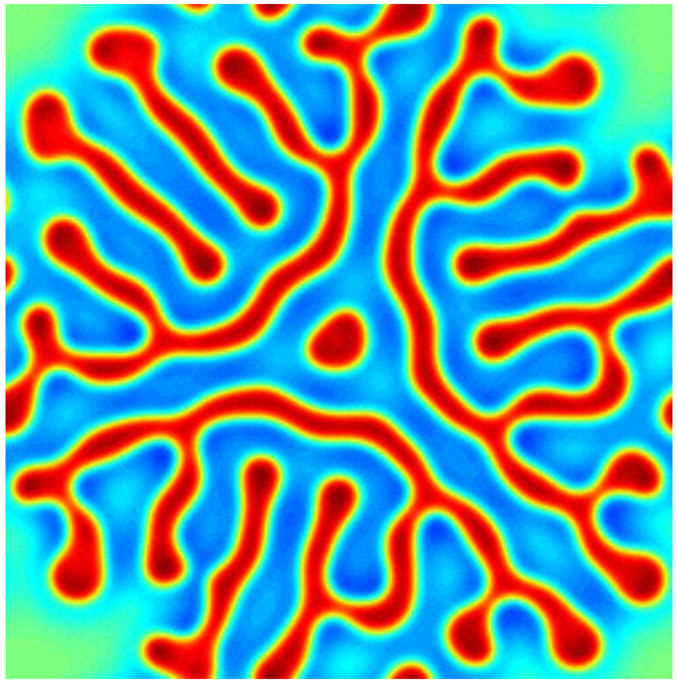
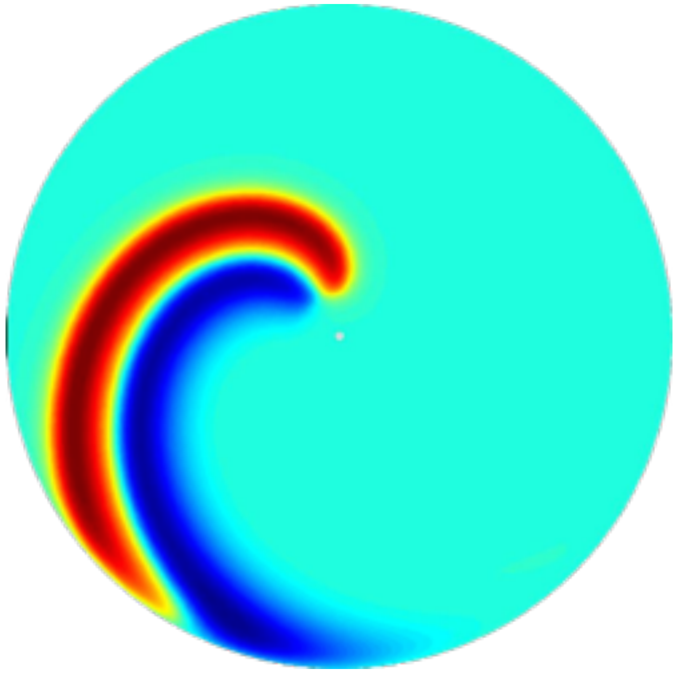
Neural interface dynamics: from spots to spirals

