau)} - e^{\zeta(\omega_0^\tau,\tilde{\omega}_1^\tau)}| \right] \right)^2 \right] \\ &\le\; C\,\mathbb{E}^{\otimes 2}\left[\,|\, a(0,x_0) - b(0,x_0)\,|^2\right] \\ &+ C\,\mathbb{E}^{\otimes 2}\left[\left(E^{Q_\infty^{0,x_0}}\left[|e^{\zeta(\omega_0^\tau,\omega_1^\tau)} - e^{\zeta(\omega_0^\tau,\tilde{\omega}_1^\tau)}| \right] \right)^2 \right], \end{aligned}$$

where on the first term we used a similar estimate to that of Lemma 7 and Proposition 2, and on the second term the boundedness of $b(0,x_0)$. Further, the second term can be estimated by

$$
\begin{aligned}
& \mathbb{E}^{\otimes 2}\left[\left(E^{Q_\infty^{0,x_0}}\left[|e^{\zeta(\omega_0^\tau,\omega_1^\tau)}+e^{\zeta(\omega_0^\tau,\tilde\omega_1^\tau)}|\cdot|\zeta_{(\omega_0^\tau,\omega_1^\tau)}-\zeta_{(\omega_0^\tau,\tilde\omega_1^\tau)}|\right]\right)^2\right] \\
\leq\ & \mathbb{E}^{\otimes 2}\left[E^{Q_\infty^{0,x_0}}\left[|e^{\zeta(\omega_0^\tau,\omega_1^\tau)}+e^{\zeta(\omega_0^\tau,\tilde\omega_1^\tau)}|^2\right]\cdot E^{Q_\infty^{0,x_0}}\left[|\zeta_{(\omega_0^\tau,\omega_1^\tau)}-\zeta_{(\omega_0^\tau,\tilde\omega_1^\tau)}|^2\right]\right] \\
\leq\ & C\,\mathbb{E}^{\otimes 2}\left[E^{Q_\infty^{0,x_0}}\left[|\zeta_{(\omega_0^\tau,\omega_1^\tau)}-\zeta_{(\omega_0^\tau,\tilde\omega_1^\tau)}|^2\right]\right], \qquad (4.27)
\end{aligned}
$$

and after a simple manipulation

$$
\begin{aligned}
\zeta_{(\omega_0^\tau,\omega_1^\tau)}-\zeta_{(\omega_0^\tau,\tilde\omega_1^\tau)} &= \int_{-\infty}^0\left[-\frac{3a_x^2}{a(a+x)}+\frac{6a_x}{x(x+a)}+\frac{3b_x^2}{b(b+x)}-\frac{6b_x}{x(x+b)}\right]ds \\
= \int_{-\infty}^0 [\frac{3a_x}{(x+a)(x+b)}&\left(\frac{a+b+x}{ab}a_x-\frac{2}{x}\right)(a-b) \\
&-\frac{3}{x+b}\left(\frac{a_x+b_x}{b}-\frac{2}{x}\right)(a_x-b_x)]ds,
\end{aligned}
$$

where, as usual, the integrand is evaluated over the path $(s,x(s))$. Using the estimates in Proposition 25 it is easy to see that the last quantity is bounded by

$$
C\int_{-\infty}^0\frac{\|a-b\|_{L^\infty\left(B_{rx(s)}(s,x(s))\right)}}{(x(s))^3}\,ds,
$$

and the estimate we are after follows by inserting the previous quantity into (4.27).

We will now use this Lemma to verify that condition (4.26) is valid.

Proposition 14 *The estimate*

$$
\int_{\mathbb{R}_+}\mathbb{E}[(\xi_\omega-\xi_\omega^\tau)^2]\,\frac{d\tau}{\tau^{(1-\delta)/2}}<\infty,
$$

holds true, for δ sufficiently small.

Proof : Inequality (4.20) implies that the integral corresponding to the second term in Lemma 9 is finite.

Regarding the first term we multiply and divide the integrand with respect to x by $(1+|s|)^{\mu}$, for some $\mu > 1/2$, and using the Cauchy-Schwarz inequality we can bound it by

$$\int_{\mathbb{R}_+} \frac{d\tau}{\tau^{(1-\delta)/2}} \mathbb{E}^{\otimes 2} E^{Q_{\infty}^{0,x_0}} \left[\int_{-\infty}^{0} (1+|s|)^{2\mu} \frac{\|a-b\|^2_{L^\infty\left(B_{rx(s)}(s,x(s)) \right)}}{(x(s))^6} \, ds \right]. \quad (4.28)$$

By Fubini's Theorem, we need to control $\int_{\mathbb{R}_+} \frac{d\tau}{\tau^{(1-\delta)/2}} \|a-b\|^2_{L^\infty\left(B_{rx(s)}(s,x(s)) \right)}$, and for this we will use the L^∞ estimate of Proposition 26.

First, following the notation of that Proposition, let us denote by $\theta(t,x) \equiv a(t,x) - b(t,x)$, and let us introduce the level $x_0^r \equiv (1-2r)x_0$. Then, for any $\tau > 0$, coinciding with the one corresponding to the decomposition $(\omega_0^\tau, \omega_1^\tau)$, $t \in \mathbb{R}$, $x \geq x_0$ and t_1, $x_1 \in B_{rx}(t,x)$, we have, by Proposition 26, that

$$\begin{aligned} |\,\theta(t_1,x_1)\,| &\leq C \left(E^{Q_{x_1}} \left[|\,\theta(t_1+\tau_{x_0^r}, x_0^r)\,|^p \right] \right)^{\frac{1}{p}} \\ &\leq C \left(E^{Q_{x_1}} \left[|\,\theta(t_1+\tau_{x_0^r}, x_0^r)\,|^p ;\ \tau_{x_0^r} \in (-t_1-\tau, -t_1+\tau)^c \right] \right)^{\frac{1}{p}} \\ &\quad + C \left(E^{Q^{x_1}} \left[|\,\theta(t_1+\tau_{x_0^r}, x_0^r)\,|^p ;\ \tau_{x_0^r} \in (-t_1-\tau, -t_1+\tau) \right] \right)^{\frac{1}{p}} \\ &\leq C\, Q_{x_1}^{\frac{1}{p}} \left(\tau_{x_0^r} \in (-t_1-\tau, -t_1+\tau)^c \right) + C \sup_{-\tau \leq t \leq \tau} |\,\theta(t, x_0^r)\,|, \end{aligned}$$

or,

$$|\,\theta(t_1,x_1)\,|^2 \leq C\, Q_{x_1}^{\frac{2}{p}} \left(\tau_{x_0^r} \in (-t_1-\tau, -t_1+\tau)^c \right) + C \sup_{-\tau \leq t \leq \tau} |\,\theta(t, x_0^r)\,|^2.$$

The integral of the second term with respect to τ can be controlled by inequality (4.20) and Proposition 11. Regarding the integral of the first term, let us first of all note that for any x_1, $t_1 \in B_{rx}(t,x)$ we have that

$$Q_{x_1}\left(\tau_{x_0^r} > \tau - t_1 \right) \leq Q_{(1+r)x}\left(\tau_{x_0^r} > \tau - t_1 \right) \leq Q_{(1+r)x}\left(\tau_{x_0^r} > \tau - t \right),$$

and

$$Q_{x_1}\left(\tau_{x_0^r} <-\tau - t_1\right) \leq Q_{(1-r)x}\left(\tau_{x_0^r} <-\tau - t_1\right) \leq Q_{(1-r)x}\left(-\tau_{x_0^r} <-\tau - t + r^2\, x^2\right).$$

In other words the supremum over $x_1, t_1 \in B_{rx}\,(t,x)$ of each of the above two quantities is achieved on a corner of the parabolic cube $B_{rx}\,(t,x)$. Moreover, the last inequality, when evaluated at $x = x_0$, makes clear the reason why we introduced the level x_0^r.

We can now bound the integral $\int_{\mathbb{R}_+} \frac{d\tau}{\tau^{(1-\delta)/2}} \left\|a-b\right\|^2_{L^\infty\left(\,B_{rx(s)}(s,x(s))\,\right)}$ by

$$\begin{aligned}
& C\int_{\mathbb{R}_+} Q^{\frac{2}{p}}_{(1+r)x(s)}(\tau_{x_0^r} > \tau - s)\,\frac{d\tau}{\tau^{(1-\delta)/2}} \\
& \quad +C\int_{\mathbb{R}_+} Q^{\frac{2}{p}}_{(1-r)x(s)}(\tau_{x_0^r} < -\tau - s + r^2\,(x(s))^2)\,\frac{d\tau}{\tau^{(1-\delta)/2}} \\
= \; & C\int_{\mathbb{R}_+} Q^{\frac{2}{p}}_{1+r}\left(\tau_{\frac{x_0^r}{x(s)}} > \frac{\tau}{(x(s))^2} - \frac{s}{(x(s))^2}\right)\frac{d\tau}{\tau^{(1-\delta)/2}} \\
& \quad +C\int_{\mathbb{R}_+} Q^{\frac{2}{p}}_{1-r}\left(\tau_{\frac{x_0^r}{x(s)}} < -\frac{\tau}{(x(s))^2} - \frac{s}{(x(s))^2} + r^2\right)\frac{d\tau}{\tau^{(1-\delta)/2}} \\
= \; & C\,(x(s))^{1+\delta}\int_{\mathbb{R}_+} Q^{\frac{2}{p}}_{1+r}\left(\tau_{\frac{x_0^r}{x(s)}} > \tau - \frac{s}{(x(s))^2}\right)\frac{d\tau}{\tau^{(1-\delta)/2}} \\
& \quad +C\,(x(s))^{1+\delta}\int_{\mathbb{R}_+} Q^{\frac{2}{p}}_{1-r}\left(\tau_{\frac{x_0^r}{x(s)}} < -\tau - \frac{s}{(x(s))^2} + r^2\right)\frac{d\tau}{\tau^{(1-\delta)/2}} \\
\leq \; & C\,(x(s))^{1+\delta}\int_{\mathbb{R}_+} Q^{\frac{2}{p}}_{1+r}\left(\tau_{\frac{x_0}{x(s)}} > \tau\right)d\tau \\
& \quad +C\,(x(s))^{1+\delta}\int_0^{-\frac{s}{(x(s))^2}+r^2} Q^{\frac{2}{p}}_{1-r}\left(\tau_{\frac{x_0^r}{x(s)}} < -\tau - \frac{s}{(x(s))^2} + r^2\right)\frac{d\tau}{\tau^{(1-\delta)/2}} \\
\leq \; & C\,(x(s))^{1+\delta}\left(1 - \frac{s}{(x(s))^2} + r^2\right),
\end{aligned}$$

