Dynamics of Sound Waves in Interacting Bose Gases

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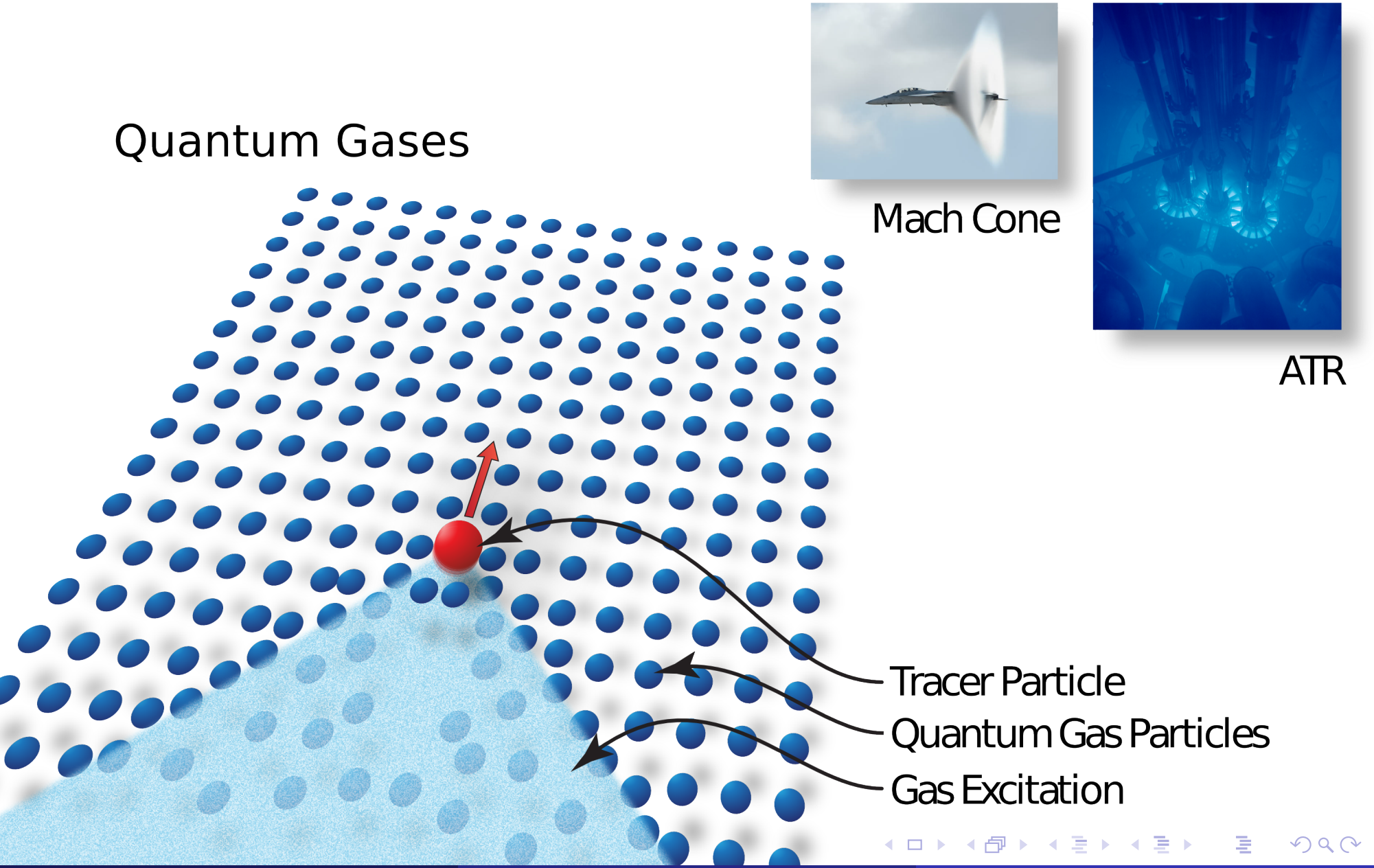
Joint work with J. Fröhlich (ETH), P. Pickl (LMU), and A. Pizzo (UCD).