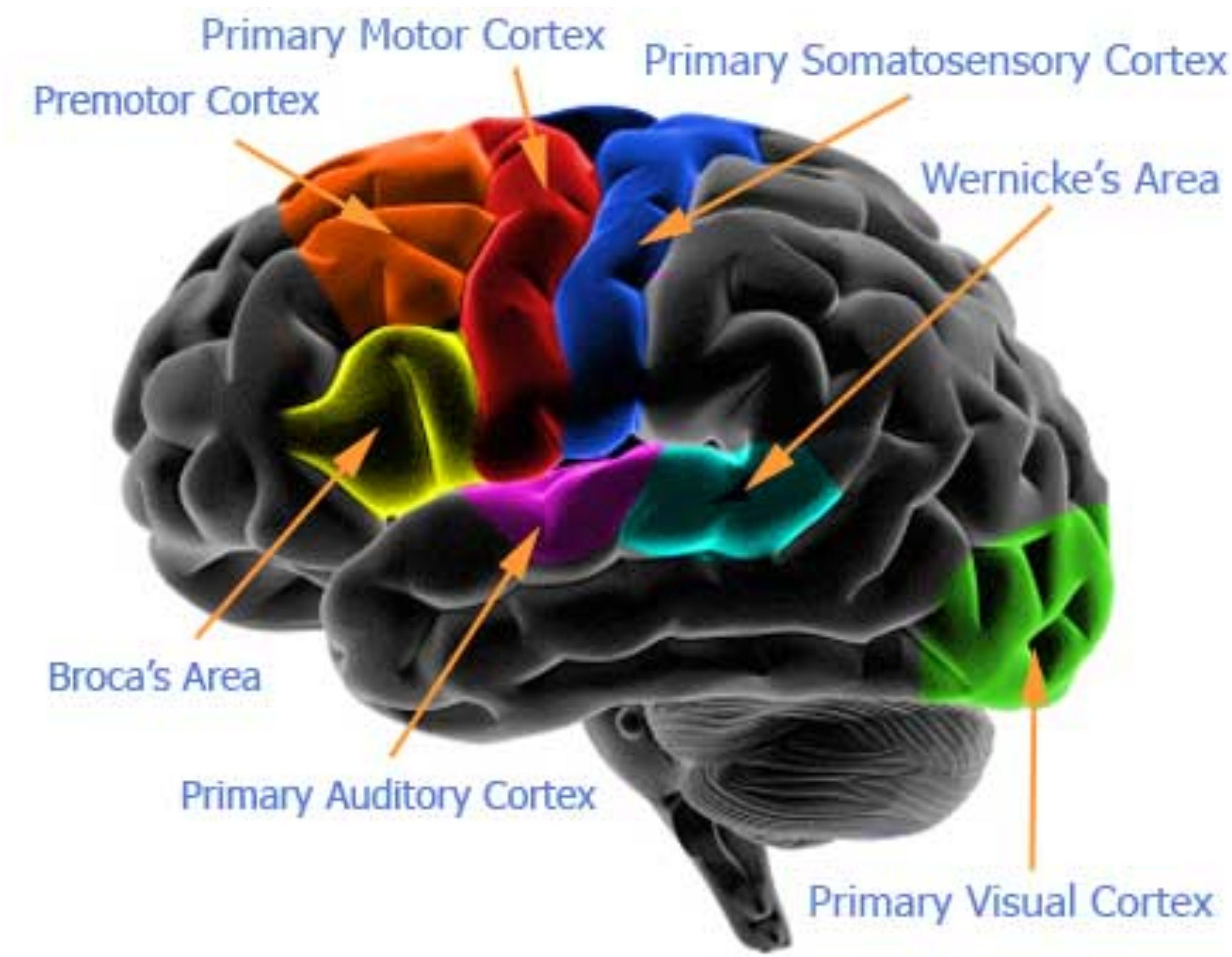


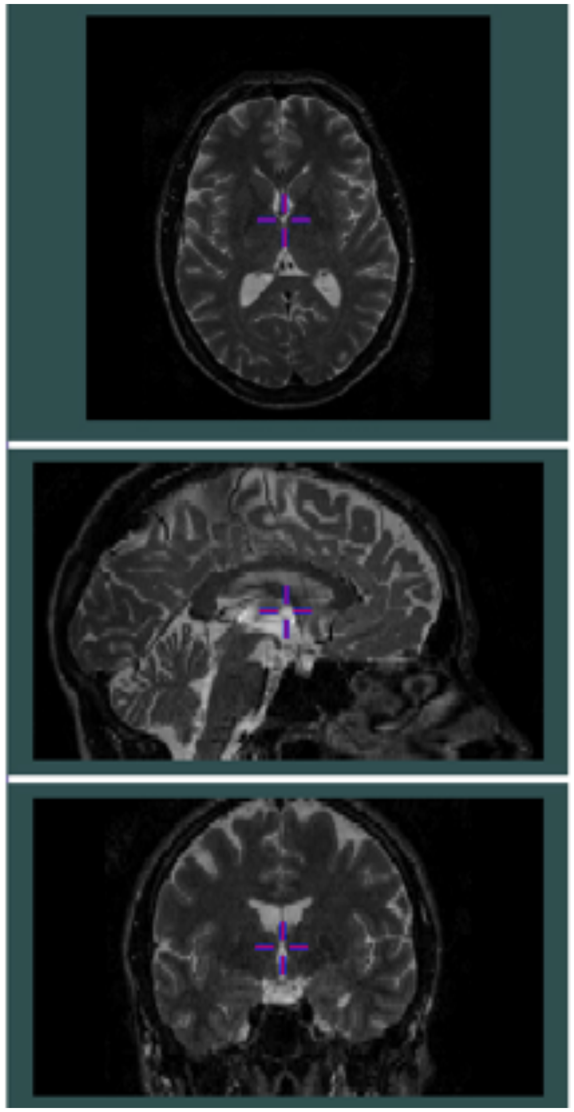


CMMB
Centre for Mathematical Medicine and Biology

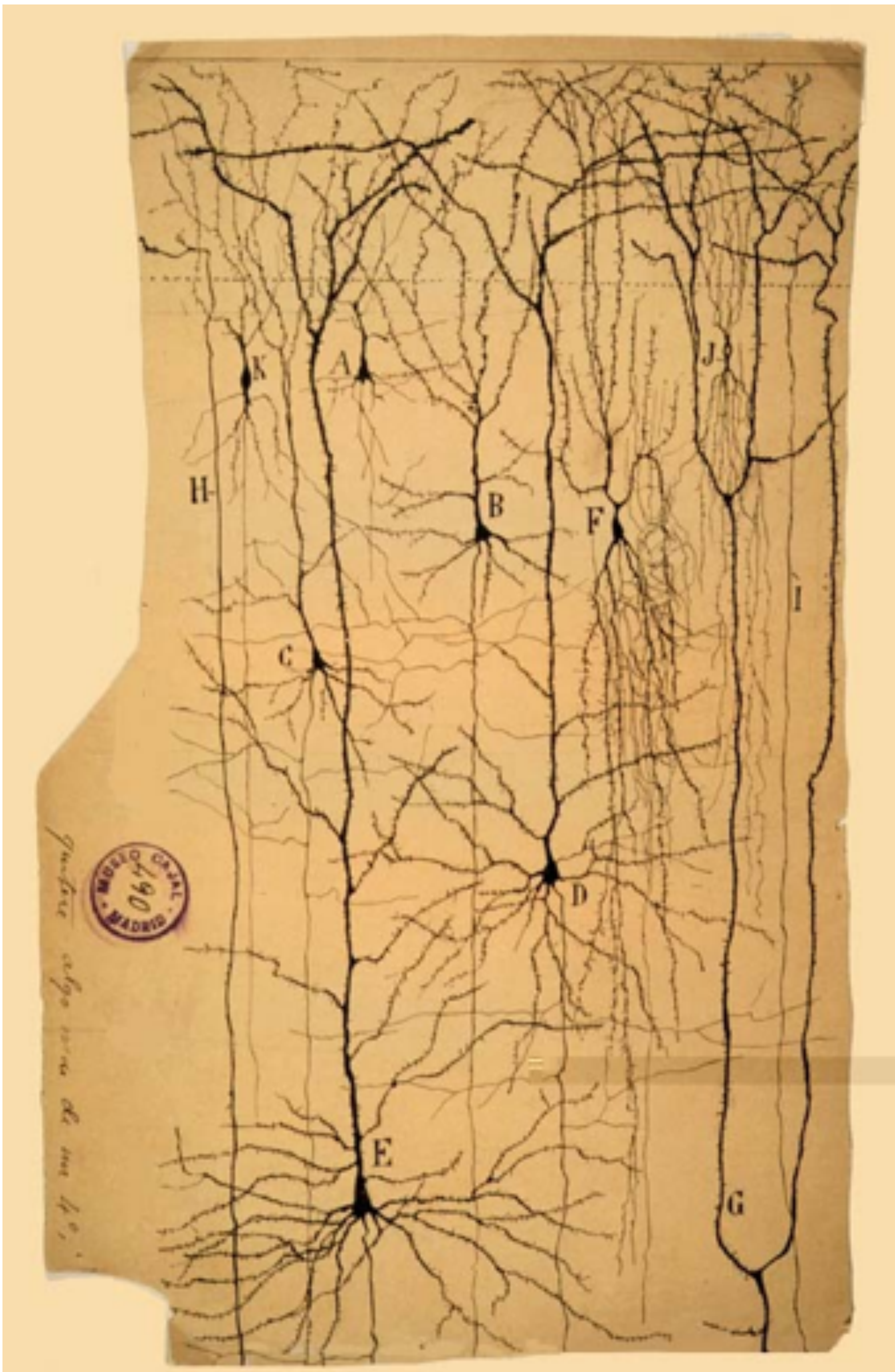
The University of Nottingham

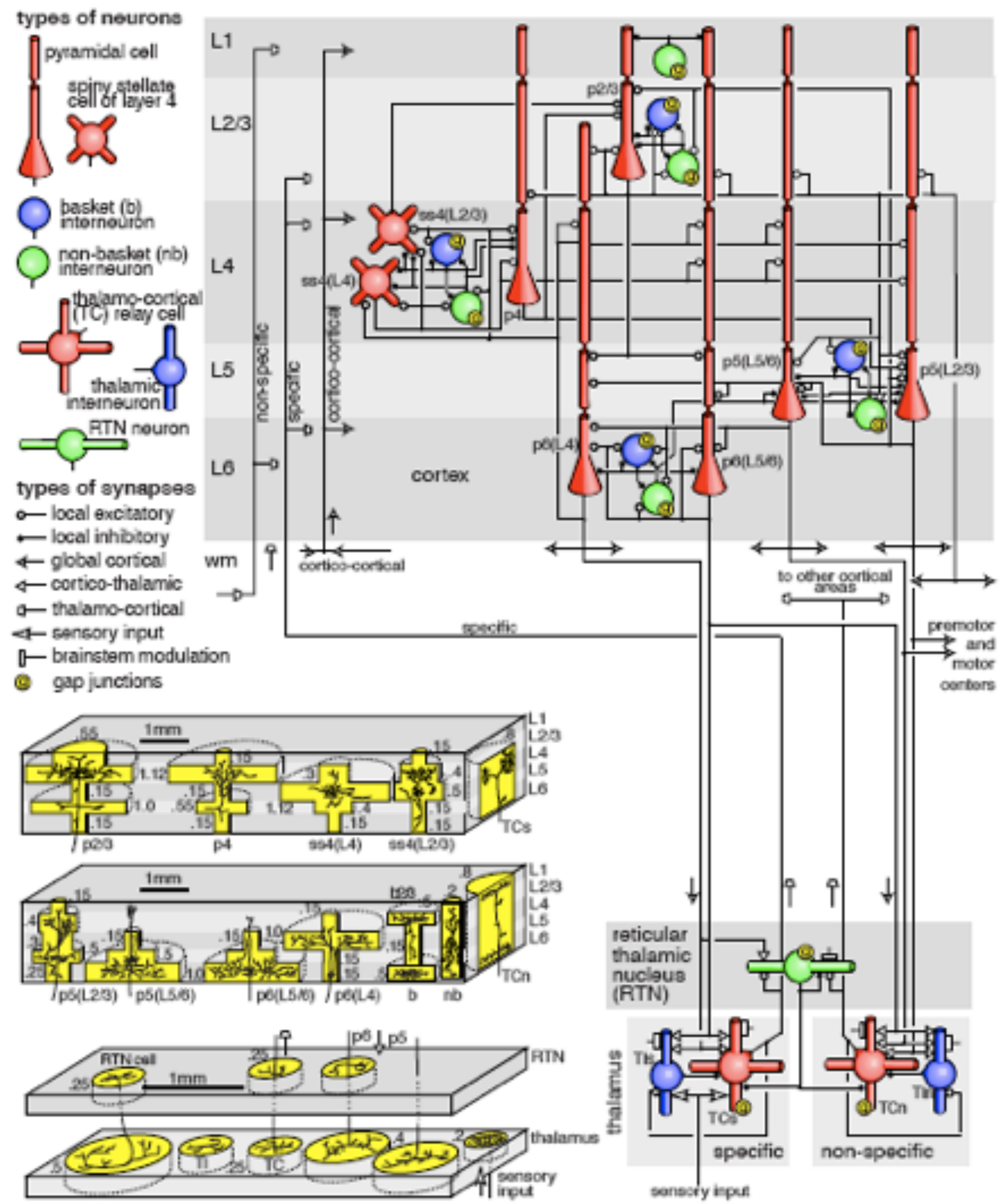
Brain and Cortex





Principal cells and interneurons

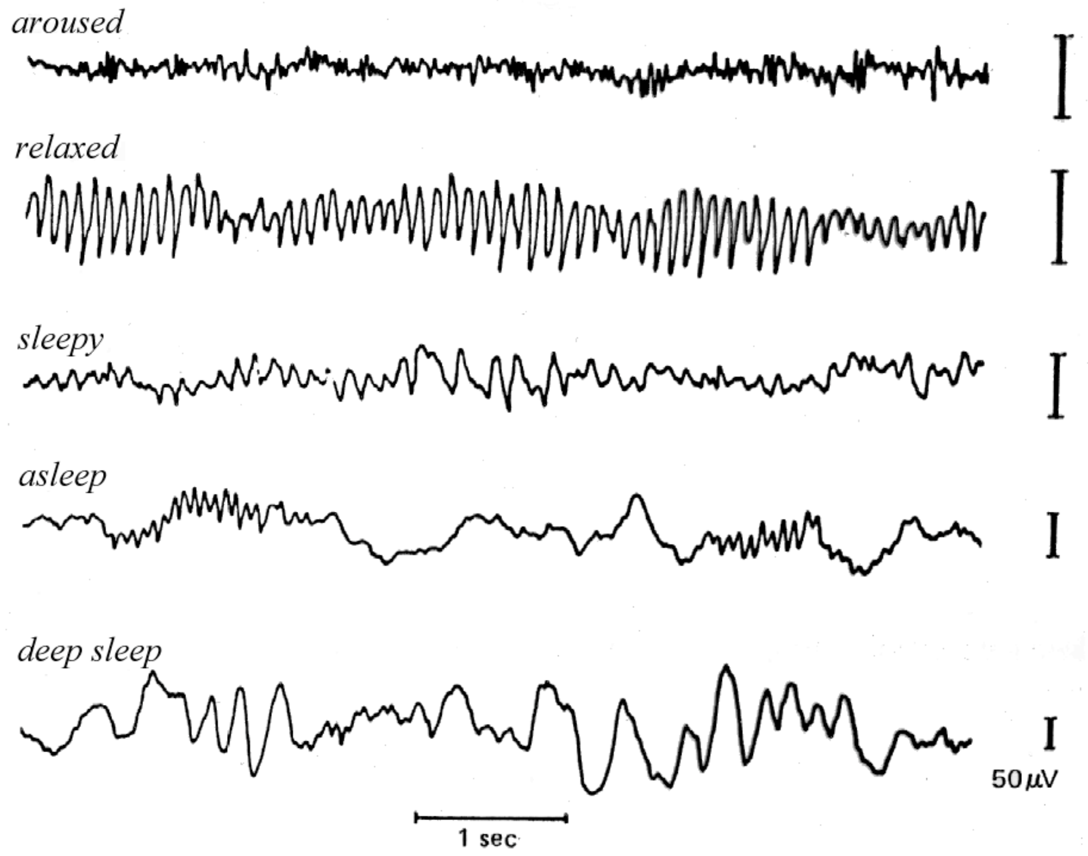


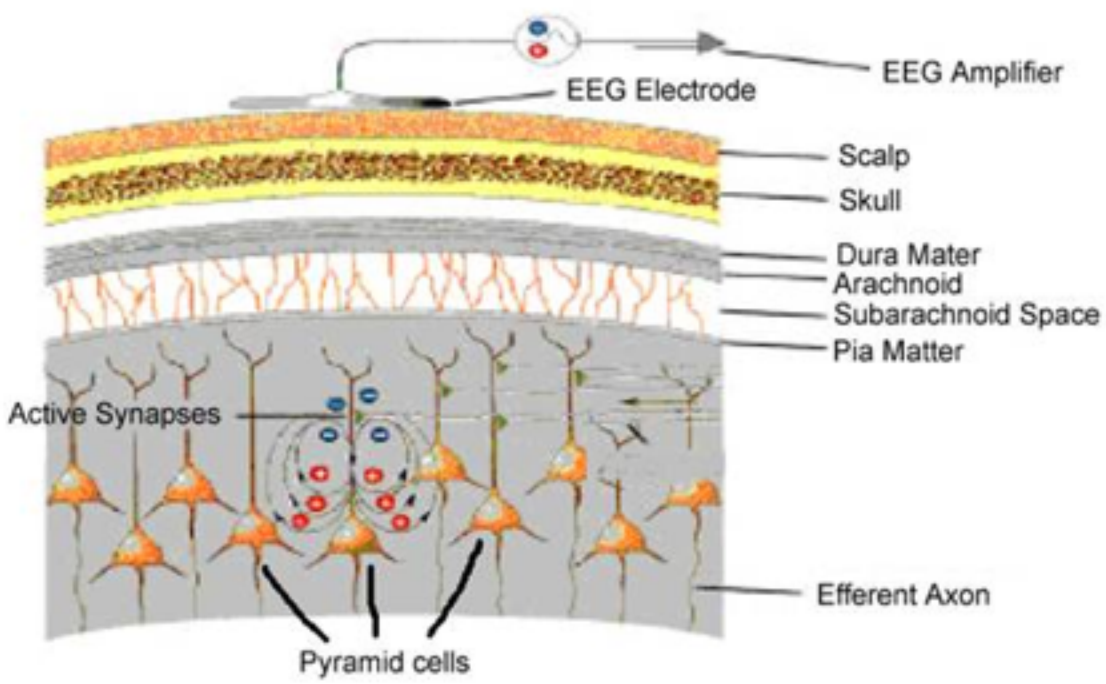


Santiago Ramón y Cajal
1900

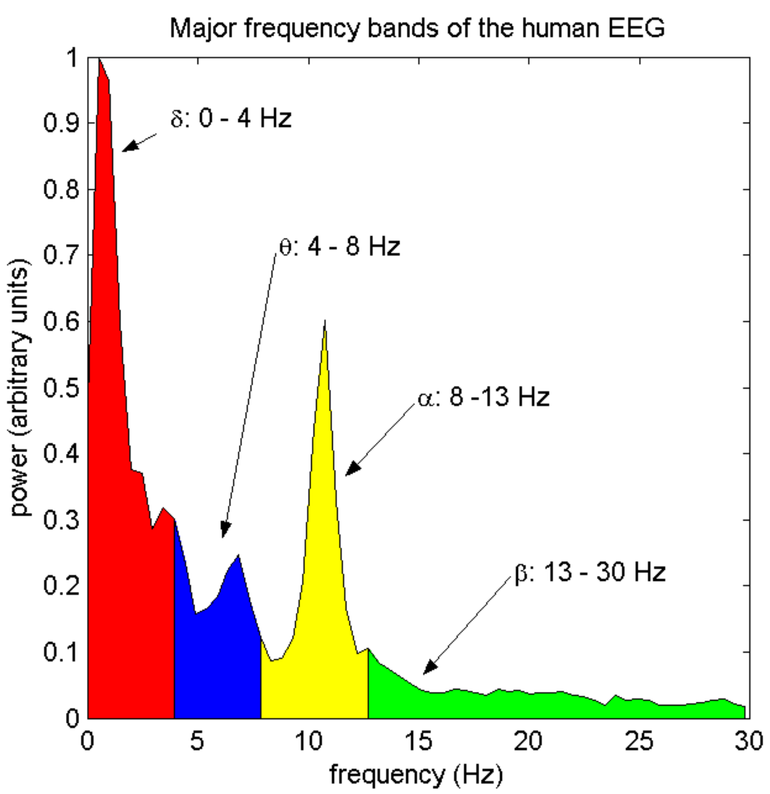
Eugene Izhikevich
2008

Electroencephalogram (EEG) power spectrum

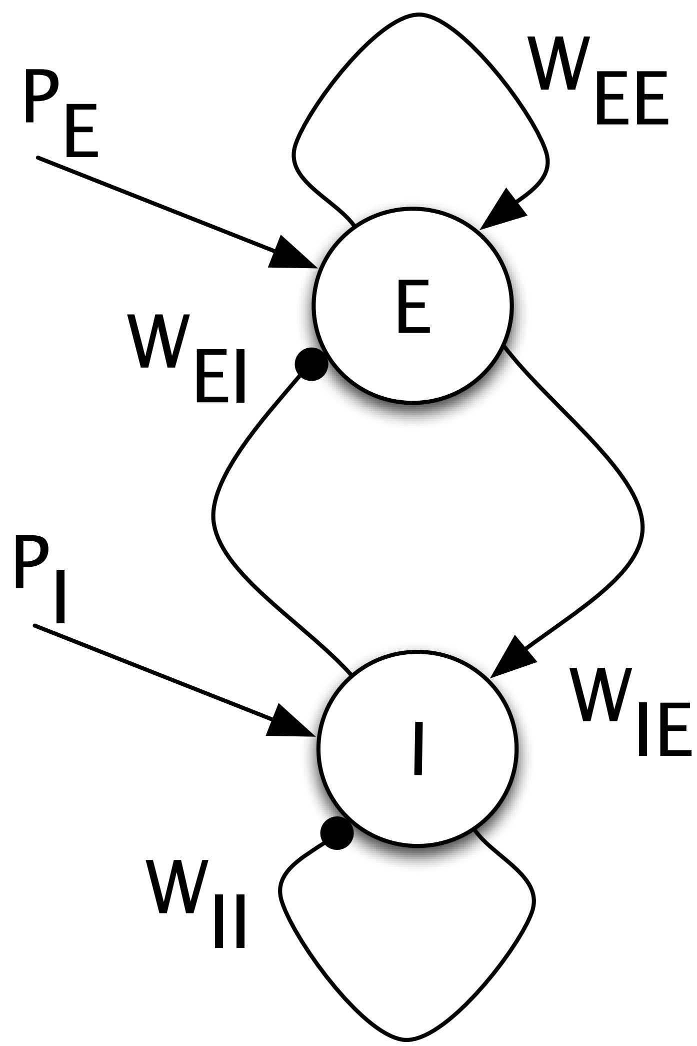




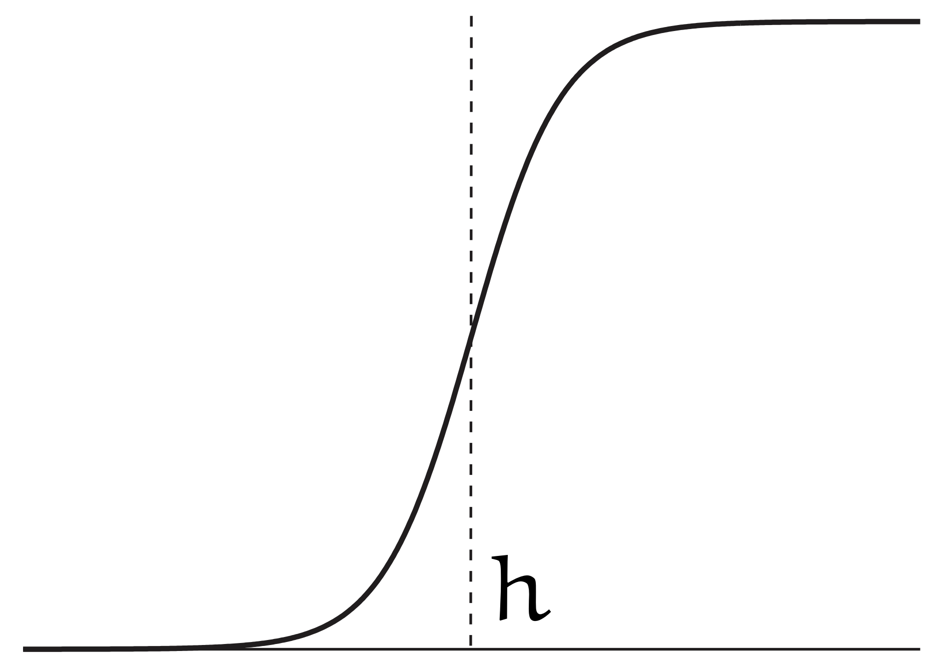
EEG records the activity of ~ 10^6 pyramidal neurons.



Population model



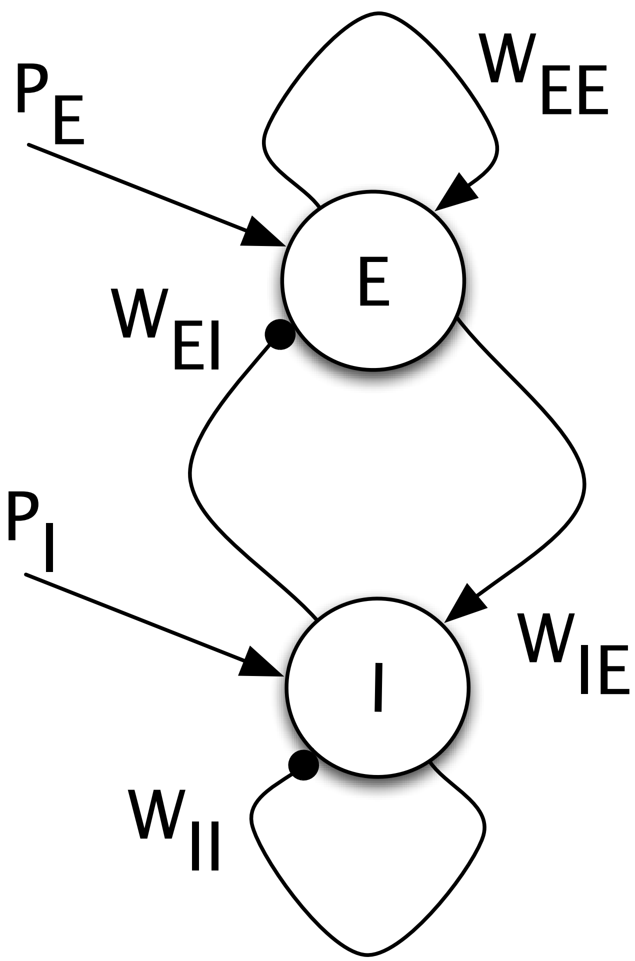
Firing rate activity $f(E)$



Firing rate activity $f(I)$

$$\dot{E} = -\frac{E}{\tau_E} + W_{EE}g_{EE}(A^+ - E) + W_{EI}g_{EI}(A^- - E) + P_E$$

$$\dot{I} = -\frac{I}{\tau_I} + W_{II}g_{II}(A^- - I) + W_{IE}g_{IE}(A^+ - I) + P_I$$



$$\eta(t) = \alpha^2 t e^{-\alpha t}$$

$$Q\eta = \delta$$

$$Q = \left(1 + \frac{1}{\alpha}\frac{d}{dt}\right)^2$$

$$Q g_{jE} = f(E) \qquad Q g_{jI} = f(I)$$

Steady state approximation

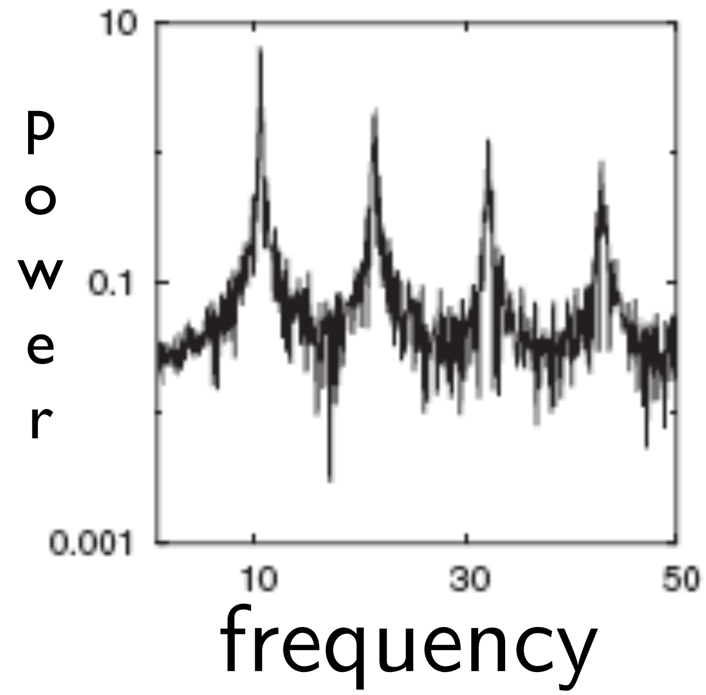
$$E = E(g_{EE}, g_{EI}) \qquad I = I(g_{II}, g_{IE})$$

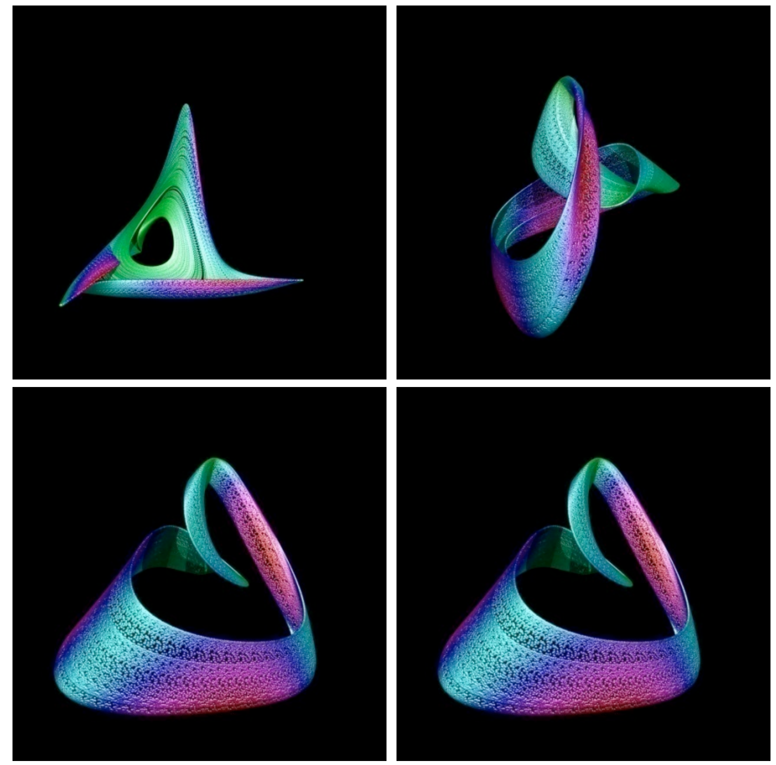
$$Qg = f$$

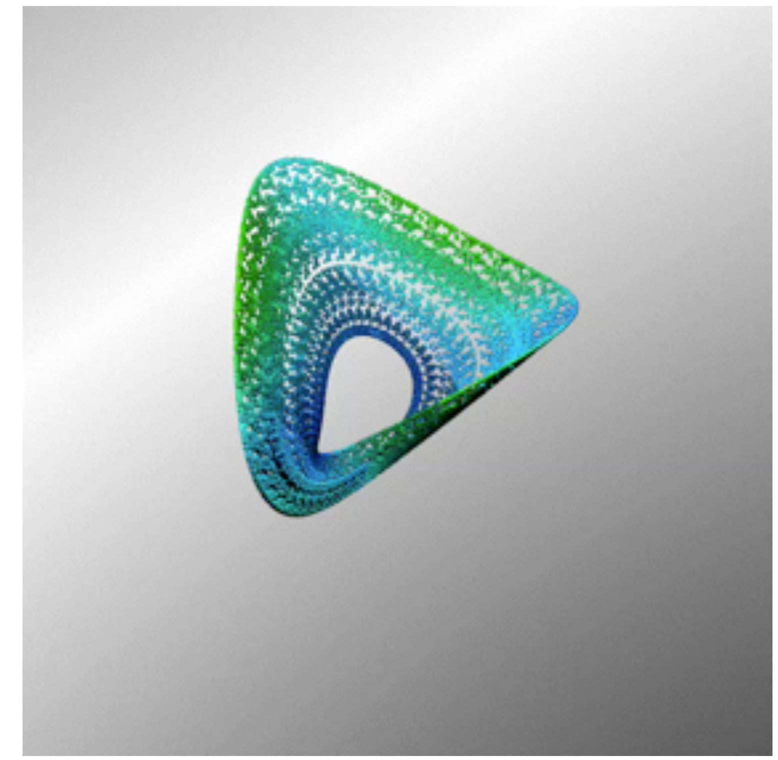
$$f = f(\{g\})$$

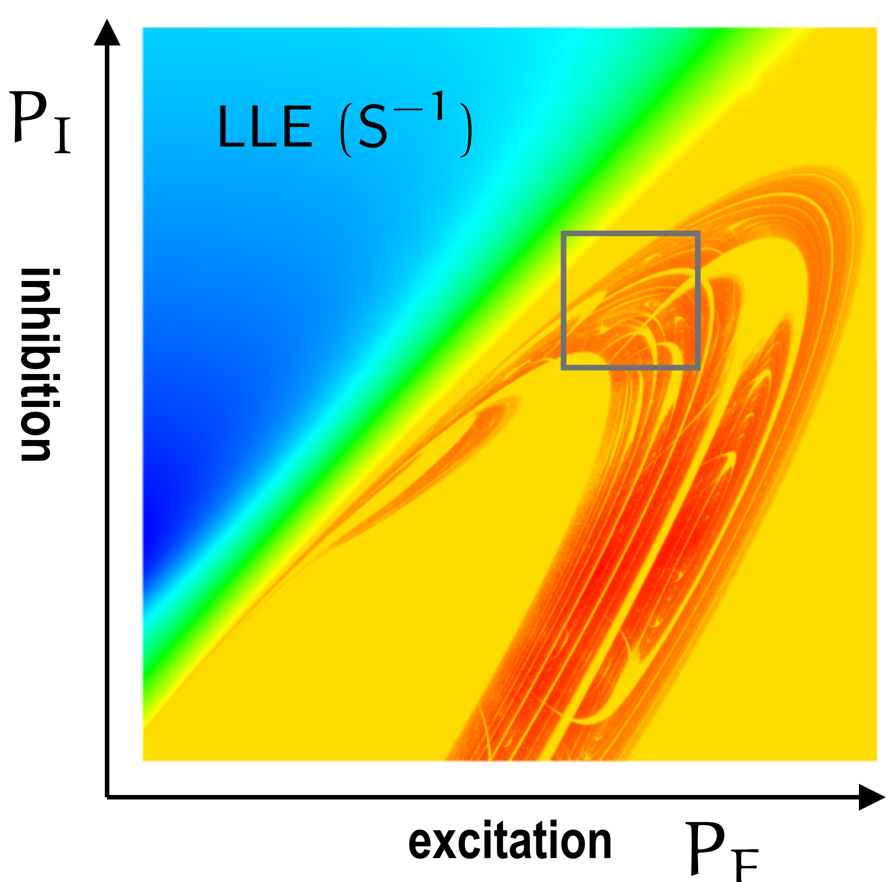
$$g = \eta * f$$

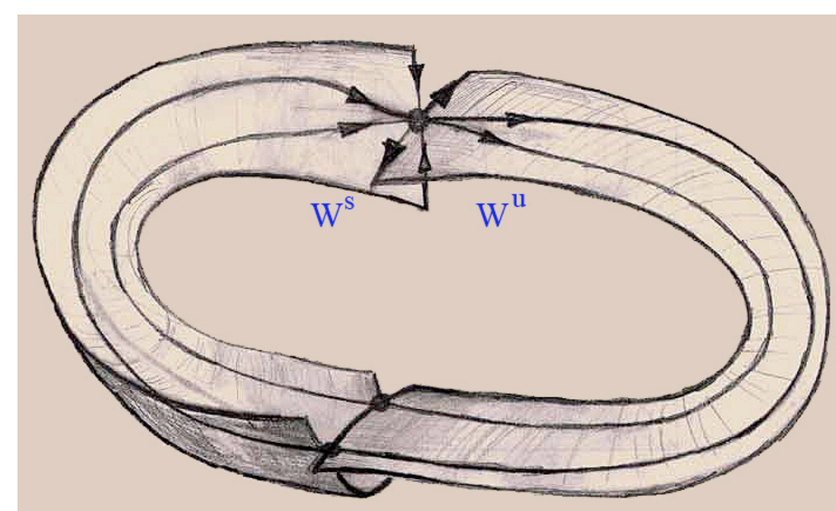
Alphoid chaos (10 D)