where in the last line we used the fact that $Q^{\frac{2}{p}}_{1+r}\left(\tau_{\frac{x_0^r}{x(s)}} > \tau\right)$ is integrable and that $Q^{\frac{2}{p}}_{1-r}\left(\tau_{\frac{x_0^r}{x(s)}} < -\tau - \frac{s}{(x(s))^2} + r^2\right) \leq 1$. Inserting this estimate into (4.28), we see that the latter is bounded by

$$CE^{Q_\infty^{0,x_0}} \int_{\mathbb{R}_-} \frac{(1+|s|)^{2\mu}}{(x(s))^{5-\delta}} \left(1 - \frac{s}{(x(s))^2} + r^2\right) ds$$
$$= C \int_{x_0}^{\infty} \frac{dy}{y^{5-\delta}} \int_{\mathbb{R}_-} (1+|s|)^{2\mu} \left(1 - \frac{s}{y^2} + r^2\right) q_\infty(s, y \,|\, 0, x_0)\, ds,$$

The marginal density $q_\infty(s, y \,|\, 0, x_0)$ is computed in Proposition 22, and so the last quantity is equal to

$$C \int_{x_0}^{\infty} \frac{dy}{y^{5-\delta}} (y - x_0) \int_{\mathbb{R}_-} (1+|s|)^{2\mu} \left(1 - \frac{s}{y^2} + r^2\right) q_y(-s; x_0)\, ds.$$

Now, using the scaling and a simple change of variable, we see that the last quantity is controlled by

$$C \int_{x_0}^{\infty} \frac{dy}{y^{5-\delta}}\, y^{4\mu} (y - x_0) \int_{\mathbb{R}_+} s^{2\mu} (1+s)\, q_1(-s; \frac{x_0}{y})\, ds.$$

This last integral is convergent, since μ can be chosen to be arbitrarily close to 1/2 (but larger than 1/2), and $q_1(s; \frac{x_0}{y}) = O(s^{-9/2})$. This completes the proof.

Now, that we know that (4.25) is valid, we will use it to prove the main estimate of Proposition 10. This is done in the following proposition.

Proposition 15 *The estimate*

$$\int_{\mathbb{R}_+} \tau^\delta\, \mathbb{E}[(\xi_\omega - \mathbb{E}\xi_\omega^\tau)^2]\, d\tau < \infty,$$

holds true, for δ sufficiently small.

Proof : Following the same steps as in the first part of the proof of Proposition 14, it follows that what we need to control is the integral

$$\int_{\mathbb{R}_+} \tau^\delta \mathbb{E}^{\otimes 2}\, E^{Q_\infty^{0,x_0}} \left[\int_{\mathbb{R}_-} s^{2\mu} \frac{\|a - b\|^2_{L^\infty\left(B_{rx(s)}(s, x(s))\right)}}{(x(s))^6}\, ds \right] d\tau$$

where μ is chosen again to be an arbitrary number greater than $1/2$. Further, we estimate one power of $\|a-b\|^2_{L^\infty\left(B_{rx(s)}(s,x(s))\right)}$ by the L^∞ estimate of Proposition 26, and the other one using (4.25) so that

$$\leq \int_{\mathbb{R}_+} d\tau\, \tau^\delta E^{Q^{0,x_0}_\infty} \int_{\mathbb{R}_-} \sup_{(\omega^\tau_0,\omega^\tau_1,\tilde{\omega}^\tau_1)} \|a-b\|_{L^\infty\left(B_{rx(s)}(s,x(s))\right)} \cdot$$

$$\cdot \mathbb{E}^{\otimes 2}\left[\, \|a-b\|_{L^\infty\left(B_{rx(s)}(s,x(s))\right)}\right] \frac{s^{2\mu}}{(x(s))^6}\, ds. \quad (4.29)$$

The term $\sup_{(\omega^\tau_0,\omega^\tau_1,\tilde{\omega}^\tau_1)} \|a-b\|_{L^\infty\left(B_{rx(s)}(s,x(s))\right)}$ can be controlled as in the previous Proposition, using the L^∞ estimate of Proposition 26. To control the term $\mathbb{E}^{\otimes 2}\left[\, \|a-b\|_{L^\infty\left(B_{rx(s)}(s,x(s))\right)}\right]$, let us recall that $a-b$ solves equation (B.4). Standard parabolic PDE estimates (see for example [L], pg.120) guarantee that

$$\|a-b\|_{L^\infty(B_{rx}(s,x))} \leq \frac{C}{|\, B_{\frac{3}{2}rx}(s,x)\,|} \int_{B_{\frac{3}{2}rx}(s,x)} |\, a(\sigma,y)-b(\sigma,y)\,|\, d\sigma dy. \quad (4.30)$$

Recall, also, that under $\mu^{\otimes 2}$, $(\omega^\tau_0,\omega^\tau_1)$ and $(\omega^\tau_0,\tilde{\omega}^\tau_1)$ have the same distribution, and thus $\mathbb{E}^{\otimes 2}[\, a(s,x(s))\,] = \mathbb{E}^{\otimes 2}[\, b(s,x(s))\,] = \mathbb{E}[\, a_\omega(\cdot,x(s))\,]$. This, combined with (4.25), implies that

$$\mathbb{E}^{\otimes 2}\left[\, |a(\sigma,y)-b(\sigma,y)|\,\right] \leq 2\,\mathbb{E}\left[|a_\omega(\cdot,y)-\mathbb{E}a_\omega(\cdot,y)|\,\right] \leq \frac{2C}{\sqrt{y}}.$$

Since this is true for any $\sigma, y \in B_{\frac{3}{2}rx}(s,x)$, (4.30) implies that

$$\mathbb{E}^{\otimes 2}\left[\, \|a-b\|_{L^\infty(B_{rx}(s,x))}\right] \leq \frac{C}{\sqrt{x}}.$$

for some positive constant C. Inserting this estimate into (4.29) we see that it is bounded by

$$C\, E^{Q^{0,x_0}_\infty} \int_{\mathbb{R}_-} ds\, \frac{|s|^{2\mu}}{(x(s))^{13/2}} \int_{\mathbb{R}_+} \tau^\delta \sup_{(\omega^\tau_0,\omega^\tau_1,\tilde{\omega}^\tau_1)} \|a-b\|_{L^\infty\left(B_{rx(s)}(s,x(s))\right)}\, d\tau,$$

and, as in Proposition 14, this is bounded by

$$
\begin{aligned}
& C\, E^{Q_\infty^{0,x_0}} \int_{\mathbb{R}_-} \frac{|s|^{2\mu}}{(x(s))^{(9-4\delta)/2}} \left(1 - \frac{s}{(x(s))^2} + r^2\right) ds \\
= \;& C \int_{x_0}^{\infty} \frac{dy}{y^{\,(9-4\delta)/2}}\, (y - x_0) \int_{\mathbb{R}_+} s^{2\mu} \left(1 + \frac{s}{y^2} + r^2\right) q_y(s; x_0)\, ds \\
= \;& C \int_{x_0}^{\infty} \frac{dy}{y^{\,(9-4\delta)/2}}\, (y - x_0) y^{4\mu} \int_{\mathbb{R}_+} s^{2\mu} (1 + s + r^2)\, q_1(s; \frac{x_0}{y})\, ds.
\end{aligned}
$$

The last integral is convergent, since μ can be taken arbitrarily close to $1/2$, and δ sufficiently small.