Dynamics of Sound Waves



Model of the Bose Gas

$$i\partial_t \Psi_t(x_1,\ldots,x_N) = H\Psi_t(x_1,\ldots,x_N)$$

$$H = \sum_{j=1}^{N}\left(-\frac{\Delta_{x_j}}{2} + \alpha\sum_{k<j} U(x_j - x_k)\right)$$

with $U \in \mathcal{C}_c^\infty$, and initially the gas particles are quite regularly arranged:

- Ψ_0 is "close" to a product state

$$\Psi_0(x_1,\ldots,x_N) = \prod_{j=1}^{N} \Lambda^{-1/2}\varphi_0(x_j), \qquad \|\Psi_0\|_2 = 1$$

for a smooth one-particle wave function φ_0 with:

$\|\varphi_0\|_\infty = 1$ and supported in a box of volume $\Lambda \subset \mathbb{R}^3$.

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A physically relevant scaling

- Gas density: $\rho = \frac{N}{\Lambda}$.
- For the product state Ψ_0 one finds

$$\left\langle \Psi_0, \alpha \sum_{k=1}^{N} U(x - x_k)\Psi_0 \right\rangle = N\,\alpha\, U * \left|\frac{\varphi_0}{\Lambda^{1/2}}\right|^2 (x) = \rho\,\alpha\, U * |\varphi_0|^2(x).$$

- For $U \in \mathcal{C}_c^\infty(\mathbb{R}^3)$ one formally has:

$$-\frac{\Delta_{x_j}}{2} + \alpha \sum_{k<j} U(x_j - x_k) = \mathcal{O}(1) + \mathcal{O}(\alpha\rho)$$

Hence, for

$$\alpha \sim \frac{1}{\rho},$$

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Microscopic dynamics:

$$i\partial_t \Psi(x_1,\dots,x_N) = H\Psi(x_1,\dots,x_N), \qquad H = \sum_{j=1}^{N}\left(-\frac{\Delta_{x_j}}{2} + \frac{1}{\rho}\sum_{k<j} U(x_j - x_k)\right)$$

Macroscopic dynamics:

$$i\partial_t \varphi_t(x) = h[\varphi_t]\varphi_t(x), \qquad h_x[\varphi_t] = -\frac{\Delta_x}{2} + 1 \cdot U * |\varphi_t|^2(x).$$

For $\rho \gg 1$ one can hope to control the micro- with the macro-dynamics in a sufficiently strong sense, e.g.,

$$\left\| \mathrm{Tr}_{x_2,\dots,x_N} |\Psi_t\rangle\langle\Psi_t| - \left|\frac{\varphi_t}{\|\varphi_t\|_2}\right\rangle\left\langle\frac{\varphi_t}{\|\varphi_t\|_2}\right| \right\| \leq C(t)\rho^{-\gamma}, \qquad \text{for } \gamma > 0, \rho \gg 1.$$

For fixed volume Λ, i.e., $\rho = \mathcal{O}(N)$, many results are available, e.g.,

Hepp '74, Spohn '80, Rodniaski & Schlein '09, Fröhlich & Knowles & Schwarz '09, Pickl '10, Erdős & Schlein & Yau '10, ...

Spectrum for large volume: Dereziński & Napiórkowski '13

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Open Key Questions

1. Can the control be maintained for large volume Λ?
2. Is the approximation good enough to be able to see $\mathcal{O}_{\Lambda,\rho}(1)$ excitations of the gas, e.g., sound waves, for large Λ?
3. Can the thermodynamic limit, $\Lambda \to \infty$ for fixed ρ, and the mean-field limit, $\rho \to \infty$, be decoupled?