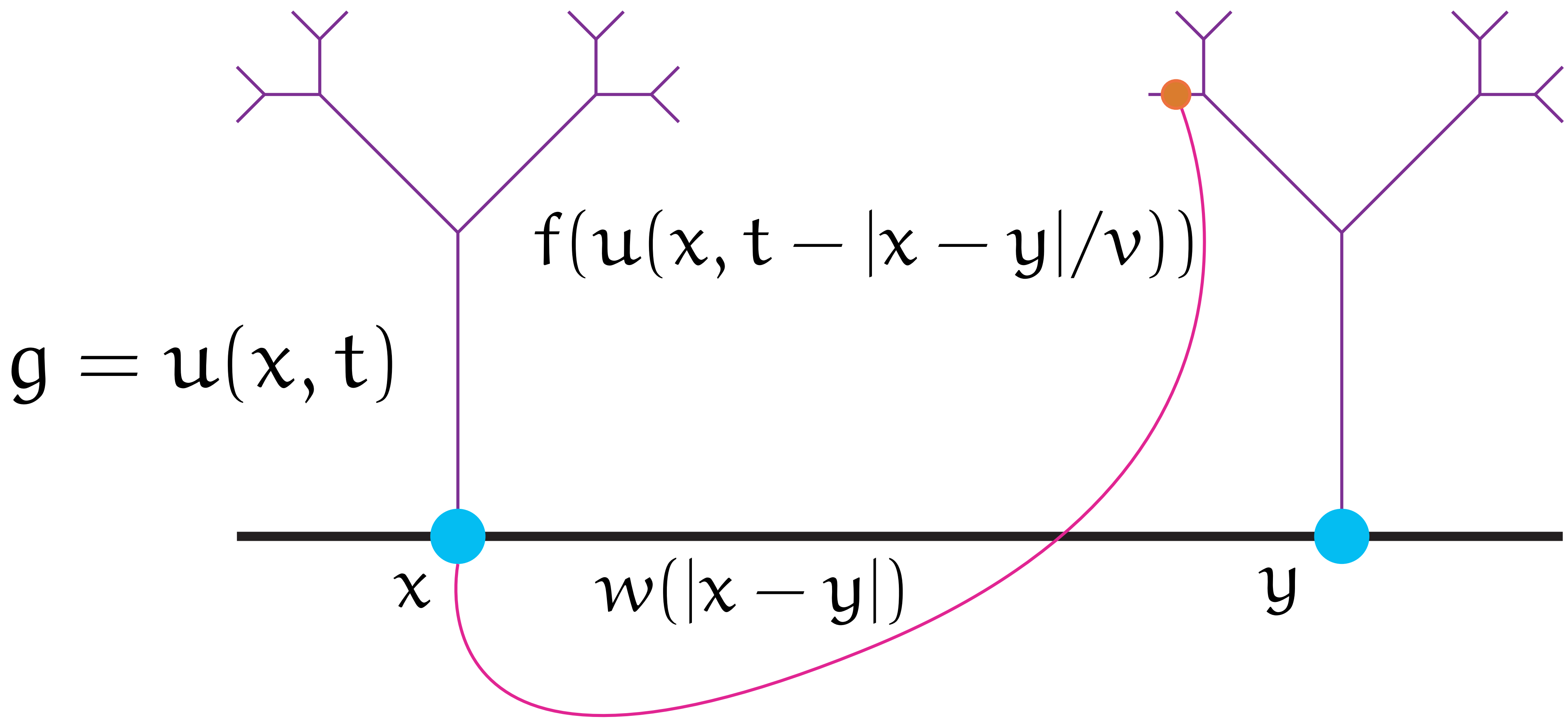
Shilnikov saddle-node route to chaos

van Veen and Liley, PRL, **97**, 208101 (2006)

Spatially extended models

$$g = w \otimes \eta * f$$

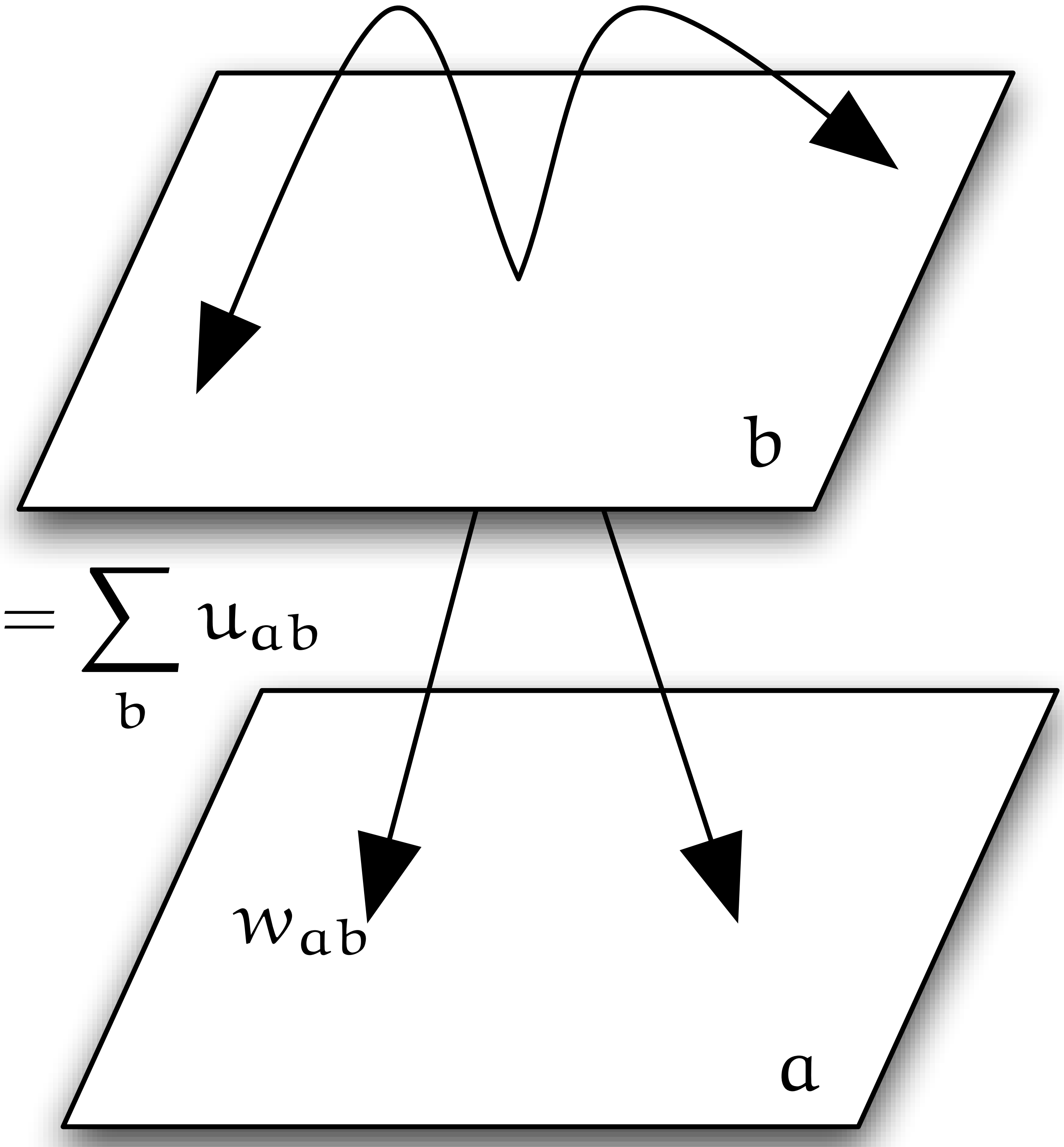
Simplest neural field model: Wilson-Cowan ('72), Amari ('77)



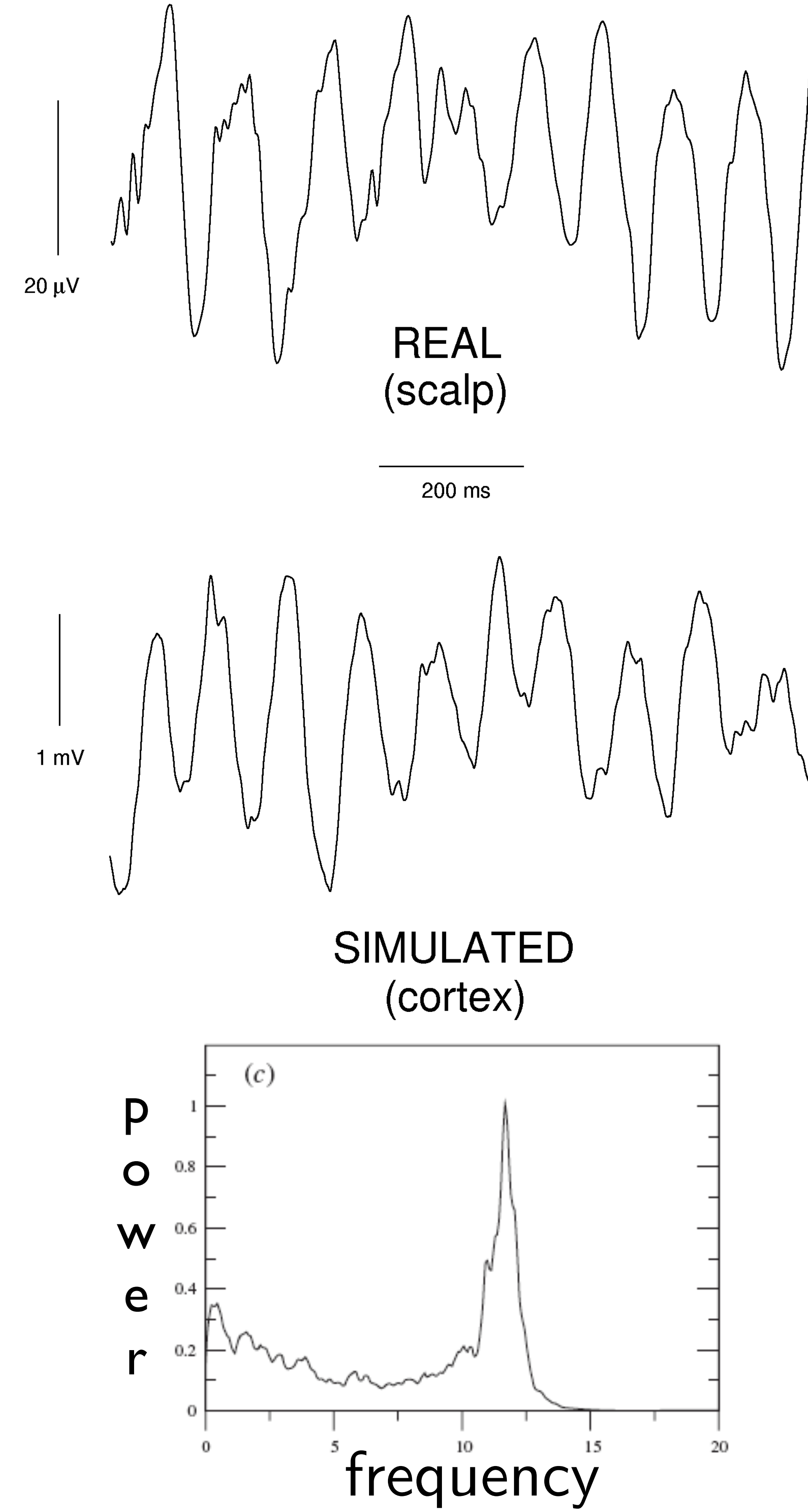
$$u(x,t) = \int_{-\infty}^{\infty} dy\ w(y) \int_0^{\infty} ds\ \eta(s)\ f\left(u(x-y, t-s-|y|/v)\right)$$

2D layers

$$u_{ab} = \eta_{ab} * \psi_{ab}$$



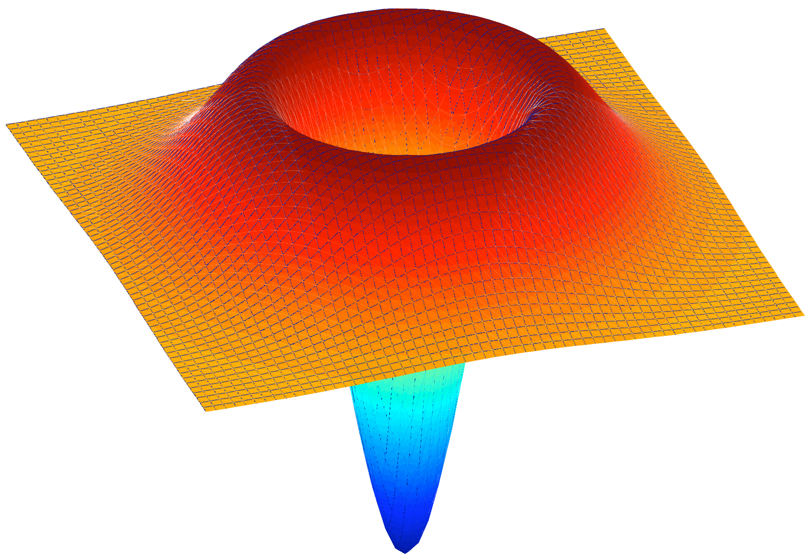
$$h_a = \sum_b u_{ab}$$



$$\psi_{ab}(\mathbf{r}, t) = \int_{\mathbb{R}^2} d\mathbf{r}' \; w_{ab}(\mathbf{r}, \mathbf{r}') f_b \circ h_b\left(\mathbf{r}', t - |\mathbf{r} - \mathbf{r}'|/\nu_{ab}\right)$$

Turing instability analysis

E layer and I layer



$$e^{i\mathbf{k}\cdot\mathbf{r}}e^{\lambda t}$$

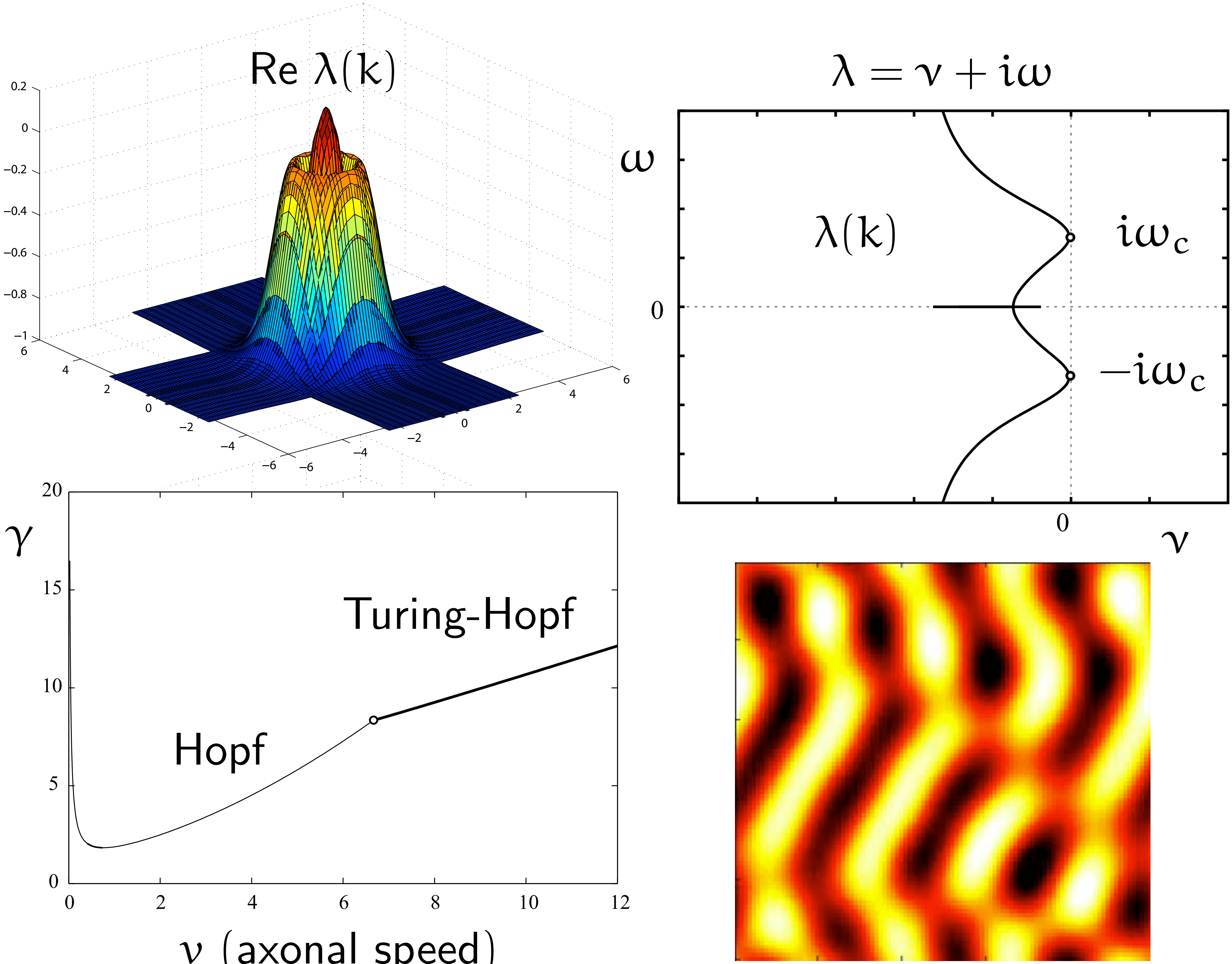
Continuous spectrum

$$\det\left(\mathcal{D}(k,\lambda)-I\right)=0$$

$$[\mathcal{D}(k,\lambda)]_{ab}=\widetilde{\eta}_{ab}(\lambda)G_{ab}(k,-i\lambda)\gamma_b$$