We have now proved that the CLT is valid for $\alpha_\omega(\cdot, x)$. In other words,

Theorem 6 *Assume that the stationary, ergodic process $\lambda_\omega(\cdot)$ satisfies assumption (A), and that there are constants λ_1, λ_2 such that $0 < \lambda_1 \leq \lambda(\cdot) \leq \lambda_2 < \infty$. If $\alpha_\omega(t,x)$ is defined from the solution of equation (1.1) by the relation $u_\omega(t,x) = \frac{6}{(x+\alpha_\omega(t,x))^2}$, then for any fixed $t \in \mathbb{R}$, $x\,(\,\alpha_\omega(t,x) - \mathbb{E}\,\alpha_\omega(t,x)\,)$ converges in distribution, as x tends to infinity, to a Gaussian random variable. By stationarity, the distribution of the limiting random variable is independent of t.*

Remark : Let us mention that the cited proposition in [HH], regarding the validity of the CLT for stationary processes with short range correlations , corresponds to standard averages of the form $\frac{1}{x}\int_0^x \zeta_\omega(\tau)\, d\tau$, while we are interested in averages of the form $\int \zeta_\omega(\tau)\, q_x(\tau)\, d\tau$ ($\zeta_\omega(\cdot)$ is a mean zero stationary process and $q_x(\tau)$ a probability density). However, an inspection of the proof of that proposition shows that the CLT is also valid when dealing with the latter form of average. This is because the proof is based on constructing a martingale approximation. That is, $\zeta_\omega(\tau)$ is written in the form $z(\tau) + \eta(\tau)$, where $\{z(\tau)\colon \tau \in \mathbb{R}\}$ is a martingale difference and, $\eta(\tau)$ is such that $\frac{1}{\sqrt{x}}\int_0^x \eta(\tau)\, d\tau$ is negligible in the limit $x \to \infty$. An inspection, now, of the CLT for martingales shows that it is also valid for $x \int z(\tau)\, q_x(\tau)\, d\tau$, as long as $x^2 \int q_x^2(\tau)\, d\tau$ converges to a strictly positive number. On the other hand, an integration by parts, as in the Step 2 of the proof of the LLN in section 4.1, shows that $x \int \eta_\omega(\tau)\, q_x(\tau)\, d\tau$ is negligible in the limit $x \to \infty$, if $\frac{1}{\sqrt{x}} \int_0^x \eta(\tau)\, d\tau$ is so.

Appendix A

Bessel Processes & Entrance Laws.

A basic ingredient in our approach is the process with generator

$$\mathcal{L} = \frac{\partial^2}{\partial x^2} - \frac{6}{x}\frac{\partial}{\partial x}, \qquad x > 0.$$

We denote the measure corresponding to this process staring at time 0, from position $x > 0$ by Q_x. This process is a particular case of a Bessel process, i.e. process with generator

$$\mathcal{L} = \frac{\partial^2}{\partial x^2} + \frac{(2\nu + 1)}{x}\frac{\partial}{\partial x}, \qquad x > 0.$$

Our case corresponds to $\nu = -7/2$. Bessel processes have been widely studied. For an overview of their properties one can consult [BS]. This will also be our main reference for this section. We will not give an extensive account of their properties here, but rather concentrate on those in relation to our PDE. One objective is to present the necessary background that will lead to the proof of the estimate in Proposition 17. Furthermore, we will define and study the basic properties of two more processes that emerge from the Bessel process, and play a central role in our approach. The first one is the Bessel process starting at time σ from position x and conditioned to first hit level x_0 at time τ. We will slightly abuse the common terminology and call this process Bessel Bridge, and we will denote it by $Q^{\tau,x_0}_{\sigma,x}$. We will also suppress the index σ whenever it is equal to 0. The second one is the entrance law of a Bessel process starting at infinity and conitioned to first hit x_0 at time τ. We will denote this process by Q^{τ,x_0}_{∞}.

A.1 Bessel Process

Starting with the Bessel process, let us first note the singular behaviour of it regarding the boundary point 0, the nature of which depends on the parameter ν. In our case, $\nu = -7/2$, and in general for $\nu \leq -1$, 0 is an exit but a not entrance boundary. This means that we can only start the diffusion at $x > 0$, and run it until it hits 0, which happens with probability 1, and after that the diffusion dies.

The transition density $q^{(\nu)}(t,x,y)$ of the Bessel process has been computed and is

$$C_\nu \frac{1}{2t}(xy)^{-\nu} \exp\left(-\frac{x^2+y^2}{4t}\right) I_{|\nu|}\left(\frac{xy}{2t}\right) y^{2\nu+1}\, 1_{xy>0}, \tag{A.1}$$

where I_ν are the modified Bessel functions of first order, and C_ν is a constant. For convenience we will drop the index ν from the densities.

One thing that needs to be stressed is that Bessel processes possess the same scaling properties as Brownian motion. This is immediatelly seen if ν is a positive integer, since then Bessel process is just the modulus of a $(2\nu+2)$-

dimensional Brownian motion. In the general case this can be also easily verified via the generator.

A particularly important quantity for our problem, is the density of the hitting time $\tau_{x_0}^x$ of some level x_0, for the process starting at x, which we will denote by $q_x(t;x_0)$ (for convenience we will drop the sup-index x whenever it is clear where the process starts from). Notice that for $\nu < 0$, the hitting time of level $x_0 < x$ is almost surely finite, and this is the case we will focus on.

The Laplace transform of this density can be easily computed and is

$$\frac{x^{-\nu}K_\nu(2\sqrt{\lambda}x)}{x_0^{-\nu}K_\nu(2\sqrt{\lambda}x_0)}, \tag{A.2}$$

where K_ν is the modified Bessel function of second order. On the other hand, the inverse of this transform does not have a concrete expression in general. An exception is the case when x_0 is 0. In this case the density can be written explicitly and is

$$q_x(t;0) = c_\nu \frac{x^{2|\nu|}}{t^{|\nu|+1}} \exp\left(-\frac{x^2}{4t}\right), \qquad t > 0 \tag{A.3}$$

where c_ν is a normalising constant.

The setting of our problem, though, requires some computations involving $q_x(t;x_0)$ for $x_0 \neq 0$. In this case it will be first of all helpful to have in mind that by scaling

$$q_x(t;x_0) = \frac{1}{x^2} \cdot q_1\left(\frac{t}{x^2};\frac{x_0}{x}\right) \tag{A.4}$$

This property reduces the estimates involving $q_x(t;x_0)$, with $0 < x_0 < x$, to estimates involving $q_1(t;x_0)$, with $0 < x_0 < 1$. Moreover, we can express the Fourier transform $\hat{q}_1(\kappa;x_0)$ of $q_1(t;x_0)$ in terms of the Fourier transform $\hat{q}_1(\kappa;0)$ of $q_1(t;0)$. To see this note that $\tau_0^1 = \tau_{x_0}^1 + \tau_0^{x_0}$. Then the strong Markov property implies that $\hat{q}_1(\kappa;0) = \hat{q}_1(\kappa;x_0)\,\hat{q}_{x_0}(\kappa;0)$, and by scaling it is easy to see that this is equal to $\hat{q}_1(\kappa;x_0)\,\hat{q}_1(x_0^2\,\kappa;0)$.

The distribution of τ_0^1 is infinitely divisible. This is because , for any sequence of levels $0 < x_1 < \cdots < x_n < 1$, $\tau_{x_0}^1 = \tau_{x_n}^1 + \cdots + \tau_0^{x_1}$. Since the Bessel process is not translation invariant, the random variables $\tau_{x_{i-1}}^{x_i}$ do not have the same distribution but, nevertheless, they are independent, and this establishes the infinite divisibility. The Lévy-Khintchine representation is

$$\hat{q}_1(\kappa;0) = \exp\left(\int \frac{e^{i\kappa\lambda} - 1}{\lambda} m(d\lambda)\right), \tag{A.5}$$

where the measure $m(d\lambda)$ is concentrated on the the interval $(0,\infty)$. The Lévy-Khintchine representation has this special form, because the distribution $q_1(t;0)$ is concentrated on the positive real numbers. Moreover the absence of the centering, $ib\kappa$, in the expression is because $\inf\{t\colon q_1(t;0) > 0\} = 0$. The positivity of the density $q_1(t;0)$ also guarantees that $m(d\lambda)$ assigns finite mass to finite intervals (see [F], pg 570-572). What is more important to our approach is the fact that the asymptotics of the tail of the measure $m(d\lambda)$ are related to the asymptotics of the tail of the density $q_1(t;0)$. More precisely,

Proposition 16 *Let $m(d\lambda)$ be the Lévy measure of the Fourier transform of $q_1(t;0)$, as this is defined in (A.5). Then $\int m(d\lambda)$ is finite.*

Proof: By (A.3) it is clear that $E^{Q_1}\tau_0$ is finite and it is equal to $-i\,\frac{\partial \hat{q}_1(\kappa;0)}{\partial\kappa}|_{\kappa=0}$. But by inspection of formula (A.5) we see that $-i\,\frac{\partial\,\hat{q}_1(\kappa;0)}{\partial\kappa}|_{k=0} = \int m(d\lambda)$, and so the result follows.

Let us remark that, at first sight, this digression, regarding the Lévy-Khintchine representation of $\hat{q}_1(\kappa;0)$, might seem redundant, since we have at hand an explicit formula of it, as given in (A.2), up to replacing λ with $-i\lambda$. The reason we have chosen to do so, is that computations involving Bessel functions are tedious, while the resource of the Lévy-Khintchine representation provides a good simplification.

We now have the basic tools to prove the main estimate of Proposition 17. First, we need the following two lemmas.