We demonstrate how to control the microscopic dynamics of excitations in the following large volume regime:

$$\Lambda, \rho \gg 1 \qquad \text{such that} \qquad \frac{\Lambda}{\rho} \ll 1 \qquad \text{and} \qquad N = \rho\Lambda.$$

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Tracking Excitations for large Λ

Coherent excitation of the gas:

$$\Psi_0(x_1, \ldots, x_N) = \prod_{j=1}^{N} \Lambda^{-1/2} \varphi_0(x_j), \qquad \text{for } \varphi_0 = \Omega_0 + \epsilon_0, \qquad \text{given:}$$

- A smooth and flat reference state Ω_0:
  $\operatorname{supp} \Omega_0 = \mathcal{O}(\Lambda)$, $\|\Omega\|_\infty = 1$ with sufficiently regular tails;
- A smooth and localized coherent excitation ϵ_0:
  $\operatorname{supp} \epsilon_0 = \mathcal{O}(1)$.

Splitting of the dynamics: given the macroscopic dynamics φ_t use

$$i\partial_t \Omega_t = \left(-\frac{\Delta_x}{2} + U * \left(|\Omega_t|^2 - 1 \right) \right) \Omega_t. \qquad \text{(reference)}$$

as reference state and define the excitation by

$$\epsilon_t = \varphi_t e^{i\|U\|_1 t} - \Omega_t \qquad \text{(excitation)}$$

Control of approximation: define

$$\rho_t^{(\mathrm{micro})} = \mathrm{proj}_{\Omega_t}^{\perp} \mathrm{Tr}_{x_2,\ldots,x_N} \left| \Lambda^{1/2} \Psi_t \right\rangle \left\langle \Lambda^{1/2} \Psi_t \right| \mathrm{proj}_{\Omega_t}^{\perp},$$

$$\rho_t^{(\mathrm{macro})} = |\epsilon_t\rangle\langle\epsilon_t|,$$

and control $\left\| \rho_t^{(\mathrm{micro})} - \rho_t^{(\mathrm{macro})} \right\|$ for large Λ, ρ.

...which turns out to be not so easy.

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New mode of approximation: Though weaker, it is strong enough to answer most physical questions in a satisfactory way.

Theorem (Micro approximated by Macro)

Suppose $\|\varphi_t\|_\infty$ is bounded on $[0, T]$. There is a trajectory $t \mapsto \widetilde{\Psi}_t$ with corresponding reduced density matrix $\widetilde{\rho}_t^{(\mathrm{micro})}$ such that:

- $\left\|\Psi_t - \widetilde{\Psi}_t\right\|_2^2 \leq C(t)\frac{\Lambda}{\rho}$;
- $\left\|\widetilde{\rho}_t^{(\mathrm{micro})} - \rho_t^{(\mathrm{macro})}\right\| \leq C(t)\left(\frac{\Lambda}{\rho}\right)^{1/2}$,

for all times $t \in [0, T]$ and sufficiently large Λ and ρ.

This implies that the actual quantity

$$\left\|\rho_t^{(\mathrm{micro})} - \rho_t^{(\mathrm{macro})}\right\|$$

is <u>*typically*</u> small, provided $\Lambda, \rho \gg 1$ such that $\frac{\Lambda}{\rho} \ll 1$.

Strategy of Proof

- Similar to Pickl's '10 technique we count "bad" particles, i.e., particles that do not follow the macroscopic dynamics given by φ_t.
- For this we employ the orthogonal one-particle projectors:

$$p^{\varphi_t} = \frac{1}{\|\varphi_t\|^2}|\varphi_t\rangle\langle\varphi_t|, \qquad q^{\varphi_t} = 1 - p^{\varphi_t},$$

  and define the N-particle operators:

$$P_k^{\varphi_t} = (q^{\varphi_t})^{\odot k} \odot (p^{\varphi_t})^{\odot(N-k)}, \qquad \sum_{k=0}^{N} P_k^{\varphi_t} = \mathrm{id},$$

  which projects onto wave functions with exactly k "bad" particles.
- A quantity that was successfully used to control fixed-volume mean-field limits is

$$\langle \Psi_t, q_1^{\varphi_t}\Psi_t\rangle = \left\langle \Psi_t, \sum_{k=0}^{N} \frac{k}{N} P_k^{\varphi_t}\Psi_t \right\rangle,$$

  the expected ratio of "bad particles" over N.