$\widetilde{\eta}=\text{LT}\ \eta$ $\quad G=\text{FLT}\ w(r)\delta(t-r/v)$ $\quad \gamma=f'(ss)$

S Coombes et al., PRE, **76**, 05190 (2007)



Re λ(k)
λ = ν + iω
ω
λ(k)
iω_c
−iω_c
0
ν
γ
20
15
10
5
0
2
4
6
8
10
12
Turing-Hopf
Hopf
ν (axonal speed)

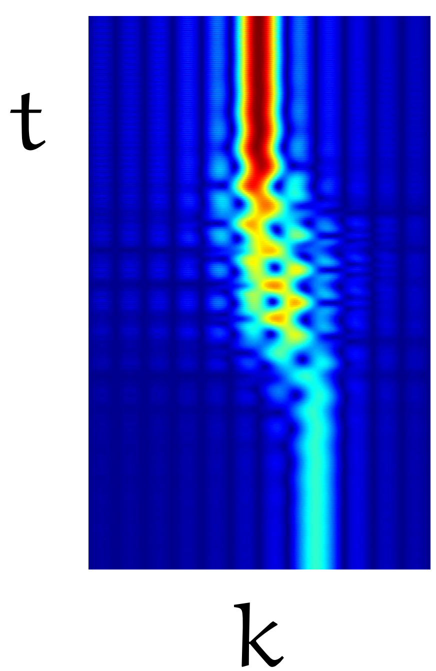
Amplitude Equations (one D)

Coupled mean-field Ginzburg–Landau equations describing a Turing–Hopf bifurcation with modulation group velocity of $O(1)$.

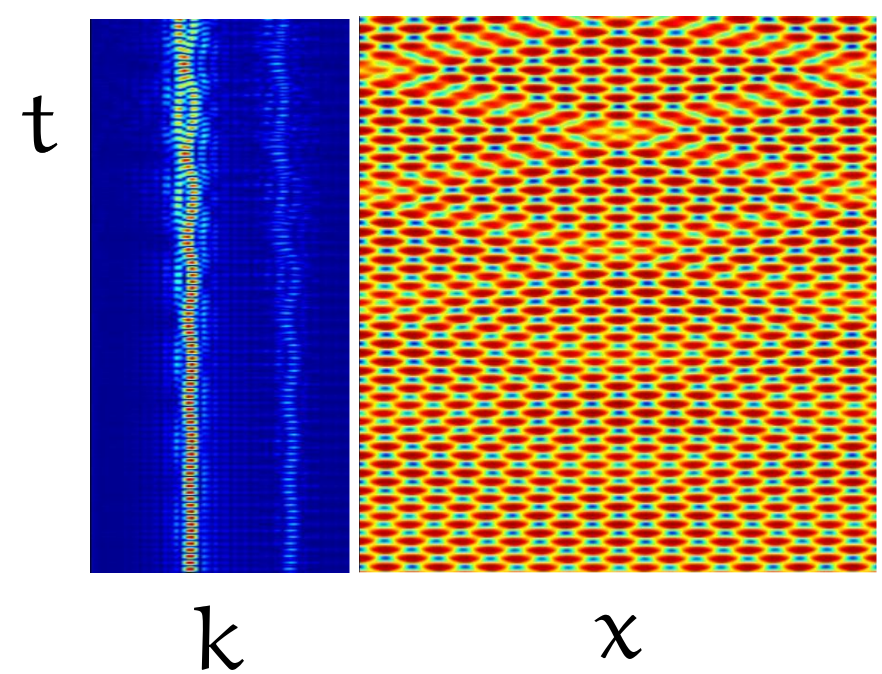
$$\frac{\partial A_1}{\partial \tau} = A_1\left(a + b|A_1|^2 + c\langle |A_2|^2 \rangle\right) + d\frac{\partial^2 A_1}{\partial \xi_+^2}$$

$$\frac{\partial A_2}{\partial \tau} = A_2\left(a + b|A_2|^2 + c\langle |A_1|^2 \rangle\right) + d\frac{\partial^2 A_2}{\partial \xi_-^2}$$

Benjamin–Feir (BF)



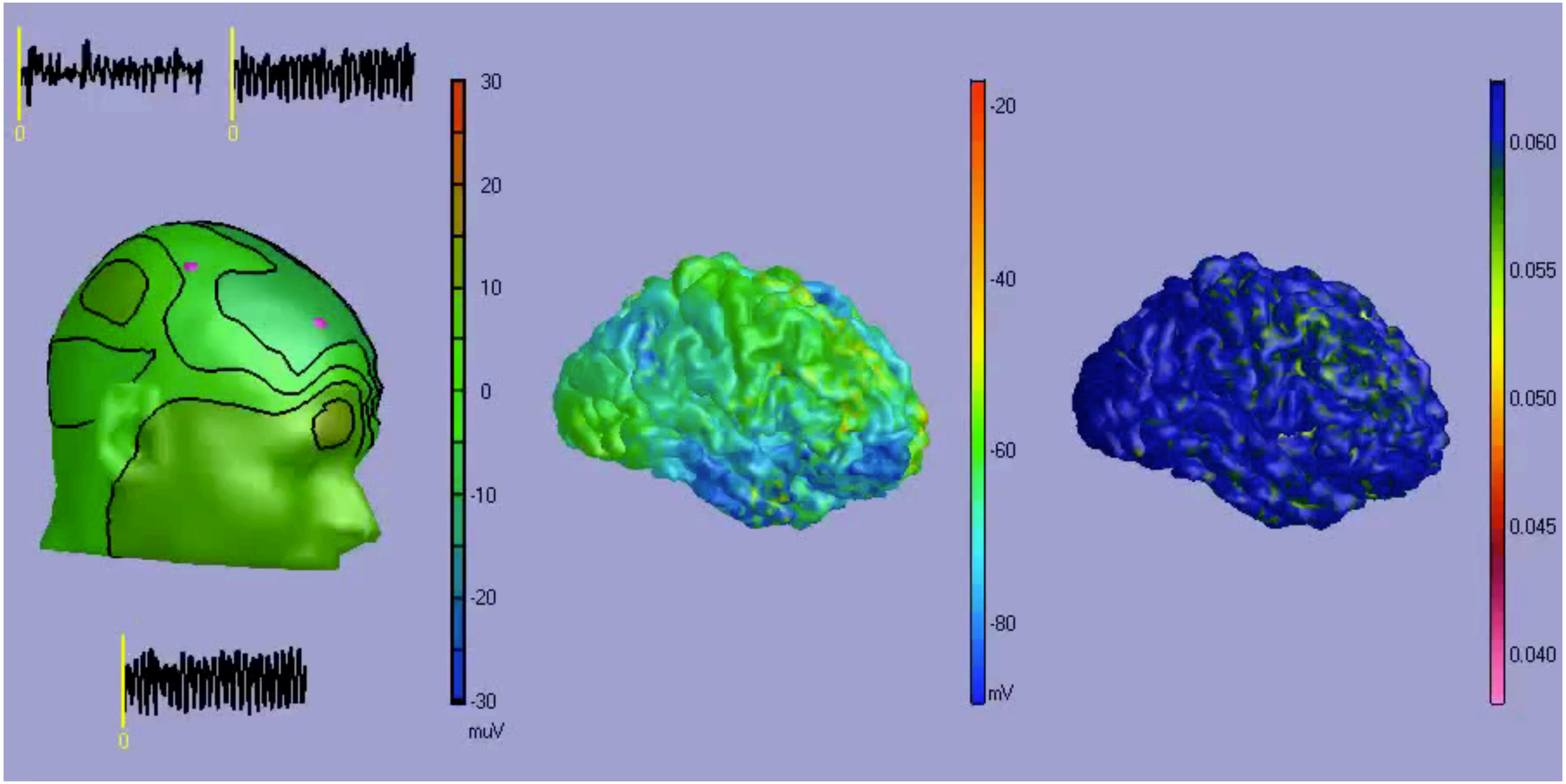
BF-Eckhaus instability



Coefficients in terms of integral transforms of w and η .

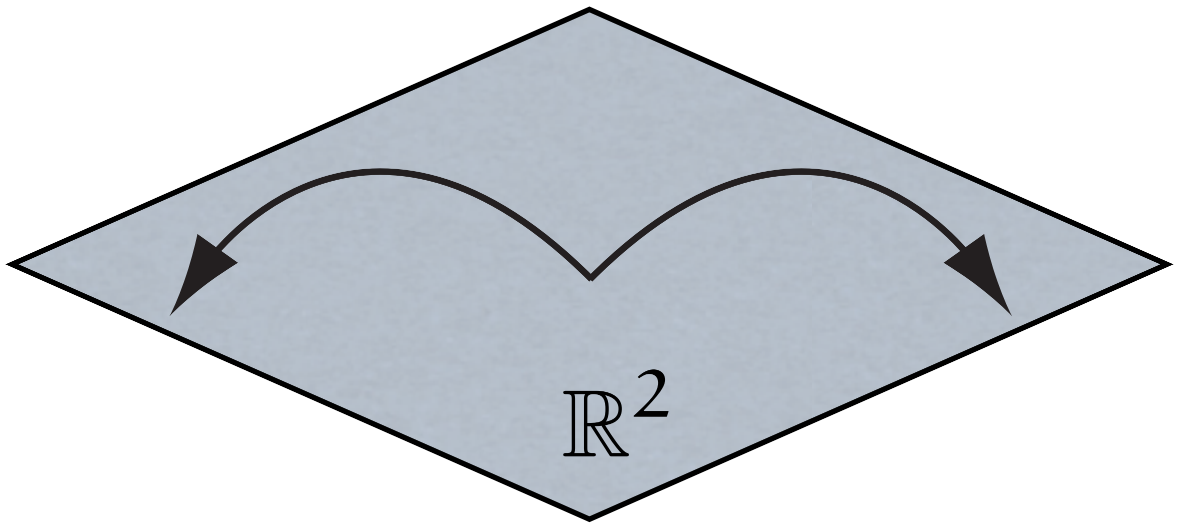
Applications to co-registered EEG/fMRI

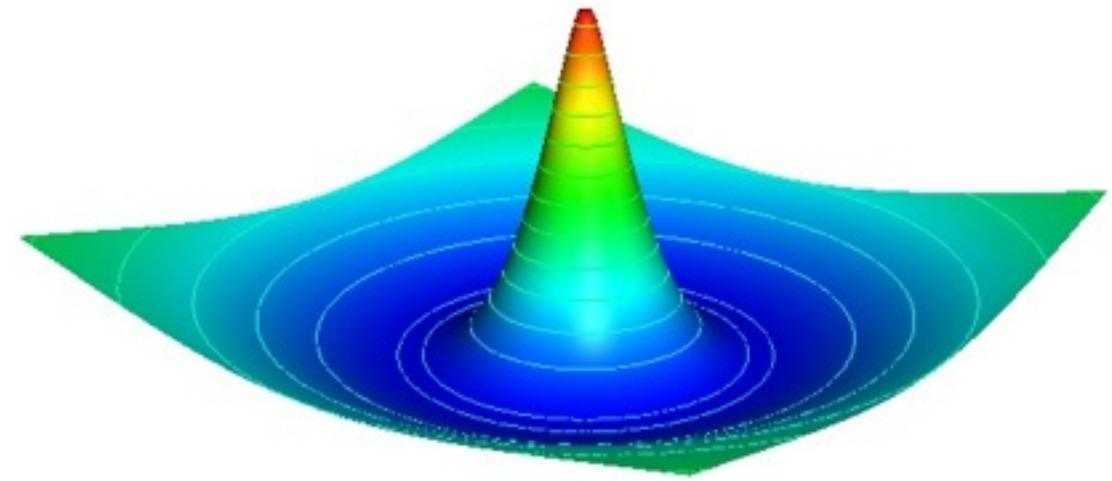
Ingo Bojak



Bojak, I., Oostendorp, T. F., Reid, A. T., Kotter, R., 2009. Realistic mean field forward predictions for the integration of co-registered EEG/fMRI. BMC Neuroscience 10, L2.

A simple 2D neural field model

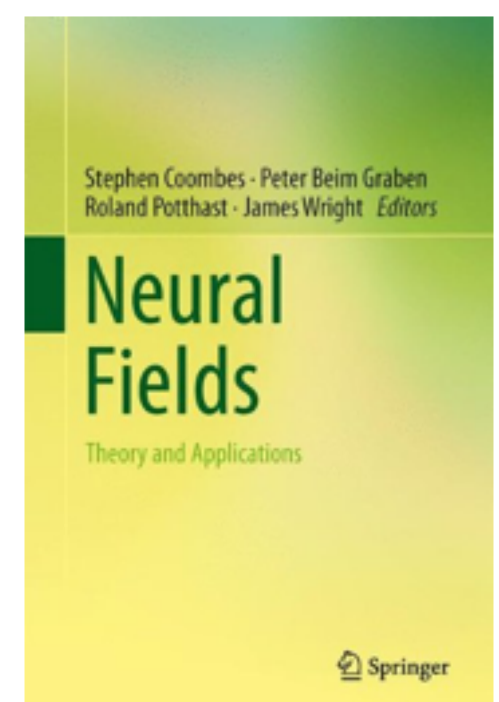




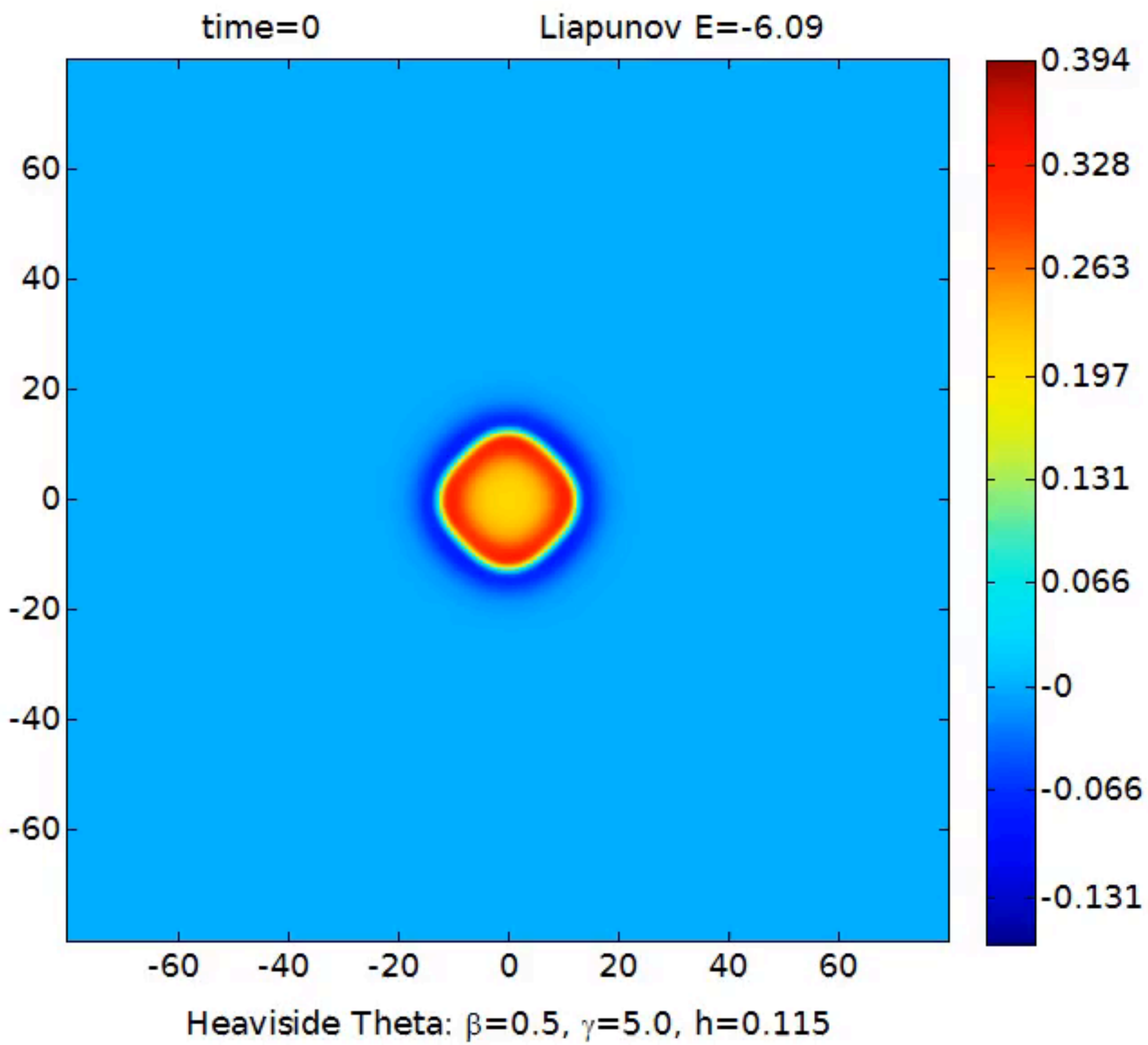
$$u_t(\mathbf{x},t) = -u(\mathbf{x},t) + \int_{\mathbb{R}^2} w(\mathbf{x}-\mathbf{x}')H[u(\mathbf{x}',t)-h]d\mathbf{x}'$$

2D Amari model

Neural Fields: Theory and Application, (531 pages)
Ed. S Coombes, P beim Graben, R Potthast
and J J Wright, Springer Verlag, June 2014