Lemma 10 *Let x be any arbitrary number between* 0 *and* 1/2*. Moreover, let $q_1(t;x)$, $q_1(t;0)$ be defined as above, and $\hat{q}_1(\kappa;x)$, $\hat{q}_1(\kappa;0)$ denote their respective Fourier transforms. Then there exists a positive constant C, such that*

$$|q_1(\cdot;x) - q_1(\cdot;0)|_{L^1} \leq C\left(|\frac{\partial}{\partial\kappa}(\hat{q}_1(\cdot;x) - \hat{q}_1(\cdot;0))|_{L^2} + |\hat{q}_1(\cdot;x) - \hat{q}_1(\cdot;0)|_{L^2}\right).$$

Proof: Let us multiply and divide the difference by $t+1$, and use Cauchy-Schwarz inequality. The decay of the tails of the densities guarantees that the right hand side of the following inequality is well defined:

$$\begin{aligned}
|\, q_1(\cdot;x) - q_1(\cdot;0)|_{L^1} &\leq \left(\int \frac{dt}{(t+1)^2}\right)^{1/2} \left(\int (t+1)^2 |q_1(t;x) - q_1(t;0)|^2 \, dt\right)^{1/2} \\
&\leq C\left(\int t^2 |q_1(t;x) - q_1(t;0)|^2 \, dt\right)^{1/2} \\
&\quad + C\left(\int |q_1(t;x) - q_1(t;0)|^2 \, dt\right)^{1/2} \\
&= C\left(|\frac{\partial}{\partial\kappa}(\hat{q}_1(\cdot;x) - \hat{q}_1(\cdot;0))|_{L^2} + |\,\hat{q}_1(\cdot;x) - \hat{q}_1(\cdot;0)|_{L^2}\right),
\end{aligned}$$

where in the last step we used the fact that for a function f, $|tf(t)| = |\left(\frac{\partial}{\partial\kappa}\hat{f}(\kappa)\right)\check{}(t)\,|$, in combination with Parseval's identity.

Lemma 11 *Following the notation of Lemma 10, there is a constant C such that for any $0 < x < 1/2$*

$$|\frac{\partial}{\partial\kappa}(\hat{q}_1(\cdot;x) - \hat{q}_1(\cdot;0))\,|_{L^2} \leq C\,x^2.$$

Proof: The identity $\hat{q}_1(\kappa;0) = \hat{q}_1(\kappa;x)\,\hat{q}_1(x^2\,\kappa;0)$, and the triangle inequality imply that

$$\begin{aligned} |\,\frac{\partial}{\partial\kappa}(\hat{q}_1(\kappa;x) - \hat{q}_1(\kappa;0))\,| &\leq |\,\frac{\partial \hat{q}_1(\kappa;0)}{\partial\kappa}\left(\hat{q}_1(x^2k;0)^{-1} - 1\right)| \\ &+ |\,\hat{q}_1(\kappa;0)\frac{\partial}{\partial\kappa}\left(\hat{q}_1(x^2\kappa;0)^{-1}\right)|. \end{aligned}$$

By (A.5) it is easy to check that $\frac{\partial}{\partial\kappa}(\hat{q}_1(\kappa;0))^{\pm1} = \pm i\left(\int e^{i\kappa\lambda}m(d\lambda)\right)\hat{q}_1(\kappa;0)^{\pm1}$. This, along with the inequality

$$\left|\exp\{\pm\int\frac{e^{ix^2\kappa\lambda}-1}{\lambda}m(d\lambda)\} - 1\right| \leq \left|\int\frac{e^{ix^2\kappa\lambda}-1}{\lambda}m(d\lambda)\right| |1 + \hat{q}_1(x^2\kappa;0)^{\pm1}|,$$

yields the bound

$$\begin{aligned} &\int m(\lambda)\,d\lambda\left|\int\frac{e^{ix^2\,\kappa\lambda}-1}{\lambda}m(d\lambda)\right|\,|\,\hat{q}_1(\kappa;0)\,|\,\left|1 + \hat{q}_1^{-1}(x^2\kappa;0)\right| \\ &\qquad + x^2\,|\,\hat{q}_1(\kappa;0)\;\hat{q}_1^{-1}(x^2\kappa;0)\,|\int m(d\lambda) \\ &\leq\; x^2\kappa\left(\int m(d\lambda)\right)^2(|\,\hat{q}_1(\kappa;0)\,| + |\,\hat{q}_1(\kappa;x)\,|) + x^2\,|\,\hat{q}_1(\kappa;x)\,|\int m(d\lambda), \end{aligned}$$

where in the last line we used again the identity $\hat{q}_1(\kappa;0) = \hat{q}_1(\kappa;x)\,\hat{q}_1(x^2\,\kappa;0)$.

Since $m(d\lambda)$ is integrable, what we need in order to finish the proof is to check that the L^2 norms of $\kappa|\,\hat{q}(\kappa;0)\,|$, $\kappa|\,\hat{q}_1(\kappa;x)\,|$ and $|\,\hat{q}_1(\kappa;x)\,|$ are finite. Parseval's identity takes care of the first quantity, since the L^2 norm of it equals the L^2 norm of $\partial\,q_1(t;0)/\partial\,t$, and the result follows by (A.3). For the other two we will use (A.2), where we replace λ by $-i\kappa$. Noting that

$$K_{7/2}(z) = \sqrt{\frac{\pi}{2z}}e^{-z}\left(\frac{15}{z^3} + \frac{15}{z^2} + \frac{6}{z} + 1\right),$$

we have that

$$\hat{q}_1(\kappa;x) = e^{-(1-x)z}\,\frac{15 + 15z + 6z^2 + z^3}{15 + 15xz + 6x^2z^2 + x^3z^3},$$

with $z = \sqrt{2\kappa}\,(1-i)$. Since $x < 1/2$, we see that $|\,\hat{q}_1(\kappa; x)\,|$ has exponential decay as $k \to \infty$. Also it is not hard to see that $\hat{q}_1(\kappa; x)$ has no pole of the form $\sqrt{2\kappa}\,(1-i)$. In particular, if we denote by z_1, z_2, z_3, the roots of the polynomial $z^3 + 6z^2 + 15z + 15$, we see that no one of them lies on the contour $z = \sqrt{2\kappa}\,(1-i), \kappa \in \mathbb{R}$. The roots of the polynomial $x^3z^3 + 6x^2z^2 + 15xz + 15$ will be of the form $z_i^* = x^{-1}z_i$, for $i = 1, 2, 3$, and this means that the distance of the roots z_i^* from the contour $z = \sqrt{2\kappa}\,(1-i), \kappa \in \mathbb{R}$, increases as $x \to 0$. Since we are interested in the case that $x < 1/2$ (and in particular in the case that $x \to 0$) we see that the denominator of $\hat{q}_1(\kappa; x)$, $\kappa \in \mathbb{R}$, is uniformly bounded away from 0.

This implies that $\kappa|\,\hat{q}_1(\kappa; x)\,|$ and $|\,\hat{q}_1(\kappa; x)\,|$ have finite L^2 norms.

Lemma 12 *Following the notation and the assumptions of Lemma 10, there is a constant C such that, for any x between 0 and $1/2$*

$$|\,\hat{q}_1(\cdot; x) - \hat{q}_1(\cdot; 0)|_{L^2} \leq Cx^2.$$

Proof: The identity $\hat{q}_1(\kappa; 0) = \hat{q}_1(\kappa; x)\,\hat{q}_1(x^2\,\kappa; 0)$ implies that

$$|\hat{q}_1(\kappa; x) - \hat{q}_1(\kappa; 0)| = \hat{q}_1(\kappa; 0)\,|\,\hat{q}_1(x^2\kappa; 0)^{-1} - 1|.$$

The rest follows the steps of Lemma 11.

Proposition 17 *For arbitrary $0 < \tau_1$ and $0 < x_0 < x_1 < \frac{1}{2}x$, there are constants c_1, c_2, such that*

$$\int |q_x(\tau - \tau_1; x_1) - q_x(\tau; x_0))|d\tau \leq c_1\,(\frac{x_1}{x})^2 + c_2\,(\frac{\tau_1}{x^2}).$$

Proof: By the scaling property and a change of variables in the integral, what we need to estimate is $\int |q_1(\tau - \tau_1/x^2; x_1/x) - q_1(\tau; x_0/x)|d\tau$. Further, add and subtract in the difference, the quantity $q_1(\tau;0)$, and use the triangle inequality. Then the integral of the difference $|q_1(\tau;x_0/x)-q_1(\tau;0)|$ is directly estimated by Lemmas (10), (11), and (12). For the integral of the difference $|q_1(\tau-\tau_1/x^2;x_1/x)-q_1(\tau;0)|$, the same method works after some small changes. In particular, the only thing we need to note is that the Fourier transform of $q_1(\cdot-\tau_1/x;x_1/x)$ is $e^{-i\tau_1\kappa/x^2}\cdot\hat{q}_1(\kappa;0)/\hat{q}_1((x_1^2\kappa/x^2;0)$. The differentiation, as in Lemma (11), of the exponential will give the extra term τ_1/x^2.

We will finish this paragraph by computing the Green's function for the Bessel process $-7/2$ in a domain $\{(t,x)\colon x>x_0>0,\ t\in\mathbb{R}\}$.

Proposition 18 *The Green's funtion for the Bessel process* $-7/2$, $G(x,y;x_0)=\int Q_x(x(s)=y;\tau_{x_0}>s)\,ds$, *in a domain* $\{(t,x)\colon x>x_0>0,\ t\in\mathbb{R}\}$ *is equal to*

$$y^{-6}((x\wedge y)^7-x_0^7)\,1_{\{y>x_0\}}$$

Proof: Consider the boundary value problem

$$\begin{aligned} u_{xx}-\frac{6}{x}u_x+f(x)=0, &\qquad x>x_0, t\in\mathbb{R}, \\ u(\cdot,x_0)\equiv 0, &\qquad t\in\mathbb{R}, \end{aligned}$$

where f is an arbitrary bounded function. The solution of this problem has the representation

$$\begin{aligned} u(x) &= E^{Q_x}\int_0^{\tau_{x_0}} f(x(s))\,ds \\ &= \int_{x_0}^{\infty} f(y)\int Q_x(x(s)=y;\tau_{x_0}>s)\,ds\,dy. \end{aligned}$$

On the other hand the boundary value problem is an ODE which we can solve and yields the solution $\int_{x_0}^{x} z^6\int_z^\infty y^{-6}f(y)\,dy$.