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Strategy of Proof

- However, to silhouette the $\mathcal{O}(1)$ excitation ϵ_t against the reference state Ω_t we need much finer control on the ratio of bad particles over ρ:

  Only less than ρ particles may behave badly!

- Therefore, we regard

$$\langle \widehat{m}^{\varphi_t} \rangle_t := \left\langle \Psi_t, \sum_{k=0}^{N} m(k) P_k^{\varphi_t} \Psi_t \right\rangle, \qquad m(k) = \begin{cases} \frac{k}{\rho} & \text{for } 0 \leq k \leq \rho \\ 1 & \text{otherwise} \end{cases} .$$

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One can show:

Lemma

– same conditions – Define:

$$\widetilde{\Psi}_t := \sum_{0 \le k \le \rho} P_k^{\varphi_t} \Psi_t$$

$$= \Psi_t \text{ projected onto the subspace with at most } \rho \text{ bad particles.}$$

Then:

- $\left\| \Psi_t - \widetilde{\Psi}_t \right\|_2^2 \le \langle \widehat{m}^{\varphi_t} \rangle_t$;
- $\left\| \widetilde{\rho}_t^{(\mathrm{micro})} - \rho_t^{(\mathrm{macro})} \right\| \le C(t) \sqrt{\langle \widehat{m}^{\varphi_t} \rangle_t}$.

Hence, it is sufficient to prove:

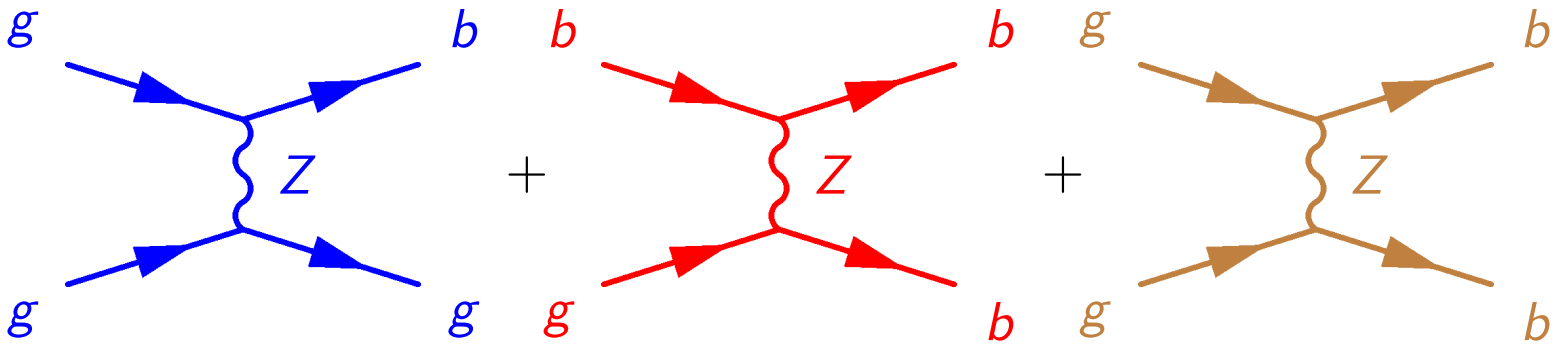
Lemma (Expected ratio of bad particles over ρ)