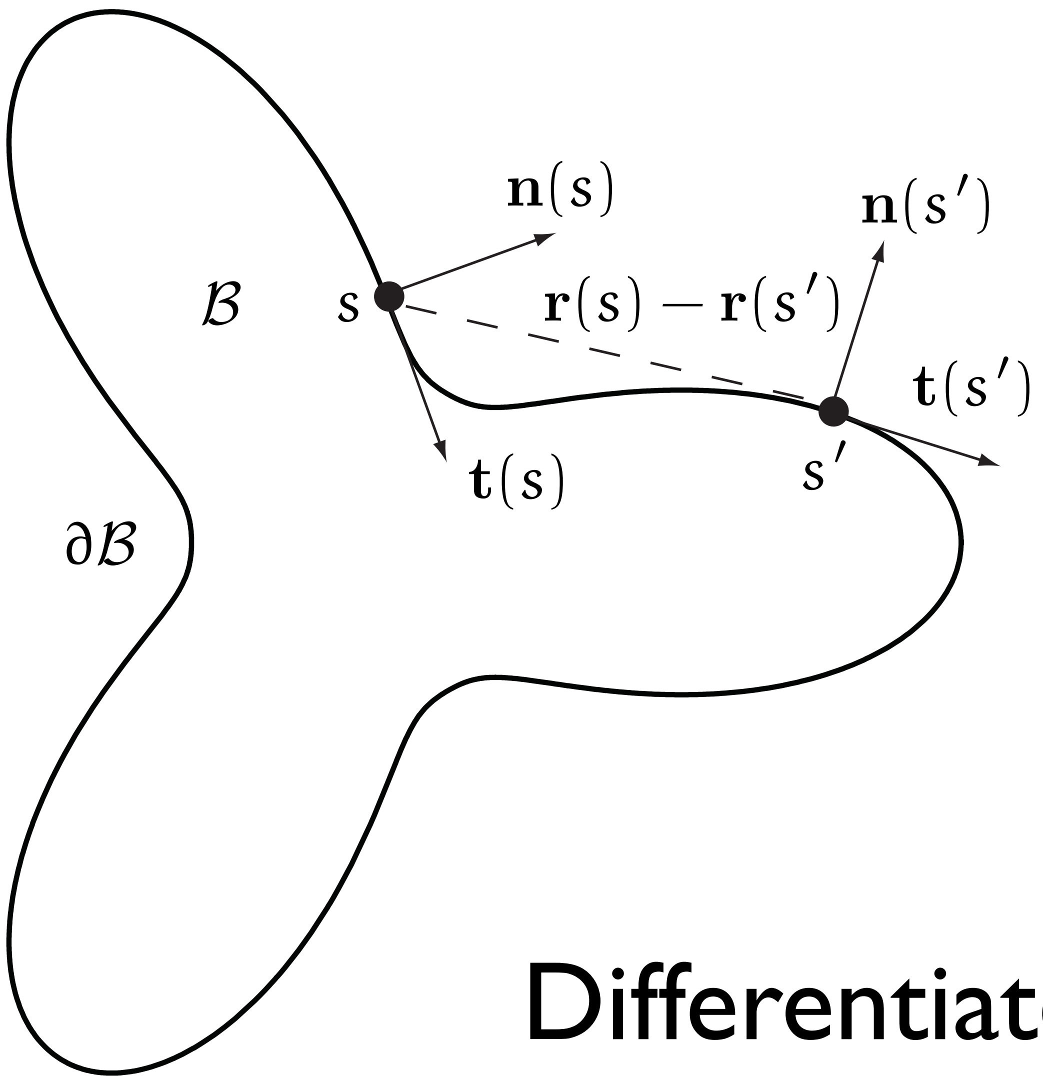


A simulation



An *interface* is easily identified

Interface dynamics in 2D



$$\mathbf{n} = -\nabla_{\mathbf{x}} u / |\nabla_{\mathbf{x}} u|$$

$$u_t(\mathbf{x}, t) = -u(\mathbf{x}, t) + \psi(\mathbf{x}, t)$$

$$\psi(\mathbf{x}, t) = \int_{\mathcal{B}(t)} w(|\mathbf{x} - \mathbf{x}'|) d\mathbf{x}'$$

Differentiate $u(\mathbf{x}, t) = h$ along $\partial\mathcal{B}(t)$

$$\nabla_{\mathbf{x}} u \cdot \frac{d\mathbf{r}}{dt} + \frac{\partial u}{\partial t} = 0$$

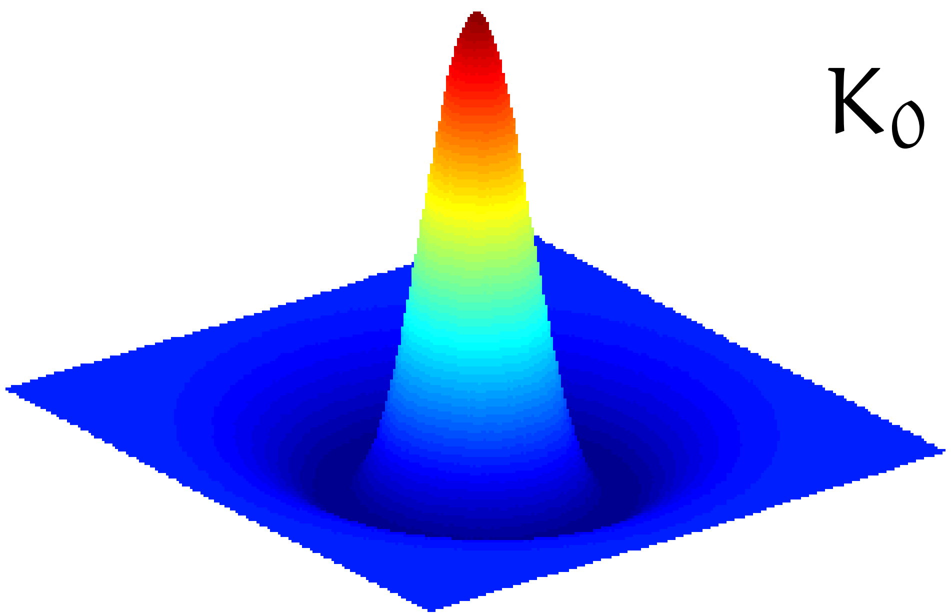
Normal velocity

$$\mathbf{n} \cdot \frac{d\mathbf{r}}{dt} = \frac{u_t}{|z|} \qquad z \equiv \nabla_{\mathbf{x}} u(\mathbf{x}, t)|_{\mathbf{x}=\mathbf{r}}$$

$$u_t = -h + \int_{\mathcal{B}} d\mathbf{x}' w(|\mathbf{r} - \mathbf{x}'|),$$

$$z_t = -z + \nabla_{\mathbf{x}} \int_{\mathcal{B}} d\mathbf{x}' w(|\mathbf{x} - \mathbf{x}'|) \Big|_{\mathbf{x}=\mathbf{r}}$$

$$\int_{\mathcal{B}} \nabla \Psi = \oint_{\partial \mathcal{B}} \mathbf{n} \Psi$$



K_0 - Bessel function of the second kind

$$(1 - \nabla^2) K_0(\mathbf{x}) = 2\pi \delta(\mathbf{x})$$

$$w(r) = \sum_{i=1}^{N} A_i K_0(\alpha_i r)$$

$$\int_{\mathcal{B}} d\mathbf{x}' \nabla_{\mathbf{x}} w(|\mathbf{x} - \mathbf{x}'|) = -\oint_{\partial \mathcal{B}} ds \mathbf{n}(s) w(|\mathbf{x} - \mathbf{x}'(s)|)$$

$$\int_{\mathcal{B}} d\mathbf{x}' K_0(\alpha |\mathbf{x} - \mathbf{x}'|) = -\frac{1}{\alpha} \oint_{\partial \mathcal{B}} ds \mathbf{n}(s) \cdot \frac{\mathbf{x} - \mathbf{r}(s)}{|\mathbf{x} - \mathbf{r}(s)|} K_1(\alpha |\mathbf{x} - \mathbf{r}(s)|) + C \frac{2\pi}{\alpha^2}$$

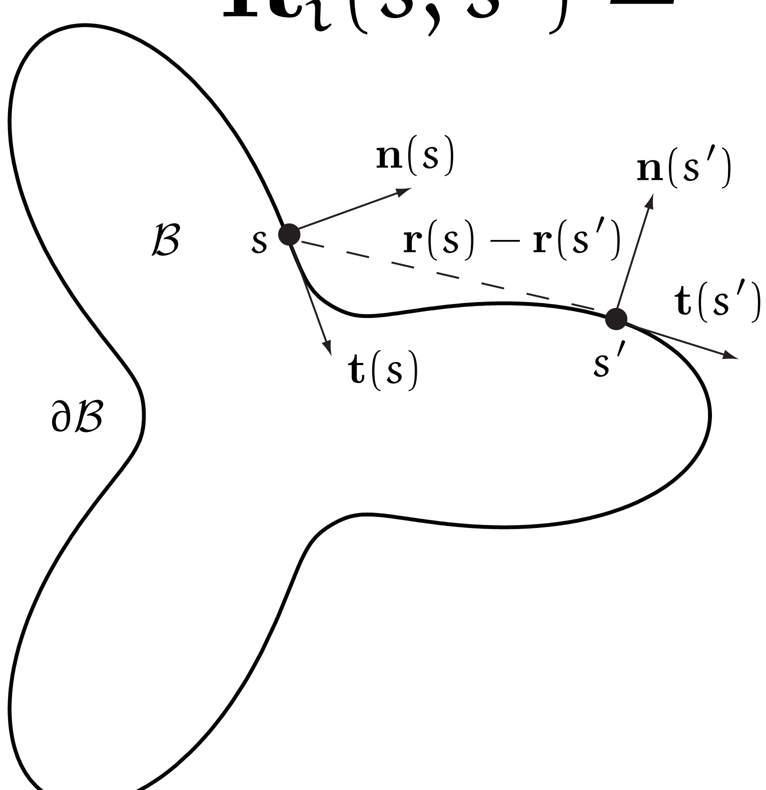
Dynamics from data on the boundary only

For points on the boundary parametrised by s

$$u_t(s) = -h + \sum_{i=1}^{N} A_i \left\{ \oint_{\partial\mathcal{B}} ds'\mathbf{n}(s') \cdot \mathbf{R}_i(s,s') + \frac{\pi}{\alpha_i^2} \right\}$$

$$z_t(s) = -z(s) - \oint_{\partial\mathcal{B}} ds'\mathbf{n}(s') w(|\mathbf{r}(s) - \mathbf{r}(s')|)$$

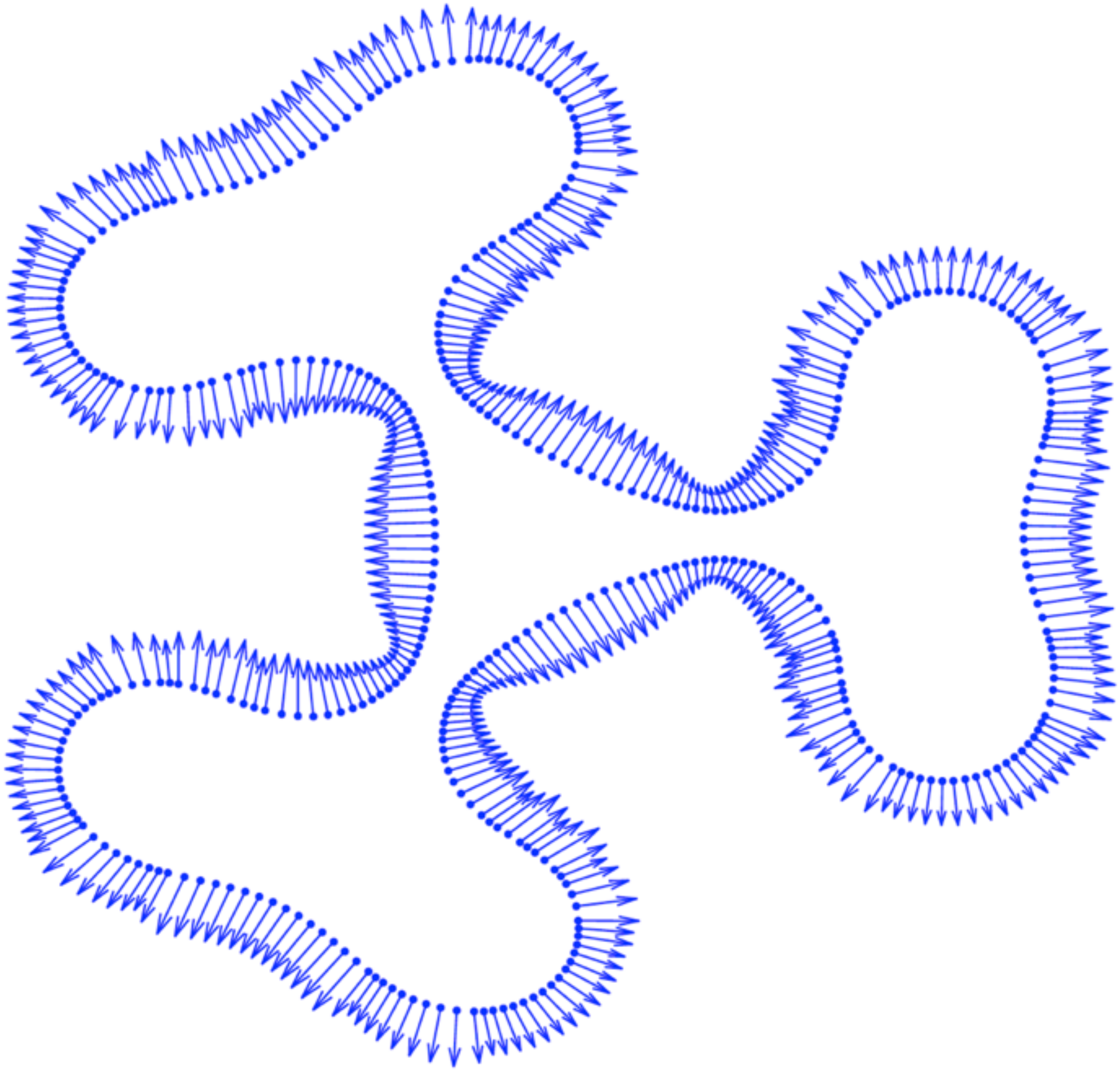
$$\mathbf{R}_i(s,s') = -\frac{1}{\alpha_i} \frac{\mathbf{r}(s) - \mathbf{r}(s')}{|\mathbf{r}(s) - \mathbf{r}(s')|} K_1(\alpha_i |\mathbf{r}(s) - \mathbf{r}(s')|)$$