The result now follows by using Fubini's theorem in the last expression and comparing with the first.

A.2 Bessel Bridge

We will call Bessel Bridge $Q^{\tau,x_0}_{\sigma,x}$ the Bessel process starting at some time σ at some position $x > 0$ and conditioned to first hit $x_0 > 0$ at time $\tau > \sigma$.

The Bessel Bridge can be defined using Doob's h-transform (see [BS]) , as the Markov process in the time interval (σ, τ) with marginal distribution at time σ, $Q^{\tau,x_0}_{\sigma,x}(x(\sigma) \in dy) = \delta_x(dy)$, and transition probabilities

$$q^{\tau,x_0}(t_1, y_1; t_2, y_2) \equiv Q_{y_1}\left(x(t_2 - t_1) = y_2; \tau_{x_0} > t_2 - t_1\right) \frac{q_{y_2}(\tau - t_2; x_0)}{q_{y_1}(\tau - t_1; x_0)}, \quad \text{(A.6)}$$

where $\sigma < t_1 < t_2 < \tau$ and $Q_{y_1}\left(x(t_2 - t_1) = y_2; \tau_{x_0} > t_2 - t_1\right)$ is the transition density for the Bessel process killed at x_0.

The function $(t, x) \to q_x(t; x_0)$ is harmonic for the generator $-\frac{\partial}{\partial t} + \frac{\partial^2}{\partial x^2} - \frac{6}{x}\frac{\partial}{\partial x}$, in the domain $x > x_0, t \in \mathbb{R}$, and twice the logarithmic derivative of it will give the extra drift that needs to be added to the Bessel process, in order to produce the Bessel Bridge. In summary, the generator of the Bessel Bridge is

$$\mathcal{L}^{\,\tau,x_0} = \frac{\partial^2}{\partial x^2} + \left(-\frac{6}{x} + 2\frac{\partial}{\partial x}\log q_x(\tau - t; x_0)\right)\frac{\partial}{\partial x}, \qquad x > x_0,\ \sigma < t < \tau. \quad \text{(A.7)}$$

Since $q_x(t; x_0)$ does not have a concrete expression, we cannot write a more concrete formula for the generator, except when $x_0 = 0$, when the generator takes the form:

$$\frac{\partial^2}{\partial x^2} + \left(\frac{8}{x} - \frac{x - x_0}{\tau - t}\right)\frac{\partial}{\partial x}, \qquad x > x_0,\ \sigma < t < \tau. \quad \text{(A.8)}$$

We will denote by $q^{\tau,x_0}_{\sigma,x}(t_1; x_1)$ the density of the hitting time of the level x_1 for the Bessel bridge. By the strong markov property it is easy to see that this is equal to

$$\frac{q_x(t_1 - \sigma; x_1)\, q_{x_1}(\tau - t_1; x_0)}{q_x(\tau - \sigma; x_0)} \quad \text{(A.9)}$$

A.3 Entrance Law

In the representation (4.11) of the limit of the solution to our PDE there appears a functional with respect to a Bessel process starting at infinity and conditioned to first hit level x_0 at time 0, which we denote by Q_{∞}^{0,x_0}. In the previous paragraph we saw how to define a conditional process. The only ambiguity lies on how to define the process starting at infinity. By this we mean that the process starts at time $-\infty$, and from position ∞, too, in a way that will be made clear in what follows.

In a situation like this, we cannot speak of initial distribution, but nevertheless such a Markov process can be defined as long as we have at hand marginal distributions $\mu_t(dx)$ and transition probabilities $q(t, x, y)$. If, for example, we would like to write down the finite dimensional distributions $Q(x(t_1) \in A_1, \cdots, x(t_k) \in A_k)$ of such a process for times $-\infty < t_1 < \cdots < t_k < 0$ these would be

$$\int_{A_1} \mu_{t_1}(dx_1) \int_{A_2} q(t_2 - t_1, x_1, dx_2) \cdots \int_{A_k} q(t_k - t_{k-1}, x_{k-1}, dx_k). \tag{A.10}$$

It is important, for the process to be well defined, that the marginals satisfy the following consistency condition :

For every $-\infty < t_1 < t_2 < 0$

$$\int \mu_{t_1}(dx_1) \int_A q(t_2 - t_1, x_1, dx_2) = \int_A \mu_{t_2}(dx).$$

This is easy to see by inspection of the form of the finite dimensional distributions in (A.10)

The family of marginals defined as above is called *Entrance Law* : we do not know how the process starts, but the prescription of the marginals describes how it 'enters'. We will abuse the terminology again and call the whole process Q^{0,x_0}_{∞} entrance law.

In our case, the transition probabilities will be those of the Bessel process conditioned to first hit x_0 at time 0, as defined in the previous paragraph. For the marginals we will follow a slightly different approach than the one described above. Instead of specifying how the process enters the horizontal lines $\mathcal{J}_t = \{(t,x)\colon x > x_0\}$, $t < 0$, we will specify how it enters the vertical lines $\mathcal{I}_x = \{(t,x)\colon t < 0\}$, $x > x_0$. In other words we think of the process starting afresh at some random time τ_x, after it hits level x, instead of some random position. This approach is more natural in our problem due to the geometry of it, and leads to easier computations.

To be more specific, let us define the family of marginals $\{q_{\infty}(t;x)\}_{x>x_0} \equiv \{q_x(-t;x_0)\}_{x>x_0}$, $t \leq 0$, where, as usual, $q_x(t;x_0)$ is the density of the hitting time of level x_0 for a Bessel process starting at x.For any $\tau < 0$, let $\mathcal{G}_{\tau}$ denote the σ-algebra generated by $\{x(s)\colon \tau < s < 0; x(0) = x_0\}$ and let us also denote by $\mathcal{G}_{\tau_x^{>}}$ the σ-algebra generated by $\cup_{t<0}\mathcal{G}_{t\vee\tau_x}$, where as usual $\tau_{x_0} = \inf\{s\colon x(s) = x_0\}$. Then for any set A, measurable with respect to $\mathcal{G}_{\tau_x^{>}}$, we define

$$Q^{0,x_0}_{\infty}(A) \equiv \int_{\mathbb{R}_-} q_{\infty}(t;x)\, Q^{0,x_0}_{t,x}(A)\, dt = \int_{\mathbb{R}_-} q_x(-t;x_0)\, Q^{0,x_0}_{t,x}(A)\, dt. \qquad \text{(A.11)}$$

The following proposition establishes the consistency of Q^{0,x_0}_{∞}:

Proposition 19 *For any levels x_1, x_2, such that $x_2 > x_1 > x_0$, and every set A measurable with respect to $\mathcal{G}_{\tau_{x_1}^{>}}$ the following is true:*

$$\int_{\mathbb{R}_-} q_{x_2}(-t;x_0)\, Q^{0,x_0}_{t,x_2}(A)\, dt = \int_{\mathbb{R}_-} q_{x_1}(-t;x_0)\, Q^{0,x_0}_{t,x_1}(A)\, dt.$$

Proof: The strong Markov property of Q^{0,x_0}_{t,x_2} implies that we can write the left hand side as

$$\int_{\mathbb{R}_-} q_{x_2}(-t;x_0) \int_{\mathbb{R}_-} q^{0,x_0}_{t,x_2}(t_1;x_1)\, Q^{0,x_0}_{t_1,x_1}(A)\, dt_1\, dt.$$

Further, (A.9) and Fubini's theorem imply that the above expression is equal to

$$\begin{aligned}
& \int_{\mathbb{R}_-} q_{x_2}(-t;x_0) \int_{\mathbb{R}_-} \frac{q_{x_2}(t_1-t;x_1)\, q_{x_1}(-t_1;x_0)}{q_{x_2}(-t;x_0)}\, Q^{0,x_0}_{t_1,x_1}(A)\, dt_1\, dt \\
= & \int_{\mathbb{R}_-} q_{x_1}(-t_1;x_0)\, Q^{0,x_0}_{t_1,x_1}(A) \int_{\mathbb{R}_-} q_{x_2}(t_1-t;x_1)\, dt\, dt_1 \\
= & \int q_{x_1}(-t_1;x_0)\, Q^{0,x_0}_{t_1,x_1}(A)\, dt_1,
\end{aligned}$$

which is what we were after.

So far, we have introduced the entrance law in a formal, but unmotivated manner. Next, we would like to explain how this emerges from the setting of our problem.