– same conditions –

$$\langle \widehat{m}^{\varphi_t} \rangle_t \le C(t) \frac{\Lambda}{\rho}.$$

$$
\begin{aligned}
\frac{d}{dt}\langle \widehat{m}^{\varphi_t}\rangle_t &= \left\langle \Psi_t, \left[H - \sum_{j=1}^{N} h_{x_j}[\varphi_t], \widehat{m}_t \right] \Psi_t \right\rangle \\
&= \left\langle \Psi_t, \sum_{j=1}^{N} \left[\frac{1}{\rho} \sum_{j<k} U(x_j - x_k) - \frac{N}{\rho} U * \frac{|\varphi_t|^2}{\Lambda}(x_j), \widehat{m}_t \right] \Psi_t \right\rangle \\
&\approx \frac{N^2}{2\rho} \left\langle \Psi_t, \left[\underbrace{U(x_1 - x_2) - U * \frac{|\varphi_t|^2}{\Lambda}(x_1) - U * \frac{|\varphi_t|^2}{\Lambda}(x_2)}_{=:Z(x_1,x_2)}, \widehat{m}_t \right] \Psi_t \right\rangle \\
&= \frac{N^2}{2\rho} \langle \Psi_t, [Z(x_1, x_2), \widehat{m}_t^{\varphi_t}] \Psi_t \rangle \\
&= \frac{N^2}{2\rho} \left\langle \Psi_t, \mathrm{id}\, \mathrm{id}\, [Z(x_1, x_2), \widehat{m}^{\varphi_t}]\, \mathrm{id}\, \mathrm{id} \Psi_t \right\rangle, \\
&\qquad\qquad \text{for } \mathrm{id} = p_1^{\varphi_t} + q_1^{\varphi_t} = p_2^{\varphi_t} + q_2^{\varphi_t}, \\
&= 16 \text{ terms.}
\end{aligned}
$$

$$\frac{d}{dt}\langle \widehat{m}^{\varphi_t}\rangle_t \leq$$



$$+\, h.c. + \text{diagrams that conserve the particle numbers}$$

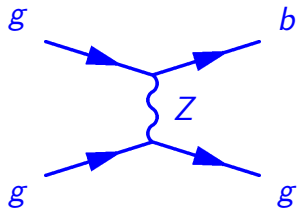
$$\leq C(t)\Big(\quad 0 \quad + \quad \langle \widehat{m}^{\varphi_t}\rangle \quad + \quad \frac{\Lambda}{\rho} \quad \Big).$$

This yields:

$$\frac{d}{dt}\langle \widehat{m}^{\varphi_t}\rangle_t \leq C(t)\left(\langle \widehat{m}^{\varphi_t}\rangle_t + \frac{\Lambda}{\rho}\right)$$

and, thanks to Grönwall's Lemma, concludes the proof.

The term determining the structure of the macroscopic equation is:



$$= \frac{N^2}{2\rho}\left\langle \Psi_t, p_1^{\varphi_t}\, p_2^{\varphi_t}\, [Z(x_1,x_2), \widehat{m}^{\varphi_t}]\, p_1^{\varphi_t}\, q_2^{\varphi_t} \Psi_t \right\rangle,$$

and one finds

$$p_1^{\varphi_t} p_2^{\varphi_t} Z(x_1,x_2) p_1^{\varphi_t} q_2^{\varphi_t}$$

$$= p_1^{\varphi_t} p_2^{\varphi_t} \left(U(x_1-x_2) - U * \frac{|\varphi_t|^2}{\Lambda}(x_2) - U * \frac{|\varphi_t|^2}{\Lambda}(x_1) \right) p_1^{\varphi_t} q_2^{\varphi_t}$$

$$= p_1^{\varphi_t} p_2^{\varphi_t} \left(\underbrace{p_1^{\varphi_t} U(x_1-x_2) p_1^{\varphi_t}}_{=\Lambda^{-1} U * |\varphi_t|^2 (x_2) p_1^{\varphi_t}} - U * \frac{|\varphi_t|^2}{\Lambda}(x_2) p_1^{\varphi_t} \right) q_2^{\varphi_t}$$

$$- p_1^{\varphi_t} U * \frac{|\varphi_t|^2}{\Lambda}(x_1) p_1^{\varphi_t} \underbrace{p_2^{\varphi_t} q_2^{\varphi_t}}_{=0}$$

$$= 0$$

Effective Equation for the Excitation

$$i\partial_t \epsilon_t = \left(-\frac{\Delta_x}{2} + U * \left(|\Omega_t|^2 - 1 \right) + U * |\epsilon_t|^2 + U * 2\Re \epsilon_t^* \Omega_t \right) \epsilon_t$$
$$+ \left(U * |\epsilon_t|^2 + U * 2\Re \epsilon_t^* \Omega_t \right) \Omega_t, \qquad \text{(excitation)}$$