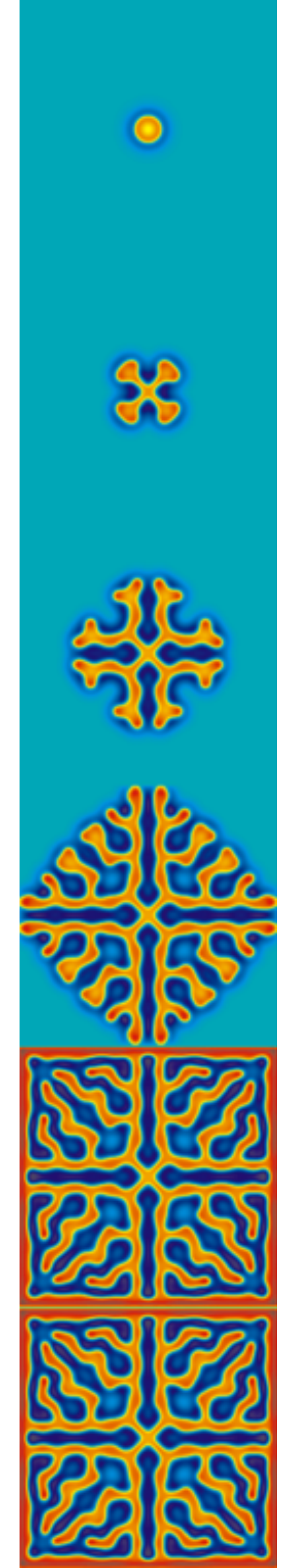


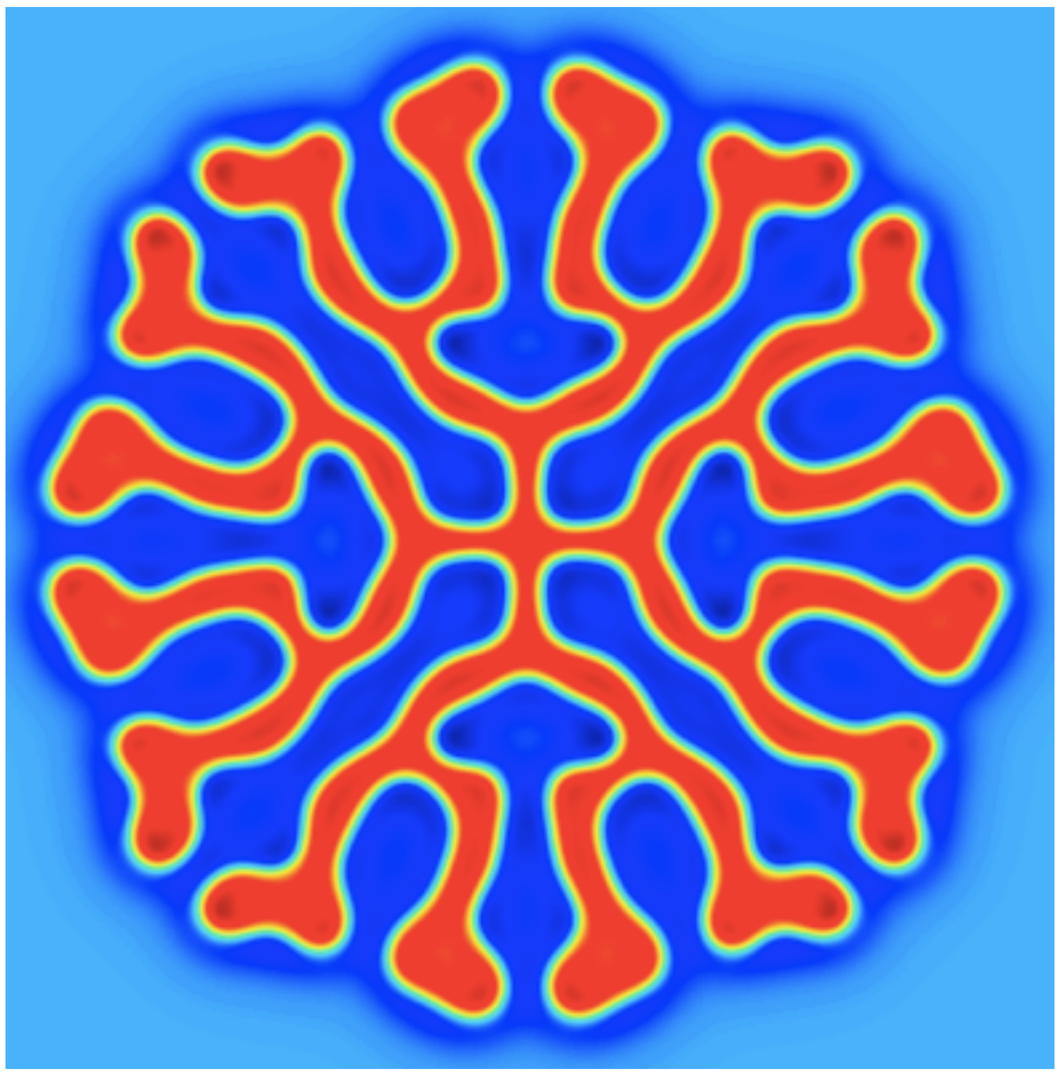
Biot-Savart style interaction

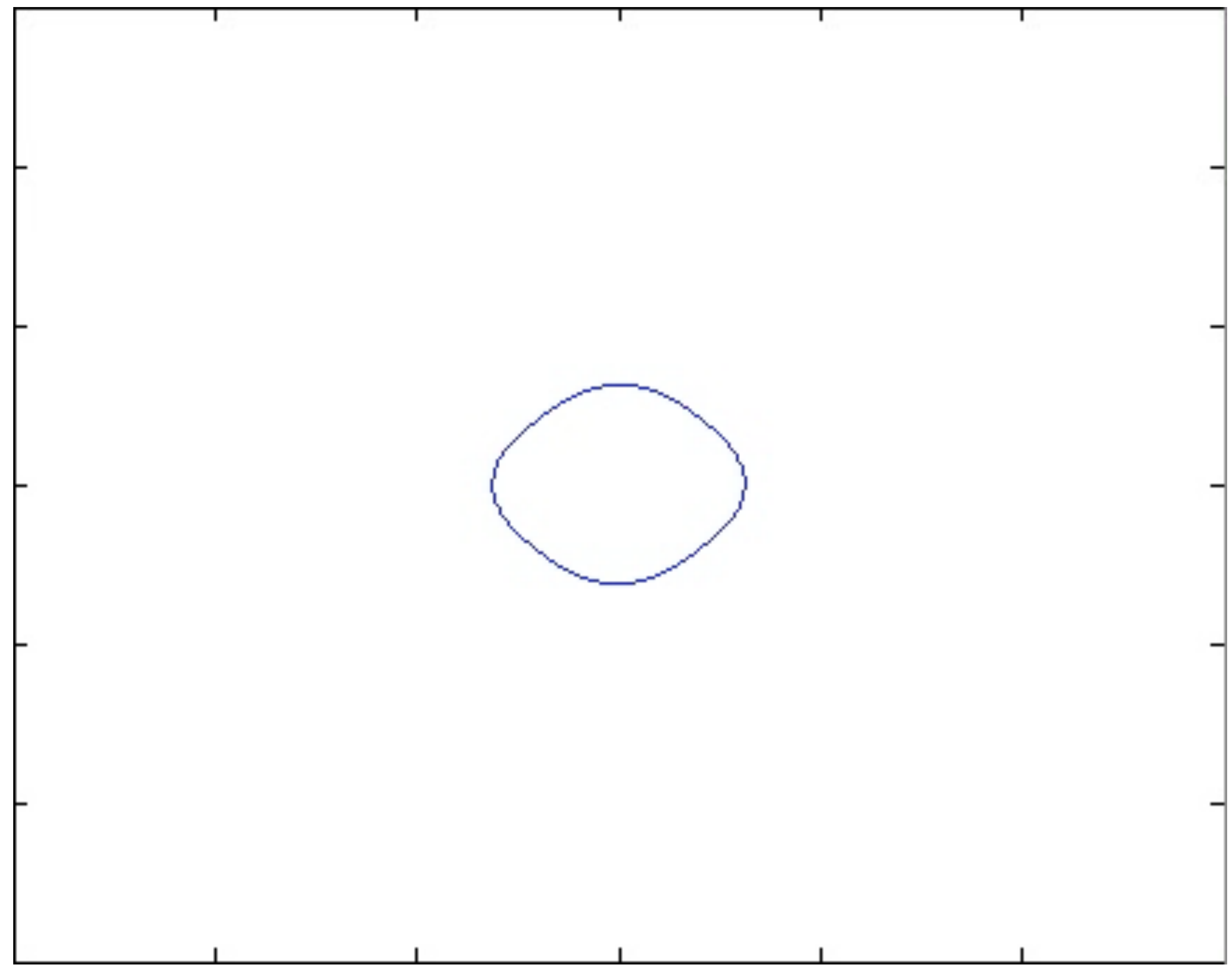
[effective repulsion between two arc length positions with anti-parallel tangent vectors]

Simple numerical scheme





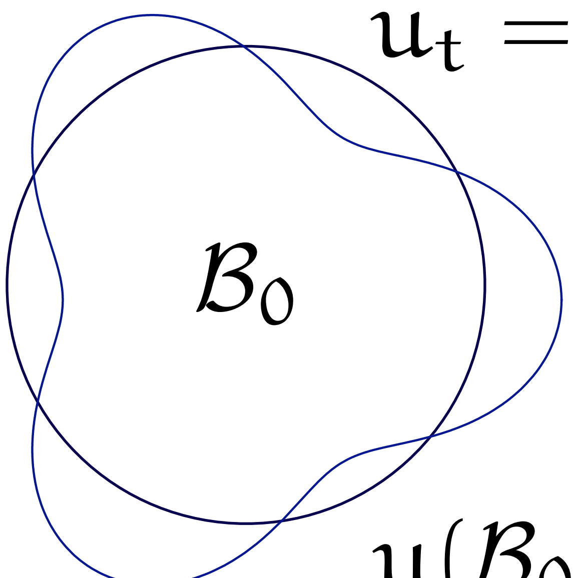




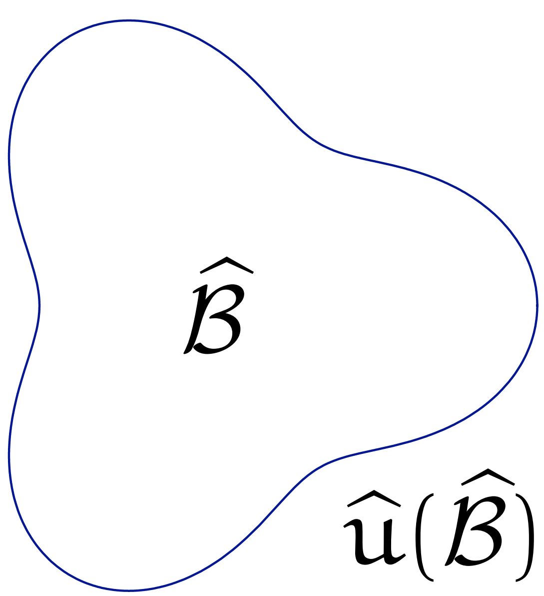
Stability of stationary states

Zero normal velocity - $u_t = 0,\ \mathcal{B}(t) \to \mathcal{B}_0$

$$h = \int_{\mathcal{B}_0} d\mathbf{x}' w(|\mathbf{r} - \mathbf{x}'|) = \sum_{i=1}^{N} A_i \left\{ \oint_{\partial\mathcal{B}_0} ds' \mathbf{n}(s') \cdot \mathbf{R}_i(s, s') + \frac{\pi}{\alpha_i^2} \right\}$$

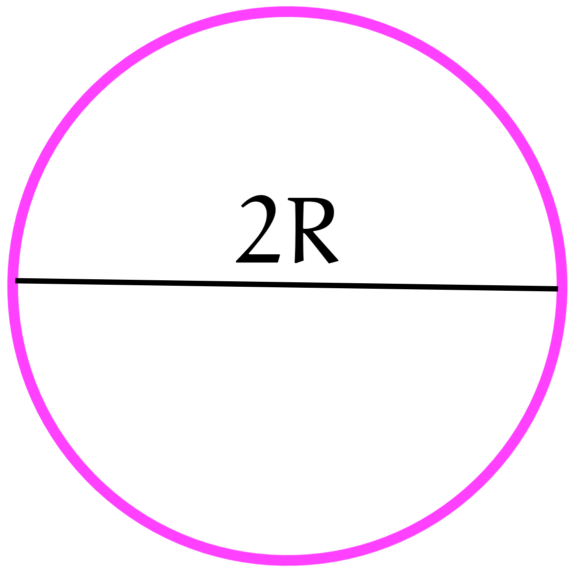


$$\delta u(t) = \widehat{u}|_{\mathbf{x} \in \widehat{\partial\mathcal{B}}} - u|_{\mathbf{x} \in \partial\mathcal{B}_0} = 0$$



defines $\widehat{R} = R + \delta R(\theta, t)$

Spots

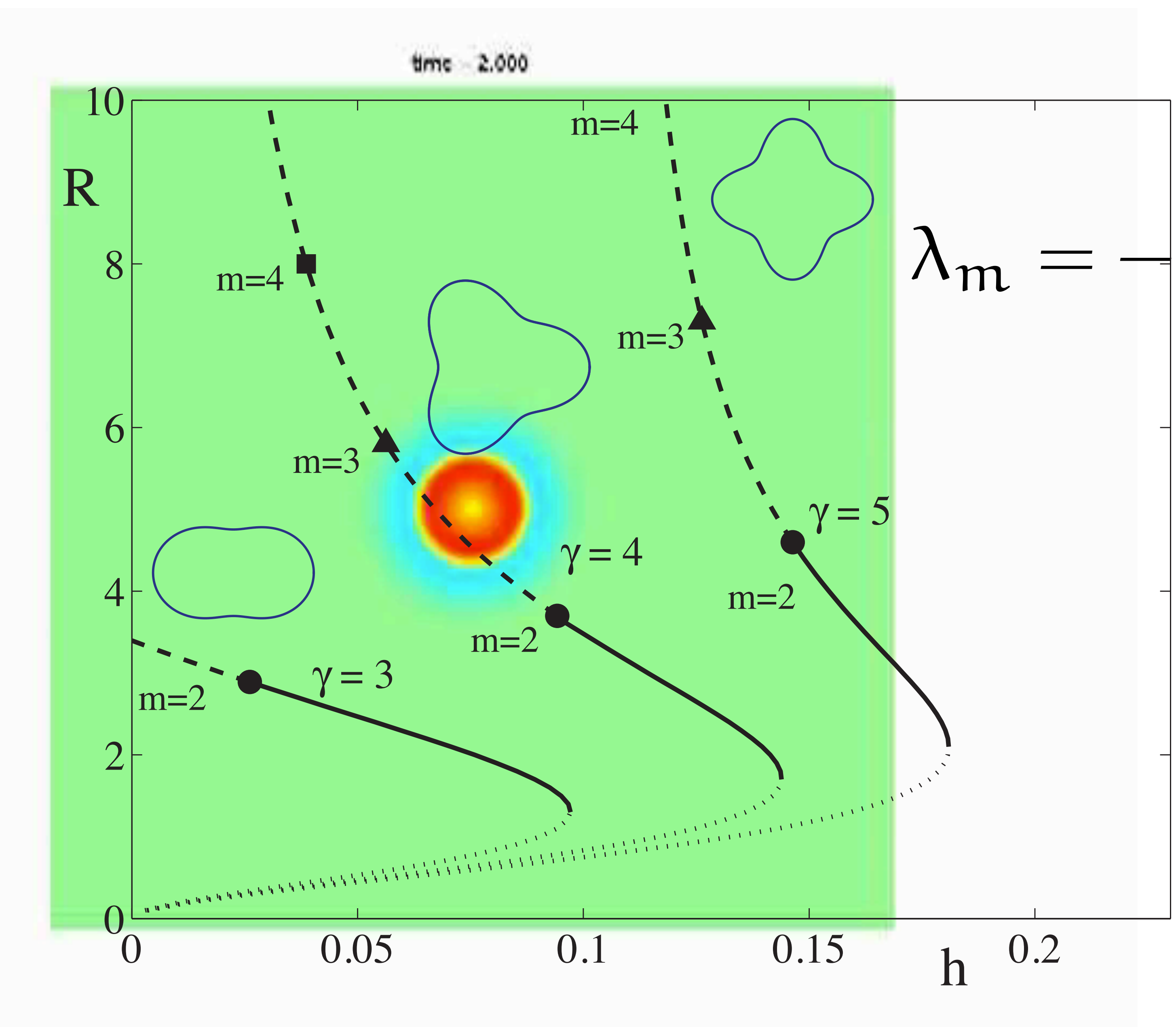


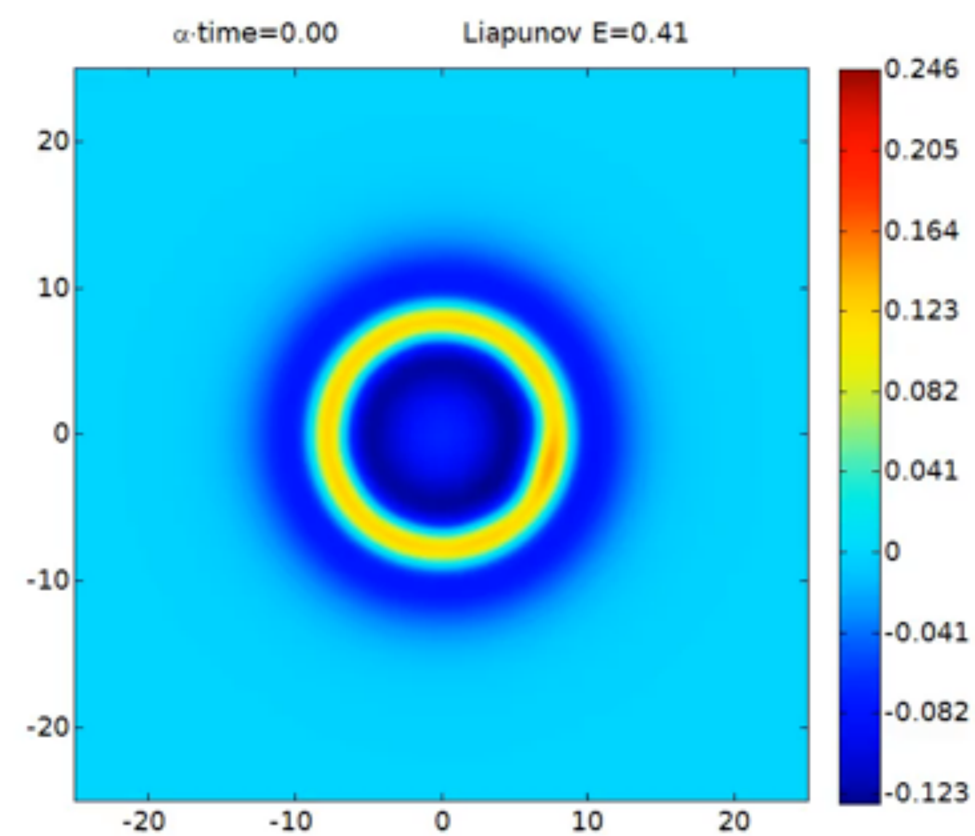
Using Graf's formula:

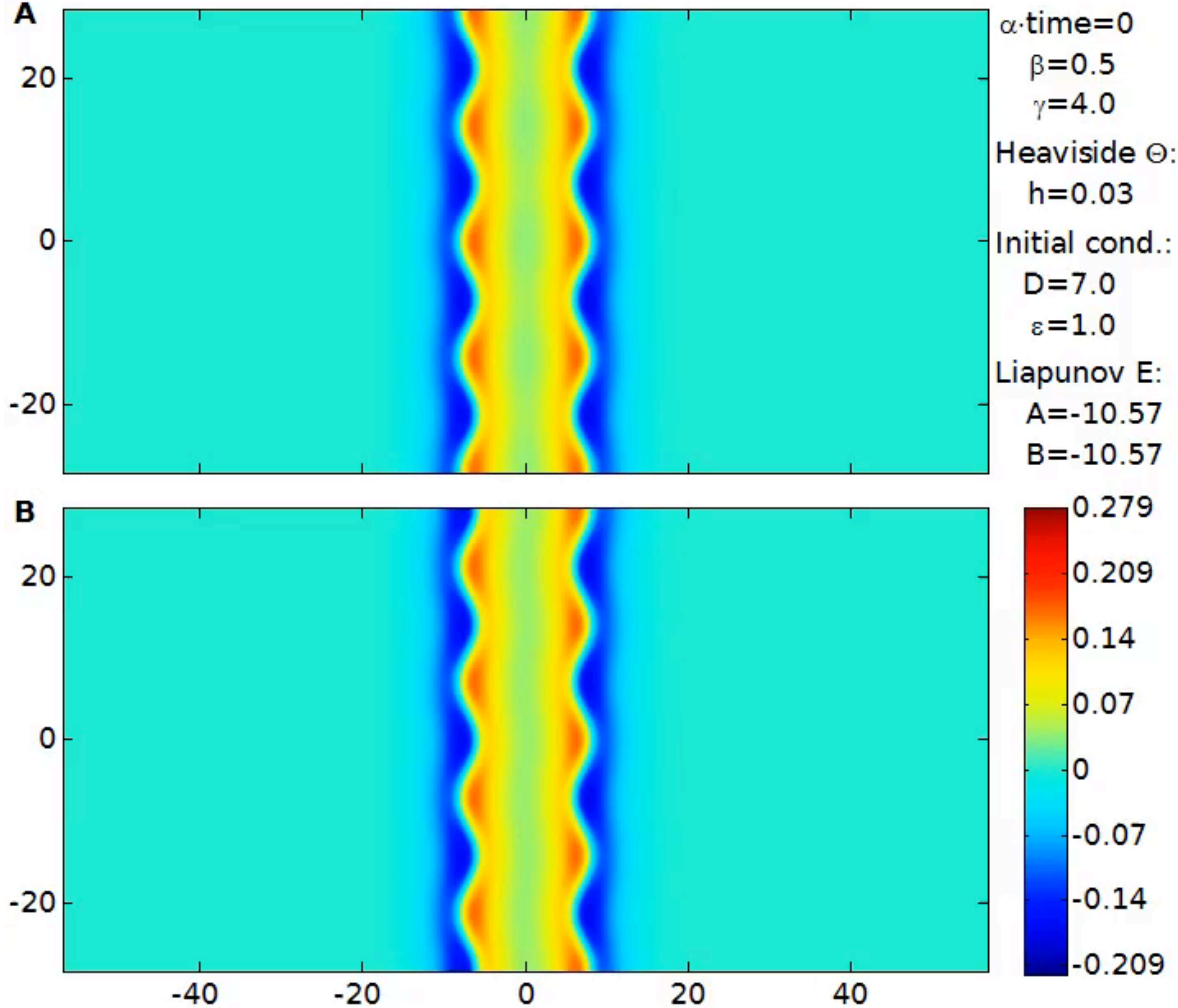
$$h = 2\pi \sum_{i=1}^{N} A_i \left\{ \frac{1}{\alpha_i^2} - \frac{R}{\alpha_i} K_1(\alpha_i R) I_0(\alpha_i R) \right\}$$

$$\delta R(\theta, t) = \cos m\theta e^{\lambda_m t}$$

$$\lambda_m = -1 + \frac{\sum_{i=1}^{N} A_i K_m(\alpha_i R) I_m(\alpha_i R)}{\sum_{i=1}^{N} A_i K_1(\alpha_i R) I_1(\alpha_i R)}$$