Let $Q^{0,x_0}_{t,x}$, $t < 0$, be a Bessel bridge. Our goal is to study what happens when $x \to \infty$. Because of the Brownian scaling, the particular case of $Q^{0,x_0}_{x^2\sigma,x}$, $\sigma < 0$, should play a central role, and it might as well have a nice limit as $x \to \infty$. In fact, it turns out that the limit of this family of processes is the process Q^{0,x_0}_{∞}, that we described above. The following two propositions show how to define the entrance laws through this procedure:

Proposition 20 *For arbitrary $\sigma < 0$ and $x_0 < x_1 < x$, consider the family of densities $\{q^{0,x_0}_{x^2\sigma,x}(\cdot;x_1)\}_{x>x_0}$, as defined in (A.9). Then, for every $t_1 < 0$,*

$$\lim_{x\to\infty} q^{0,x_0}_{x^2\sigma,x}(t_1;x_1) = q_{x_1}(-t_1;x_0).$$

Proof: By (A.9) and by the scaling property, we have

$$\begin{aligned} q^{0,x_0}_{x^2\sigma,x}(t_1;x_1) &= \frac{q_x(t_1 - x^2\sigma; x_1)\, q_{x_1}(-t_1; x_0)}{q_x(-x^2\sigma; x_0)} \\ &= \frac{q_1(\frac{t_1}{x^2} - \sigma; \frac{x_1}{x})\, q_{x_1}(-t_1; x_0)}{q_1(-\sigma; \frac{x_0}{x})}, \end{aligned}$$

and the result now follows by continuity.

We now want to show that for any $\sigma < 0$, $Q^{0,x_0}_{x^2\sigma,x} \Rightarrow Q^{0,x_0}_{\infty}$, as $x \to \infty$, where the double arrow signifies weak convergence.

First, we need to put the measures onto the same measurable space, which will be the space of continuous paths $\mathcal{S} = \{x(t)\colon -\infty < t < 0\,;\ \ x(0) = x_0\}$. To do this, we need to extend the set of continuous paths $\mathcal{C}_x = \{x(t)\colon x^2\sigma < t < 0\,;\ x(0) = x_0,\ x(x^2\sigma) = x\}$ to be equal to x, for $t < x^2\sigma$. Let us call this extension $\overline{\mathcal{C}}_x$. The σ-algebras in these spaces are considered to be generated by the coordinate mappings. We can now extend the measure $Q^{0,x_0}_{x^2\sigma,x}$ on $\mathcal{S}$ so as to be concentrated on $\overline{\mathcal{C}}_x$, and we have the following statement

Proposition 21 *Let the family of measures* $\{Q^{0,x_0}_{x^2\sigma,x}\}_{x>x_0}$*,* $\sigma < 0$*, and* Q^{0,x_0}_{∞} *be defined on the set of paths* $\mathcal{S}$ *as in (A.11). Then as* $x \to \infty$

$$Q^{0,x_0}_{x^2\sigma,x} \Rightarrow Q^{0,x_0}_{\infty}.$$

Proof: By the definition of the weak convergence we need to show that for any bounded, continuous function F on $\mathcal{S}$, $E^{Q^{0,x_0}_{x^2\sigma,x}}[F] \longrightarrow E^{Q^{0,x_0}_{\infty}}[F]$, as $x \to \infty$. By standard arguments it is enough to show this for bounded , continuous functions F measurable with respect $\mathcal{F}_{\tau_y^>}$, for arbitrary $y > x_0$.

This follows from the previous proposition, since by the strong Markov property, for any $x > y > x_0$,

$$E^{Q^{0,x_0}_{x^2\sigma,x}}[F] = \int q^{0,x_0}_{x^2\sigma;x}(t,y)\, E^{Q^{0,x_0}_{t,y}}[F]\, dt,$$

and by Proposition 20 and the bounded convergence theorem this converges to $\int_{\mathbb{R}_-} q_x(-t;y)\, E^{Q^{0,x_0}_{t,y}}[F]\, dt$, which is equal to $E^{Q^{0,x_0}_{\infty}}[F]$.

We will close this section by computing the density of the marginal of the law Q^{0,x_0}_{∞} at time $s<0$.

Proposition 22 *Let $q_\infty(s,y\,|\,0,x_0)$ denote the density of the marginal of the law Q^{0,x_0}_{∞} at time $s<0$, that is $q_\infty(s,y\,|\,0,x_0) = Q^{0,x_0}_{\infty}(x(s)=y)$, $y>x_0$. Then*

$$q_\infty(s,y\,|\,0,x_0) = c\,(y-x_0)\, q_y(-s;x_0), \tag{A.12}$$

where $q_y(-s;x_0)$ is the density of the hitting time of level x_0 for the Bessel process Q_y, and c is a normalizing constant.

Proof: The marginal $q_\infty(s,y\,|\,0,x_0)$ must solve Kolmogorov's forward equation

$$-\frac{\partial\, q_\infty}{\partial s} + \left(\mathcal{L}^{0,x_0}\right)^*\, q_\infty = 0,$$

together with the boundary conditions $q_\infty(s,x_0\,|\,0,x_0) = 0$, for $s<0$, and $q_\infty(0,y\,|\,0,x_0) = \delta_{x_0}(y)$. By (A.7), the equation reads as

$$-\frac{\partial\, q_\infty}{\partial s} + \frac{\partial^2 q_\infty}{\partial y^2} - \frac{\partial}{\partial y}\left((-\frac{6}{y} + 2\frac{\partial}{\partial y}\log q_y(-s;x_0))\, q_\infty \right) = 0 \tag{A.13}$$

We will seek a solution of the form $\phi(y)\, q_y(-s;x_0)$. Plugging this function into (A.13) we see that the equation reads as

$$\phi\left(\frac{\partial\, q_y}{\partial\, s} - \frac{\partial^2 q_y}{\partial\, y^2} + \frac{6}{y}\frac{\partial\, q_y}{\partial\, y} - \frac{6}{y^2}\, q_y\right) + q_y\left(\frac{\partial^2 \phi}{\partial\, y^2} + \frac{6}{y}\frac{\partial \phi}{\partial y}\right) = 0,$$

where we write for brevity $q_y(-s;x_0)$ as q_y. Since q_y satisfies the equation

$$-\frac{\partial\, q_y}{\partial\, s} + \frac{\partial^2 q_y}{\partial\, y^2} - \frac{6}{y}\frac{\partial\, q_y}{\partial\, y} = 0,$$

we see that ϕ must satisfy the equation

$$\frac{\partial^2 \phi}{\partial y^2} + \frac{6}{y}\frac{\partial \phi}{\partial y} - \frac{6}{y^2}\phi = 0. \tag{A.14}$$

ϕ needs to satisfy a boundary condition on x_0. To find this notice that, to obtain a probability density, we need to have that the integral $\int_{x_0}^{\infty} \phi(y)\, q_y(-s;x_0)\, dy$ is constant in time, so that we can normalize. Differentiating with respect to s we have that

$$\begin{aligned} 0 = \int_{x_0}^{\infty} \phi(y)\, \frac{\partial}{\partial s} q_y(-s;x_0)\, dy &= -\int_{x_0}^{\infty} \phi(y) \left(\frac{\partial^2 q_y}{\partial y^2} - \frac{6}{y}\frac{\partial\, q_y}{\partial\, y} \right) dy \\ &= \phi(x_0) \frac{\partial\, q_y}{\partial\, y}|_{y=x_0}. \end{aligned}$$

Equation (A.14), along with the boundary condition $\phi(x_0) = 0$, has the solutions $\phi(y) = y - x_0$ and $\phi(y) = x_0^{-6} - y^{-6}$, and since there should be no singularity as x_0 goes to 0, $\phi(y)$ must equal $y - x_0$.

The fact that $q_\infty(s, y \,|\, 0, x_0) = c\,(y - x_0) q_y(-s;x_0)$, where c is now a normalizing constant, satisfies the correct boundary conditions on $y = x_0, s < 0$ and on $s = 0, y \geq x_0$ follows from the fact that $q_{x_0}(-s;x_0) = 0$ if $s < 0$ (regarding the first condition) and from the fact that $c \int_{x_0}^{\infty} \phi(y) q_y(-s;x_0)\, dy = 1$, for any $s < 0$ and that $q_y(-s;x_0) \to 0$ exponentially fast as $s \to 0$, when $y \neq x_0$ (regarding the second condition).

Remark: The reason we chose to try a solution of $(A.13)$ of the form $\phi(y) q_y(-s;x_0)$ is because, in the case that $x_0 = 0$, we can use formulae (A.1) and (A.6) to compute $q_\infty(s, y \,|\, 0, 0)$ by letting the initial position and time, x and σ, go to infinity in such a way that $x \sim \sigma^2$. An easy computation shows that $q_\infty(s, y \,|\, 0, 0) = \frac{y^8}{(-s)^{9/2}} \exp\left(\frac{y^2}{4s}\right)$, which has the above form.

Appendix B

Estimates

In this section we collect the estimates related to the decay of α_x that are necessary in order to treat the dynamics of our model as a perturbation of the linear dynamics. The L^∞ estimates are considered to be uniform with respect to all possible boundary data $\lambda(\cdot)$. The main L^∞ estimate is the one in Proposition 23, which also allows us to obtain exponential estimates, as well as Proposition 25. The main energy estimate is the one in Propostion 28.