If there are no blow-ups for $t \in [0, T]$ and the L^2 norm is small enough, we find a simple effective equation for the excitation:

Theorem (Small Excitations)

Let η_t solve

$$i\partial_t \eta_t = -\frac{\Delta_x}{2} \eta_t + U * 2\Re \eta_t, \qquad \eta_0 = \epsilon_0.$$

Then:

$$\|\epsilon_t - \eta_t\|_2 \leq C(t) \left(\Lambda^{-1/3} + \sup_{s \in [0,T]} \|\epsilon_t\|_2 \right)$$

for all $t \in [0, T]$.

The evolution equation

$$i\partial_t \eta_t = -\frac{\Delta_x}{2}\eta_t + U * 2\Re\eta_t$$

for Fourier transform $\widehat{\eta}_t$ can be given by

$$i\partial_t \begin{pmatrix} \widehat{\eta}_t(k) \\ \widehat{\eta}_t^*(-k) \end{pmatrix} = \mathcal{H}(k) \begin{pmatrix} \widehat{\eta}_t(k) \\ \widehat{\eta}_t^*(-k) \end{pmatrix}, \quad \mathcal{H}(k) = \begin{pmatrix} \omega_0(k) + \widehat{U}(k) & \widehat{U}(k) \\ -\widehat{U}(k) & -\omega(k) - \widehat{U}(k) \end{pmatrix},$$

where $\omega_0(k) = k^2/2$. The eigenvalues $\omega(k)$ of $\mathcal{H}(k)$ fulfill

$$\omega(k)^2 = \omega_0(k)\left(\omega_0(k) + 2\widehat{U}(k)\right).$$

One can explicitly solve for the dispersion relation:

$$\omega(k) = |k|\sqrt{\frac{k^2}{4} + \widehat{U}(k)}.$$

first discovered by Bogulyubov.

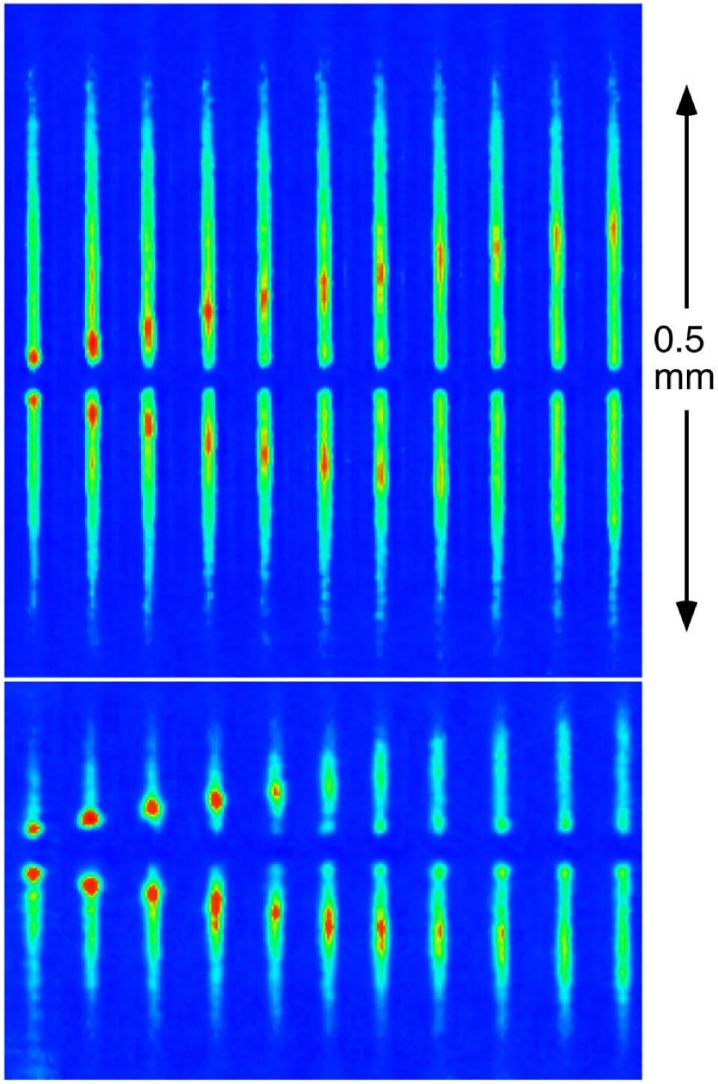
For small momenta $|k|$ we can distinguish two cases:

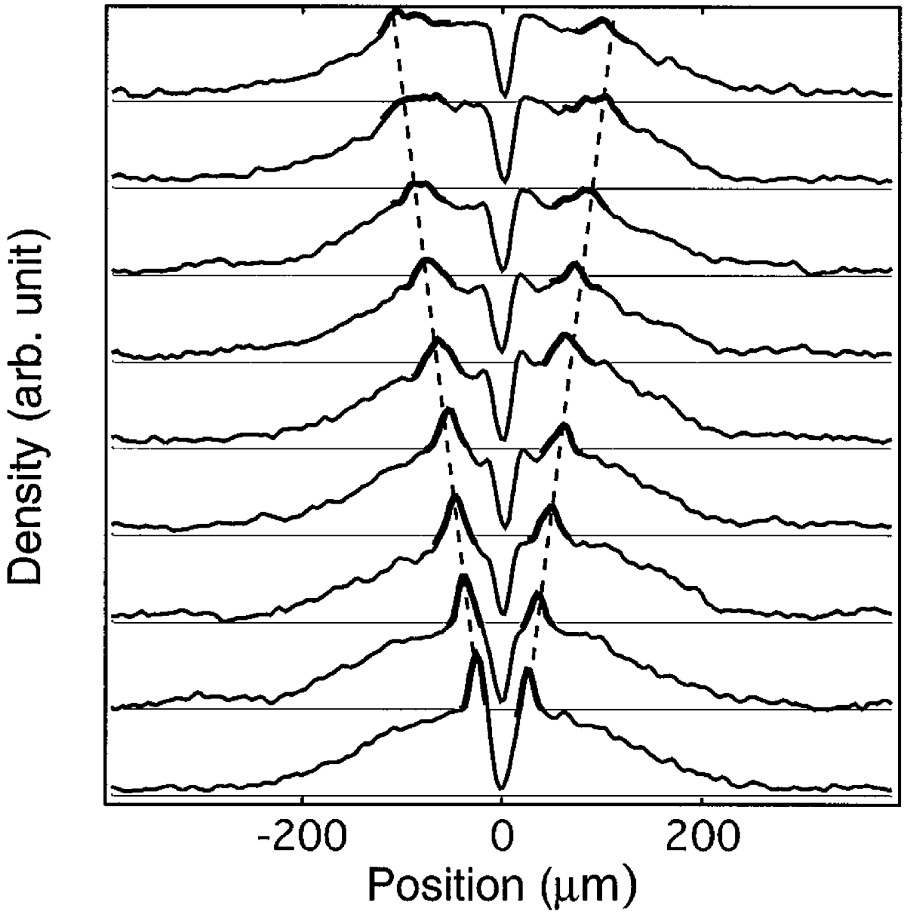
- **Repulsive potential, i.e.,** $\widehat{U}(0) > 0$:

$$v_{\text{sound}} = \frac{d}{dk}\omega(k)\bigg|_{k\to 0} = \sqrt{\widehat{U}(0)}.$$

  Sound waves with arbitrary small modes travel at a strictly positive speed.
- **Attractive potential, i.e.,** $\widehat{U}(0) < 0$**:**
  - Modes with wave vectors k such that $k^2/4 = \widehat{U}(k)$ become static.
  - Modes such that $k^2/4 < \widehat{U}(k)$ become dynamical unstable.

Laser induced sound waves in a BEC [Ketterle et. al.]





Outlook

- Show that the actual ground state of the gas is "close" to a product state much like our initial state.
- Uncouple the scales $\Lambda \gg 1$ and $\rho \gg 1$.
- Treat the case of non-zero temperature.

Thank you!