A
B
α·time=0
β=0.5
γ=4.0
Heaviside Θ:
h=0.03
Initial cond.:
D=7.0
ε=1.0
Liapunov E:
A=-10.57
B=-10.57
0.279
0.209
0.14
0.07
0
-0.07
-0.14
-0.209
20
0
-20
-40
-20
0
20
40

Linear adaptation

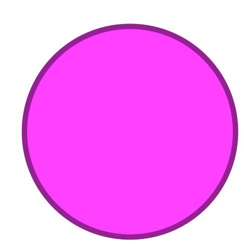
$$\frac{1}{\alpha}u_t = -u + \psi - ga, \qquad a_t = u - a$$

Exploit linearity

$$u(\cdot,t) = \int_{-\infty}^{t} ds\eta(t-s)\psi(\cdot,s), \;\; a(\cdot,t) = \int_{-\infty}^{t} dse^{-(t-s)}u(\cdot,s)$$

$$\eta(t) = \frac{\alpha}{\lambda_- - \lambda_+}\left\{(1-\lambda_+)e^{-\lambda_+ t} - (1-\lambda_-)e^{-\lambda_- t}\right\}$$

$$\lambda_\pm = \frac{1+\alpha \pm \sqrt{(1+\alpha)^2 - 4\alpha(1+g)}}{2}$$

Easy to construct stationary spots  $h \to h(1+g)$

Instabilities and travelling pulses

Eigenvalues determined by $\mathcal{E}_m(\lambda) = 0$

$$\mathcal{E}_m(\lambda) = \frac{1}{\widetilde{\eta}(\lambda)} - (1+g)W_m \qquad \widetilde{\eta}(\lambda) = \int_0^\infty e^{-\lambda s}\eta(s)ds$$

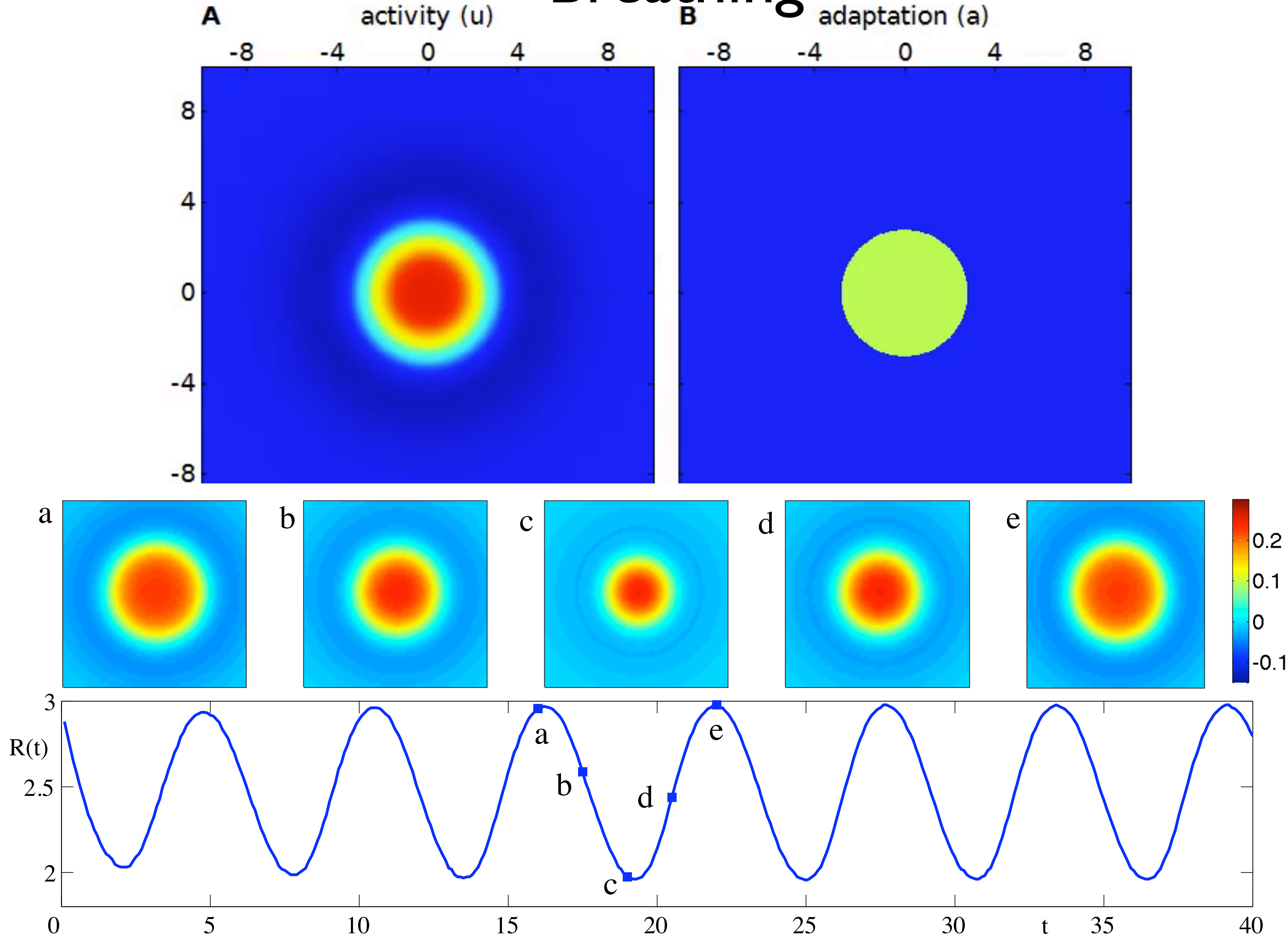
$$W_m = \frac{R}{|\psi'(R)|}\int_0^{2\pi} d\theta \cos(m\theta) w(\mathcal{R}(\theta))$$

$m = 0$ mode ($\lambda = i\omega$, breath) unstable when $g > 1/\alpha$

emergent frequency $\omega = \sqrt{\alpha g - 1}$

$m = 1$ mode ($\lambda \in \mathbb{R}$, drift) unstable when $g > 1/\alpha$

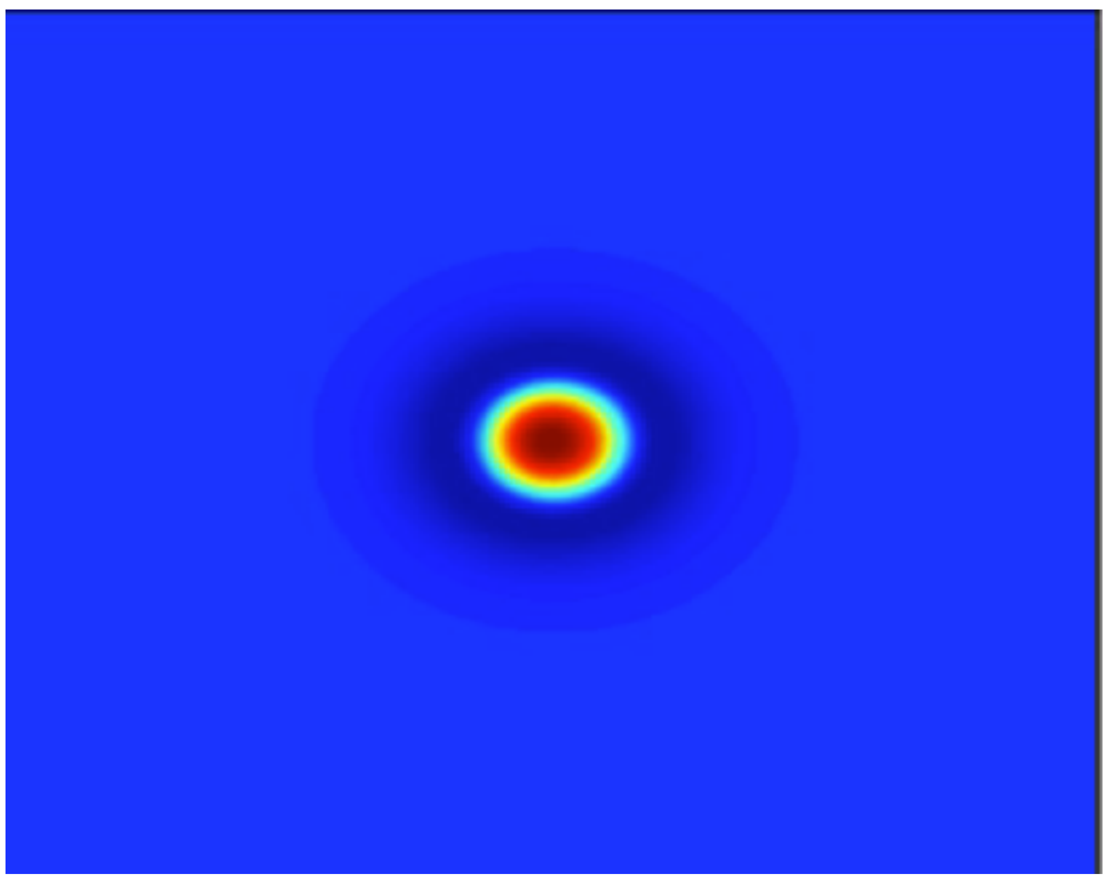
Breathing

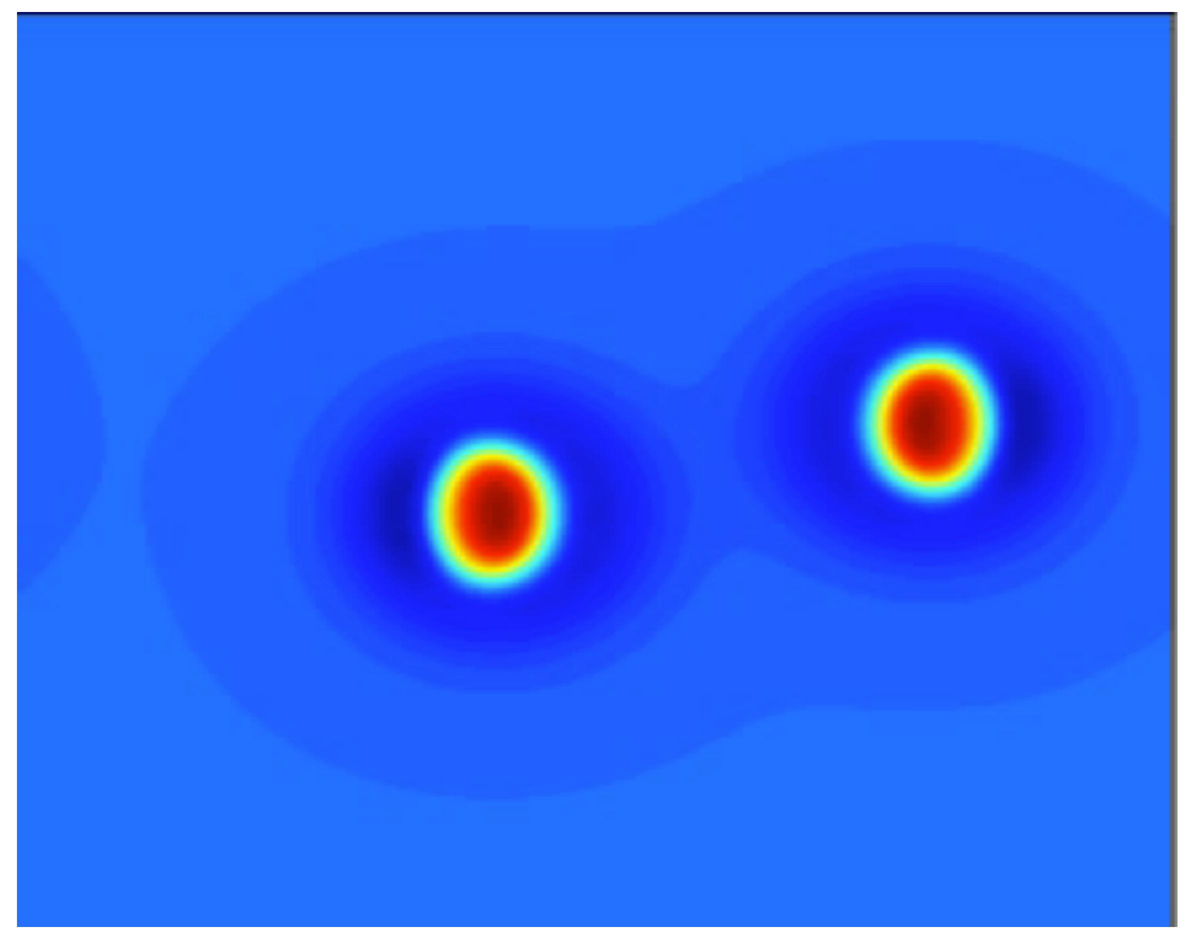

A
activity (u)
B
adaptation (a)
-8
-4
0
4
8
a
b
c
d
e
0.2
0.1
0
-0.1
R(t)
3
2.5
2
0
5
10
15
20
25
30
t
35
40

Drifting

… at the point where $g = 1/\alpha$ the shape of the spot deviates from circular with an amplitude that depends on quadratic and higher powers of c

$$R(\theta) = R + \sum_{m \geq 2} c^m a_m \cos m\theta$$





Lu Y, Amari S: Traveling bumps and their collisions in a 2D neural field.
Neural Computation 2011, 23:1248–1260

Drifting (weakly nonlinear analysis)

For any sigmoid drifting will occur when g increases through $1/\alpha$

Amplitude analysis
(translation operator and drift eigen-modes):

$$X(\mathbf{r},t) = \tau(\mathbf{p})\left[S(\mathbf{r}) + \sum_{j=1}^{2} a_j(t)\psi_j(\mathbf{r}) + \chi(\mathbf{r},t)\right]$$

$\mathbf{p}$ denotes location of spot

$$a = a_1 + ia_2$$

$$\dot{\mathbf{p}} = \mathbf{a}$$

$$\dot{a} = a(M_1|a|^2 + M_2\eta)$$

$$\pi M_1 = \frac{1}{6}\langle \mathcal{F}'''\psi_1^3 \mid \phi_1^\dagger\rangle + \langle \mathcal{F}''\psi_1 V_1^2 \mid \phi_1^\dagger\rangle + \langle \partial_{x_1} V_1 \mid \phi_1^\dagger\rangle,$$

$$\pi M_2 = \langle \mathcal{F}''\psi_1 V_4 \mid \phi_1^\dagger\rangle + \langle \gamma'(S)\psi_1 \mid \phi_1^\dagger\rangle + \langle \partial_{x_1} V_4 \mid \phi_1^\dagger\rangle$$

spare the details!

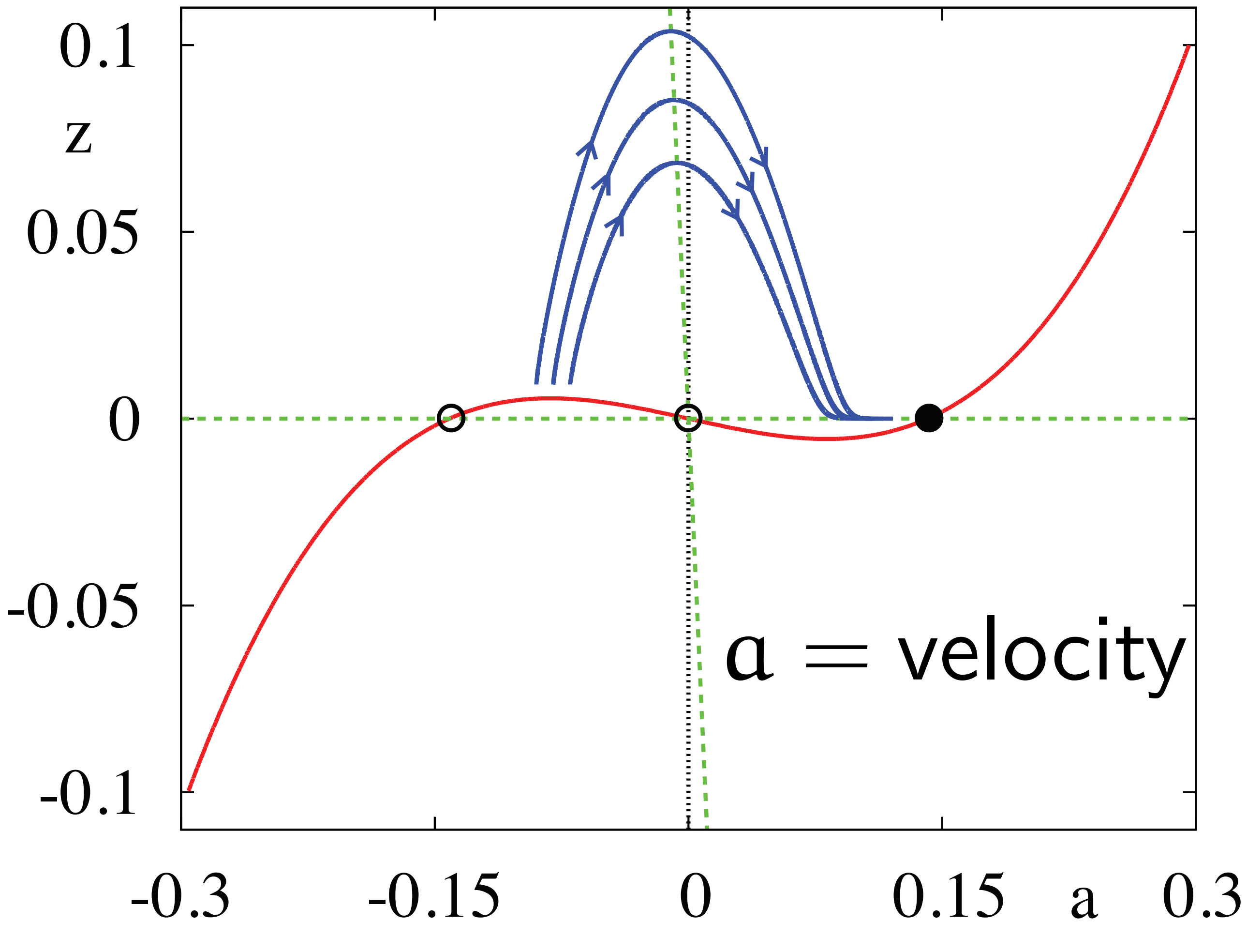
Scattering

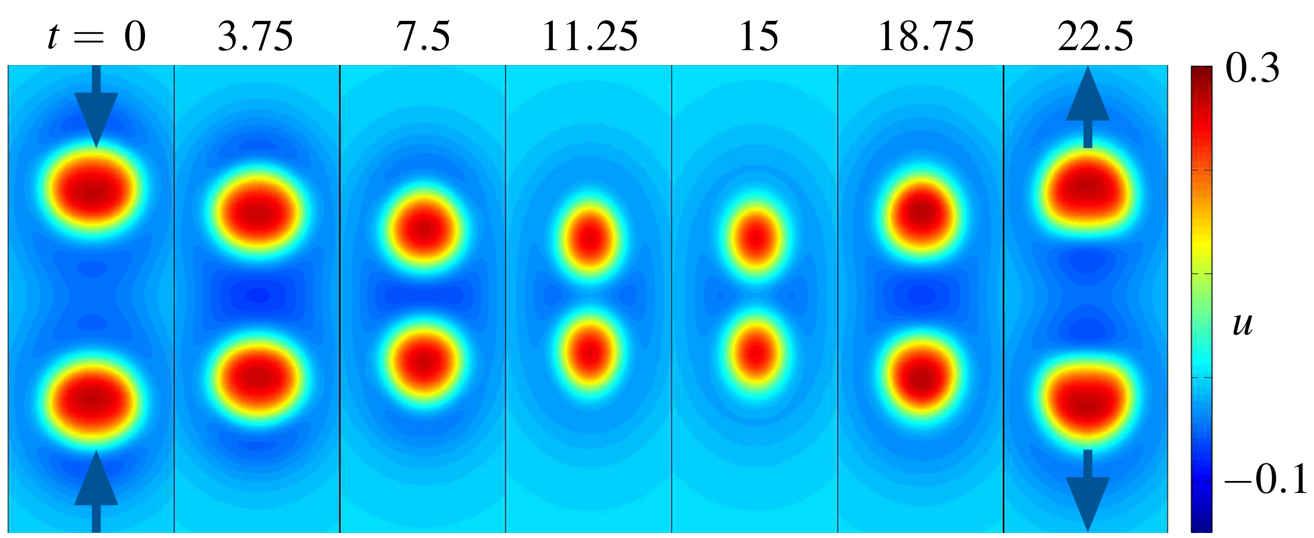
Two spots with centers offset by a vector $\mathbf{h} = 2\mathbf{p}$