Proposition 23 *If $\alpha(t,x)$ is a solution of equation (4.1), then for any $x_0 > 0$,*

$$\sup_{x>x_0,\,t} E^{Q_x} \int_0^{\tau_{x_0}} \frac{\alpha_x^2(t-s,x(s))}{x(s)+\alpha(s,x(s))}\,ds \leq \|\alpha\|_{L^\infty} + \frac{9\|\alpha\|_{L^\infty}^2}{2x_0}. \tag{B.1}$$

Proof : Using the variation of constants formula, we can write the solution of equation (4.1) using equation (4.2) in the form

$$\alpha(t,x) = E^{Q_x}[\alpha(t-\tau_{x_0},x_0)] + E^{Q_x}\int_0^{\tau_{x_0}} \left(-\frac{3\alpha_x^2}{x+\alpha} + \frac{6\alpha\alpha_x}{x(x+\alpha)}\right)(t+s,x(s))\,ds.$$

Using that $\frac{6\alpha\alpha_x}{x(x+\alpha)} \leq \frac{2\,\alpha_x^2}{x+\alpha} + \frac{9\,\alpha^2}{2\,x^2(x+\alpha)}$ we get the bound

$$\begin{aligned} E^{Q_x}\int_0^{\tau_{x_0}} \frac{\alpha_x^2\,(t-s,x(s))}{x(s)+\alpha\,(t-s,x(s))}\,ds \;&\leq\; E^{Q_x}[\alpha\,(t-\tau_{x_0},x_0)] - \alpha(t,x) \\ &+ E^{Q_x}\int_0^{\tau_{x_0}} \frac{9\,\alpha^2\,(t-s,x(s))}{2\,x^2\,(s)\,(x(s)+\alpha(t-s,x(s)))}\,ds. \end{aligned}$$

Using the positivity of α and Proposition 18 we can bound the last quantity by

$$\|\alpha\|_{L^\infty} + \frac{9}{2}\,\|\alpha\|_{L^\infty}^2\,E^{Q_x}\int_0^{\tau_{x_0}} \frac{ds}{x(s)^3} \leq \|\alpha\|_{L^\infty} + \frac{9\|\alpha\|_{L^\infty}^2}{2x_0}.$$

The following lemma is called Khasminskii's Lemma and it will play a central role in most of our uniform estimates. The proof of it is by Taylor expansion combined with the Markov property, see, for example, [D].

Lemma 13 *Let $f(x)$ be a positive function in $\mathbb{R}^d$, $d \geq 1$, and E_x denote the expectation with respect to an arbitrary diffusion starting from position x. Let, also, τ denote the exit time from an arbitrary domain, $D \subset \mathbb{R}^d$. If*

$$\sup_{x\in D} E_x \int_0^{\tau} f(x(s))ds = \kappa < 1,$$

then,

$$\sup_{x\in D} E_x \left[\exp(\int_0^{\tau} f(x(s))ds)\right] < \frac{1}{1-\kappa}.$$

Remark : In our case the dimension is $d = 2$, space and time, the diffusion is the one with generator $-\frac{\partial}{\partial t} + \frac{\partial^2}{\partial x^2} - \frac{6}{x}\frac{\partial}{\partial x}$, and τ is the exit time τ_{x_0}, from the domain $x > x_0$.

As a corollary of the previous proposition and lemma we have

Corollary 1 *If we choose $\mu > 0$ such that $\mu < \left(\|\alpha\|_{L^\infty} + \frac{9\|\alpha\|^2_{L^\infty}}{2x_0}\right)^{-1}$ for arbitrary $x_0 > 0$, then*

$$\mu \sup_{x>x_0,t} E^{Q_x} \int_0^{\tau_{x_0}} \frac{\alpha_x^2(t-s,x(s))}{x(s)+\alpha(t-s,x(s))}\, ds < 1,$$

and thus,

$$\sup_{x>x_0,t} E^{Q_x} \left[\exp\left(\mu \int_0^{\tau_{x_0}} \frac{\alpha_x^2(t-s,x(s))}{x(s)+\alpha(t-s,x(s))}\, ds\right)\right] < \infty.$$

Corollary 2 *Denote by $\mathcal{P}_{t,x}$ the measure corresponding to the diffusion with generator $\mathcal{L} = \frac{\partial^2}{\partial x^2} + \left(-\frac{6}{x} - \frac{3a_x}{(x+\alpha)}\right)\frac{\partial}{\partial x}$. If $x_0 > 0$ is large enough, then the measures $\mathcal{P}_{t,x}$ and Q_x restricted to the σ-algebra $\mathcal{F}_{\tau_{x_0}}$ are mutually absolutely continuous, and their Radon-Nikodym derivative is*

$$\frac{d\mathcal{P}_{t,x}}{dQ_x}|_{\mathcal{F}_{\tau_{x_0}}} = \exp\left(\int_0^{\tau_{x_0}} \frac{3\alpha_x}{2(x+\alpha)}\, d\overline{x}(s) - \int_0^{\tau_{x_0}} \frac{9\alpha_x^2}{4(x+\alpha)^2}\, ds\right),$$

where $d\overline{x}(s) = dx(s) + \frac{6}{x(s)}\, ds$.

Proof : The statement will follow as soon as we verify that the Girsanov transformation is legitimate. For this it suffices to have that

$$E^{Q_x} \exp\left(\int_0^{\tau_{x_0}} \frac{9\alpha_x^2}{4(x+\alpha)^2}\, ds\right) < \infty.$$

But this quantity is bounded by $E^{Q_x} \exp\left(\frac{9}{x_0}\int_0^{\tau_{x_0}} \frac{\alpha_x^2}{x+\alpha}\, ds\right)$,and is clear that we can choose x_0 large enough so that Corollary 1 is applicable.

Next, we prove the first pointwise estimate on $\alpha_x(t,x)$.

Proposition 24 *Under the assumptions of Theorem 1, there is a positive, constant C such that a.s. $|\alpha_x(t,x)| \leq C$, for any $x > 0, t \in \mathbb{R}$.*

Proof: The proof uses the maximum principle. Define

$$w(t,x) \equiv u_x(t,x) = -\frac{12}{(x+\alpha(t,x))^3}(1+\alpha_x(t,x)).$$

Differentiating the equation (3.1) with respect to x, we get that w solves the Dirichlet problem

$$w_t = w_{xx} - 2uw, \quad x > 0, t \in \mathbb{R}, \tag{B.2}$$

$$w(t,0) = -\frac{1}{2}\lambda(t). \tag{B.3}$$

Recalling that u satisfies the bound $6/(x+\alpha_1)^2 \leq u(t,x) \leq 6/(x+\alpha_2)^2$, we can show that there is a negative constant C_2, such that the function $\overline{w}(x) = C_2/(x+\alpha_2)^3$ is an upper solution for the problem (B.2). Indeed,

$$\begin{aligned} \overline{w}_t - \overline{w}_{xx} + 2u\overline{w} &= -\overline{w}_{xx} + 2u\overline{w} \\ &\geq -\overline{w}_{xx} + \frac{12}{(x+\alpha_2)^2}\overline{w} = 0, \end{aligned}$$

and on the boundary $\overline{w}(t,0) = C_2/\alpha_2^3 \geq \sup_t w(t,0)$, if C_2 is appropriatelly chosen. In the same way we can prove that $\underline{w}(x) = C_1/(x+\alpha_1)^3$ is a lower solution for the same problem, when C_1 is an appropriatelly chosen, negative constant. Hence, we have the bound

$$\left| \frac{12}{(x+\alpha(t,x))^3}(1+\alpha_x(t,x)) \right| \leq \frac{C}{(x+c)^3},$$

for some constants positive c, C. From this the boundedness of α_x follows immediately.

This bound can be improved a lot when we are refering to a stationary solution of equation (4.1), and x is large. In fact the next Proposition shows that for large x, α_x decays like $\frac{1}{x}$.

Proposition 25 *Let $\alpha(t,x)$ a stationary solution of equation (4.1) and let $B_r(t,x)$ denote the set $\{(s,y) \in \mathbb{R}^2 \colon s \in (t-r^2,t), y \in (x-r,x+r)\}$. Then, for any $r>0$, there is a positive constant C, that depends on r, such that for any $t \in \mathbb{R}$ and $x>0$*

$$\| \alpha_x \|_{L^\infty(B_{xr}(t,x))} \leq \frac{C}{x} \| \alpha \|_{L^\infty(B_{2xr}(t,x))}.$$

Proof : It is easy to check that, for any k, the function $\alpha^{(k)}(t,x) \equiv \frac{1}{k}\alpha(k^2 t, kx)$ is also a solution of equation (4.1). Standard PDE estimates (see for example [L], chap.3) show that

$$\| \alpha_x^{(k)} \|_{L^\infty(B_r(t,x))} \leq \| \alpha^{(k)} \|_{L^\infty(B_{2r}(t,x))} \qquad \text{for any} \quad t \in \mathbb{R}, x>0,$$

or,

$$\| \alpha_x(k^2\cdot, k\cdot), \|_{L^\infty(B_r(t,x))} \leq \frac{1}{k} \| \alpha(k^2\cdot, k\cdot) \|_{L^\infty(B_{2r}(t,x))},$$

which can be also written as

$$\| \alpha_x \|_{L^\infty(B_{kr}(k^2 t, kx))} \leq \frac{1}{k} \| \alpha \|_{L^\infty(B_{2kr}(k^2 t, kx))}.$$

The result now follows by choosing x to be equal to 1 and using the stationarity of $\alpha(\cdot,x)$ to replace $k^2 t$ by an arbitrary $t \in \mathbb{R}$

The next Proposition establishes an estimate on the pointwise difference between two solutions of equation (4.3) corresponding to two different Dirichlet boundary conditions, in terms of the difference between the boundary conditions.

Proposition 26 *Let $a(t,x)$ and $b(t,x)$ be two solutions of equation (4.3) corresponding to Dirichlet boundary conditions $a(\cdot,x_0)$ and $b(\cdot,x_0)$, on $x=x_0$. Let us* *denote the difference between these solutions by $\theta(t,x)$. Then there is a positive constant C, such that for $p>1$, that depends on the level x_0,*

$$|\,\theta(t,x)\,| \leq C\ \left(E^{Q_x}\left[\,|\,\theta(t+\tau_{x_0},x_0)\,|^p\,\right]\right)^{\frac{1}{p}}.$$

The larger x_0 is, the closer to 1 can p be chosen.

Proof : The equation that $\theta(t,x)$ satisfies is easily deduced from equation (4.3) and is

$$\theta_t + \theta_{xx} + \left(-\frac{6}{x} + h_1(t,x)\right)\theta_x + h_2(x)\,\theta = 0, \tag{B.4}$$

where

$$h_1(t,x) = \frac{6\,a}{x\,(x+a)} - \frac{3\,(a_x+b_x)}{x+a} \quad \text{and} \quad h_2(t,x) = \frac{6\,b_x + 3\,b_x^2}{(x+a)\,(x+b)}.$$

By Khasminskii's Lemma and a similar calculation to that of Corollary 2, we can use the Girsanov transformation and the Feynman-Kac formula to write the solution to the equation (B.4) as

$$\theta(t,x) = E^{Q_x}\left[\theta(t+\tau_{x_0},x_0)\exp\left(H(0,\tau_{x_0})\right)\right],$$

where

$$\begin{aligned} H(0,\tau_{x_0}) \ &= \ \exp\big(\frac{1}{2}\int_0^{\tau_{x_0}} h_1(s,x(s))\,d\overline{x}(s) - \frac{1}{4}\int_0^{\tau_{x_0}} h_1^2(s,x(s))\,ds \\ &\qquad + \int_0^{\tau_{x_0}} h_2(s,x(s))\,ds\,\big), \end{aligned}$$

and $d\overline{x}(s) = dx(s) + \frac{6}{x(s)}\,ds$.

Now, using Hölder's inequality with $\frac{1}{p}+\frac{1}{q}=1$ we have that

$$|\,\theta(t,x)\,| \leq \left(E^{Q_x}\left[\,|\,\theta(t+\tau_{x_0},x_0)\,|^p\,\right]\right)^{\frac{1}{p}} \cdot \left(E^{Q_x}\left[\exp\left(\,q\,H(0,\tau_{x_0})\,\right)\right]\right)^{\frac{1}{q}}.$$

The second term on the right hand side can be uniformly bounded if q is chosen, in terms of x_0, so that Khasminskii's Lemma is applicable. The same calculation as in Corollary 2 shows that q can be chosen as large as we wish if x_0 is also chosen large enough. Thus, the larger x_0 is, the closer to 1 we can choose p to be.

Remark : This Proposition provides a way to measure the dependence of a solution of equation (4.3) (or equivalently equation (4.1)) on the Dirichlet boundary values, at a level $x = x_0$. In particular, we use this estimate in the proofs of Propositions 14 and 15. There, we assume that x_0 is such that p can be chosen sufficiently close to 1 in order to ensure the integrability of $Q_{1+r}^{\frac{4}{p}}\left(\tau_{\frac{x_0^r}{x(s)}} > \tau\right)$ with respect to τ.

We now prove the first energy estimate.

Proposition 27 *Under the assumptions of Theorem 4, the following estimate holds*

$$\mathbb{E}\int_0^\infty \alpha_x^2 dx < \infty.$$

Proof: Multiplying both sides of equation (4.1) by α and taking expectations, yields

$$\mathbb{E}[\alpha\alpha_{xx}] = \mathbb{E}\left[\frac{6+3\alpha_x}{x+\alpha}\alpha_x\alpha\right].$$

Integrating both sides between two arbitrary positive numbers x_1, x_2, we get that

$$\begin{aligned}
\mathbb{E}\int_{x_1}^{x_2}\alpha_x^2 dx &= \mathbb{E}\left[\alpha\alpha_x|_{x_1}^{x_2}\right] - \mathbb{E}\int_{x_1}^{x_2}\frac{6+3\alpha_x}{x+\alpha}\alpha_x\alpha dx \\
&\leq \left|\mathbb{E}\left[\alpha\alpha_x|_{x_1}^{x_2}\right]\right| + \mathbb{E}\int_{x_1}^{x_2}\left|\frac{6+3\alpha_x}{x+\alpha}\alpha_x\alpha\right| dx \\
&\leq \left|\mathbb{E}\left[\alpha\alpha_x|_{x_1}^{x_2}\right]\right| + \frac{1}{2}\mathbb{E}\int_{x_1}^{x_2}\alpha_x^2 dx + \frac{1}{2}\mathbb{E}\int_{x_1}^{x_2}\left|\frac{6+3\alpha_x}{x+\alpha}\right|^2\alpha^2 dx.
\end{aligned}$$

Hence,

$$\frac{1}{2}\mathbb{E}\int_{x_1}^{x_2}\alpha_x^2 dx \leq |\mathbb{E}\left[\alpha\alpha_x|_{x_1}^{x_2}\right]| + \frac{1}{2}\mathbb{E}\int_{x_1}^{x_2}\left|\frac{6+3\alpha_x}{x+\alpha}\right|^2\alpha^2 dx.$$

The result is implied by the boundedness of α and α_x, and the fact that x_1, x_2 are arbitrary.

The main energy estimate we are after is given in the next proposition:

Proposition 28 *Under the assumptions of Theorem 4, the following estimate holds:*

$$\mathbb{E}\int_0^\infty x\alpha_x^2 dx < +\infty \tag{B.5}$$

Proof: Once more, multiplying the equation (4.1) by $x\alpha$, taking expectations on both sides and integrating between two arbitrary positive numbers we have that

$$\mathbb{E}\int_{x_1}^{x_2} x\alpha\alpha_{xx}dx = \mathbb{E}\int_{x_1}^{x_2}\frac{6+3\alpha_x}{x+\alpha}x\alpha\alpha_x dx$$

and integrating by parts

$$\mathbb{E}\left[x\alpha\alpha_x|_{x_1}^{x_2}\right] - \mathbb{E}\int_{x_1}^{x_2}\alpha\alpha_x dx - \mathbb{E}\int_{x_1}^{x_2}x\alpha_x^2 dx$$

$$= 6\,\mathbb{E}\int_{x_1}^{x_2}\frac{x\alpha}{x+\alpha}\alpha_x dx + 3\,\mathbb{E}\int_{x_1}^{x_2}\frac{x\alpha}{x+\alpha}\alpha_x^2 dx$$

$$= 6\,\mathbb{E}\int_{x_1}^{x_2}\alpha\alpha_x dx - 6\,\mathbb{E}\int_{x_1}^{x_2}\frac{\alpha^2}{x+\alpha}\alpha_x dx + 3\,\mathbb{E}\int_{x_1}^{x_2}\frac{x\alpha}{x+\alpha}\alpha_x^2 dx,$$

or,

$$\begin{aligned}\mathbb{E}\int_{x_1}^{x_2}x\alpha_x^2 dx &= \mathbb{E}\left[x\alpha\alpha_x|_{x_1}^{x_2}\right] - \frac{7}{2}\,\mathbb{E}[\alpha^2|_{x_1}^{x_2}] \\ &\quad +6\,\mathbb{E}\int_{x_1}^{x_2}\frac{\alpha^2}{x+\alpha}\alpha_x dx - 3\,\mathbb{E}\int_{x_1}^{x_2}\frac{x\alpha}{x+\alpha}\alpha_x^2 dx.\end{aligned}$$

Now, set $x_1 = 0$ and let x_2 be arbitrary. Then by Proposition 25 the first term on the right hand side is uniformly bounded. Moreover, using the

Cauchy-Schwartz inequality to bound the third term on the right hand side by $6\left(\mathbb{E}\int_{x_1}^{x_2}\frac{1}{(x+\alpha)^2}dx\right)^{1/2}\left(\mathbb{E}\int_{x_1}^{x_2}\alpha^4\alpha_x^2dx\right)^{1/2}$, and controlling the last term by Proposition 27, we obtain the desired result.

Bibliography

[BS] Borodin, Andrei N.; Salminen, Paavo: Handbook of Brownian Motion—Facts and Formulae. Second edition. Probability and its Applications. Birkhuser Verlag, Basel, 2002

[D] Durrett, Richard; Stochastic Calculus. A Practical Introduction.CRC Press, Boca Raton, FL, 1996.

[EMS] E, Weinan; Mattingly, J. C.; Sinai, Ya.: Gibbsian dynamics and ergodicity for the stochastically forced Navier-Stokes equation. Dedicated to Joel L. Lebowitz. Comm. Math. Phys. 224 (2001), no. 1, 83–106

[EH] Eckmann, J.-P.; Hairer, M.: Uniqueness of the invariant measure for a stochastic PDE driven by degenerate noise. Comm. Math. Phys. 219 (2001), no. 3, 523–565

[F] Feller, William: An Introduction to Probability Theory and its Applications. Vol. II. Second edition John Wiley & Sons, Inc., New York-London-Sydney

[L] Lieberman, Gary M.: Second Order Parabolic Differential Equations. World Scientific Publishing Co., Inc., River Edge, NJ

[HH] Hall, P.; Heyde, C. C.: Martingale Limit Theory and its Application. Probability and Mathematical Statistics. Academic Press, Inc.

[S] Sznitman, Alain-Sol Topics in Propagation of Chaos. École d'Été de Probabilités de Saint-Flour XIX—1989, 165–251, Lecture Notes in Math., 1464, Springer, Berlin, 1991

[V] Varadhan, S. R. S.: Probability Theory. Courant Lecture Notes in Mathematics, 7.