$$\dot{p} = a + G_0 f(p), \qquad \dot{a} = M_1 a^3 + M_2 a\eta + H_0 f(p),$$

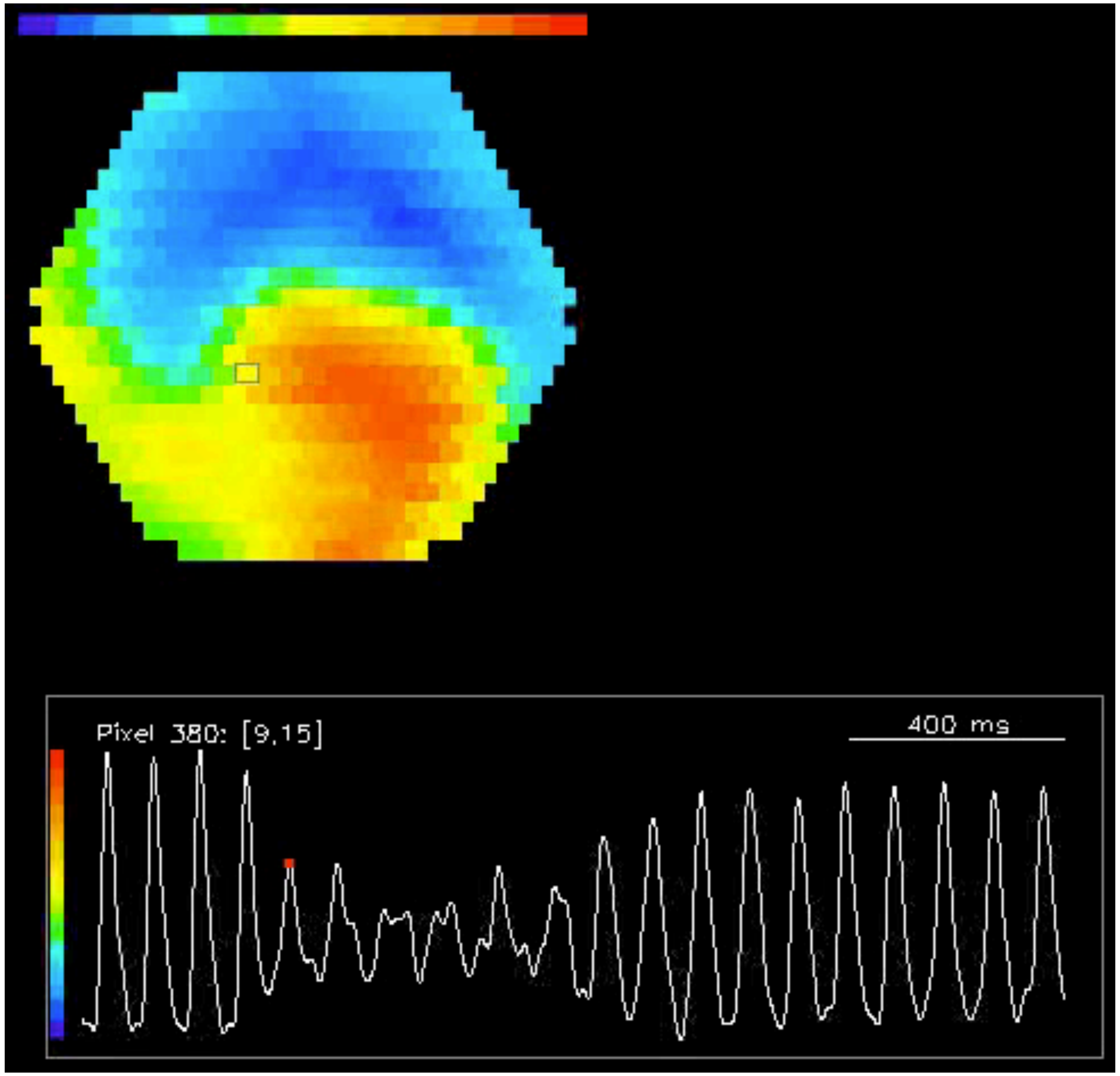
$z = f(p)$

$f(p) = e^{-2p}/\sqrt{2p}$

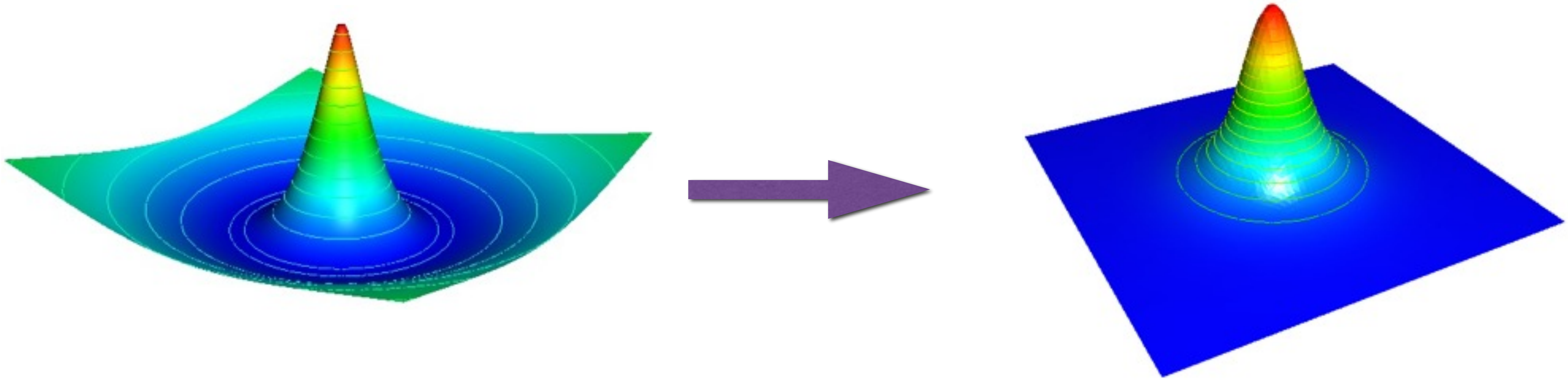


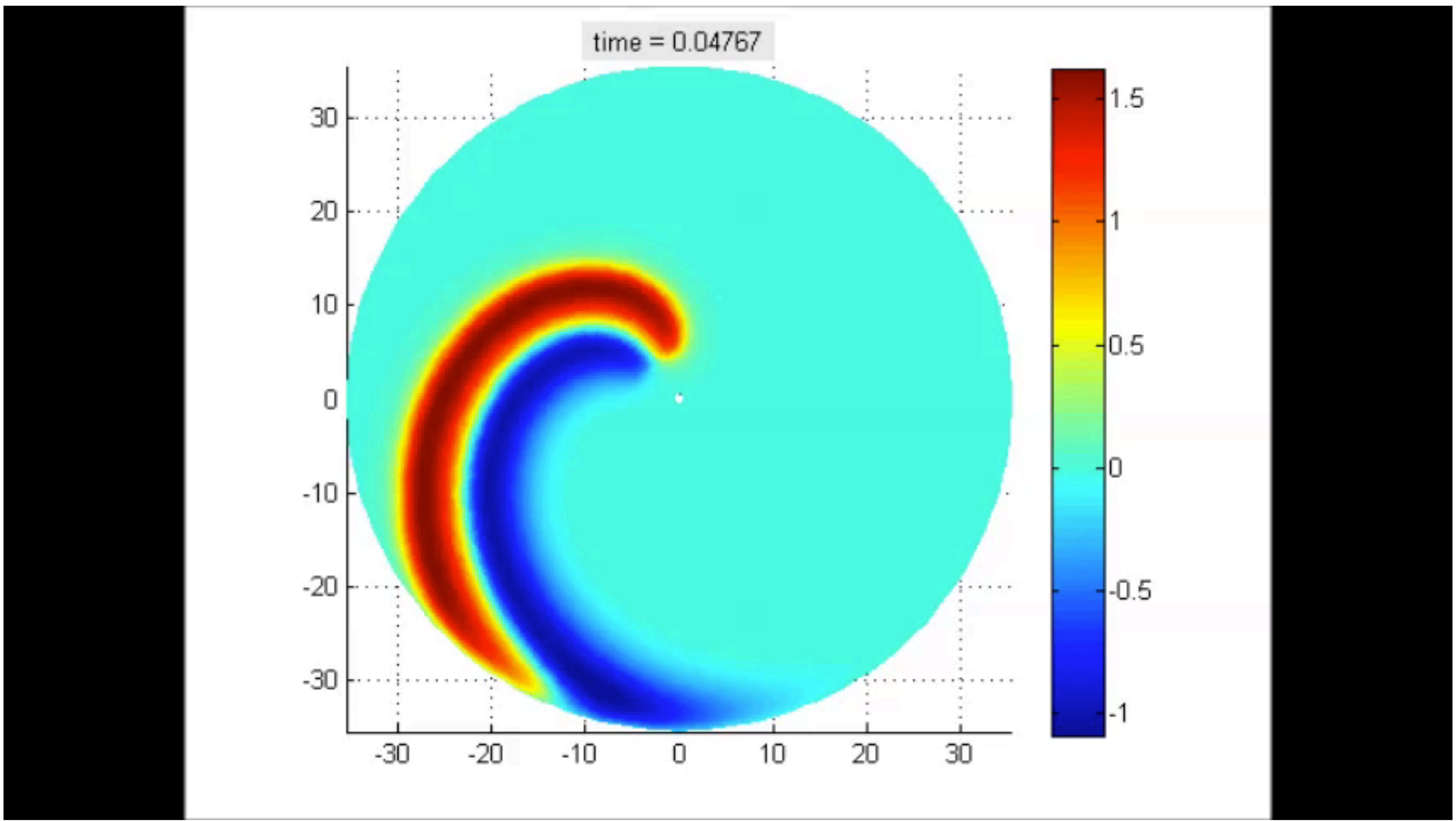


Spirals



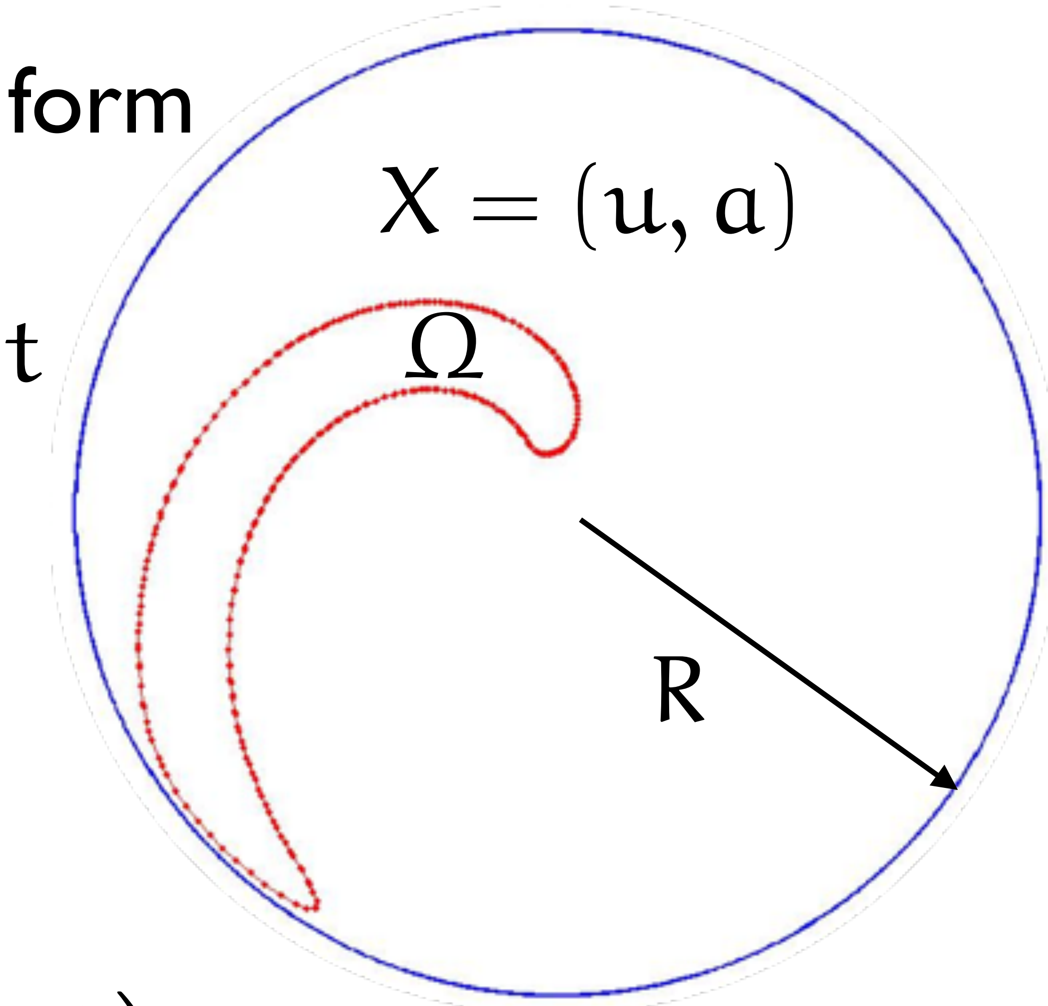
Spiral Waves in Disinhibited Mammalian Neocortex (rat slice)
Huang et al., J Neurosci. 2004




time = 0.04767
30
20
10
0
-10
-20
-30
-30
-20
-10
0
10
20
30
1.5
1
0.5
0
-0.5
-1

An interface approach (in progress)

Look for rigidly rotating solutions of the form



$$X(r, \theta, t) = X(r, \phi) \qquad \phi = \theta - \omega t$$

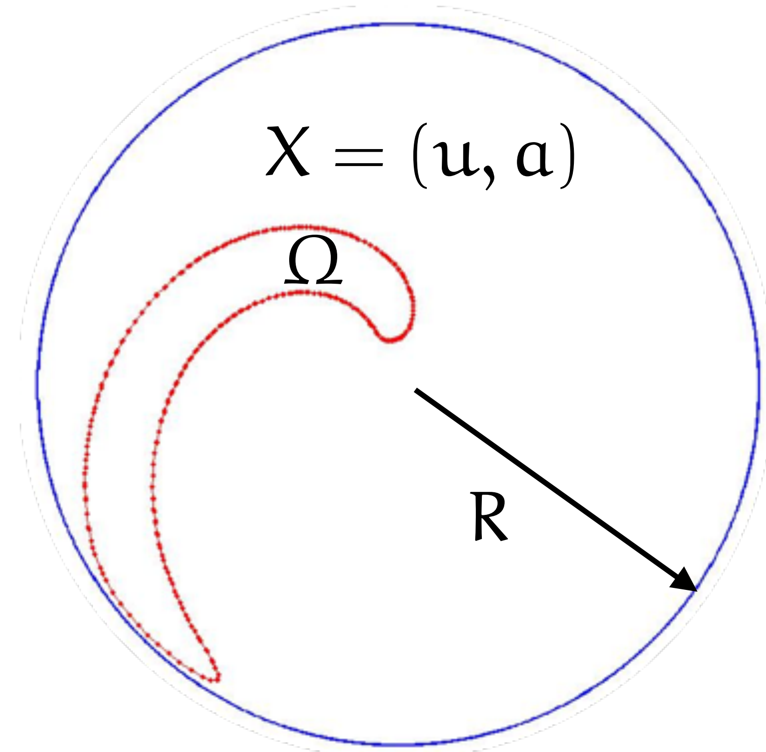
$$\psi(\mathbf{r}) = \int_{\Omega} d\mathbf{r}' w(|\mathbf{r} - \mathbf{r}'|)$$

$$G(\phi) = e^{A\phi}$$

$$X(r, \phi) = G(\phi)\left([e^{-2\pi A} - I]^{-1}\int_0^{2\pi} - \int_0^{\phi}\right) d\phi' G(-\phi') B(r, \phi')$$

$$A = \begin{bmatrix} \alpha/\omega & \alpha g/\omega \\ -1/\omega & 1/\omega \end{bmatrix}, \qquad B(r, \phi) = -\frac{\alpha}{\omega}\begin{bmatrix} \psi(r, \phi) \\ 0 \end{bmatrix}$$

Shape of the spiral arm determined by



$$u(\mathbf{r})|_{\mathbf{r}\in\partial\Omega} - h = 0$$

$$u(R, \phi) = 0, \qquad \forall\phi$$

Stability:

$$(u(r,\theta,t), a(r,\theta,t)) = (u(r,\phi), a(r,\phi)) + (\delta u(r,\phi), \delta a(r,\phi))e^{\lambda t}$$

$$\begin{bmatrix} \lambda + \alpha - \omega\partial_\phi - \alpha w\odot & \alpha g \\ -1 & \lambda + 1 - \omega\partial_\phi \end{bmatrix} \begin{bmatrix} \delta u \\ \delta a \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$[w \odot \delta u](\mathbf{r}) = \oint_{\partial\Omega} ds\, w(\mathbf{r} - \mathbf{r}(s)) \frac{\delta u(\mathbf{r}(s))}{|\nabla u(\mathbf{r}(s))|}$$

... watch this space!

In collaboration with

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Ingo Bojak (Reading)

Daniele Avitabile and Aytül Gökçe (Nottingham)

S Coombes, H Schmidt and I Bojak 2012 [Interface dynamics in planar neural field models]().