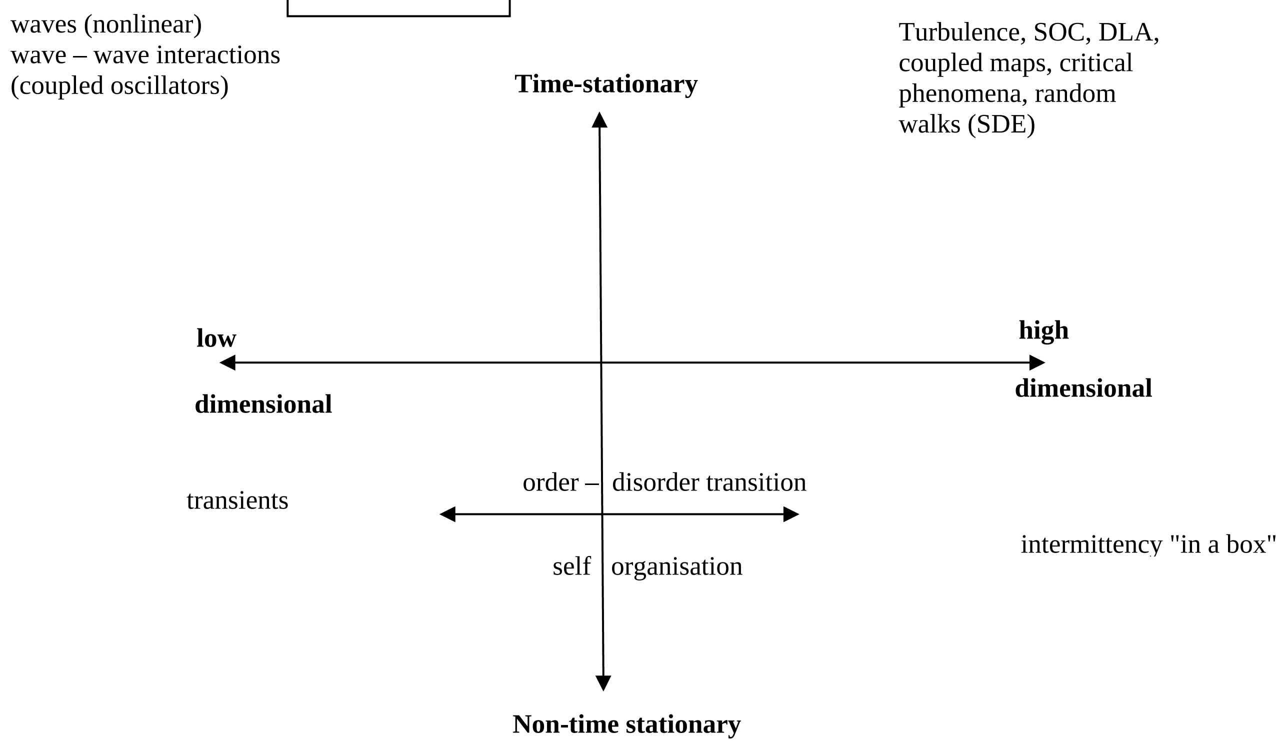
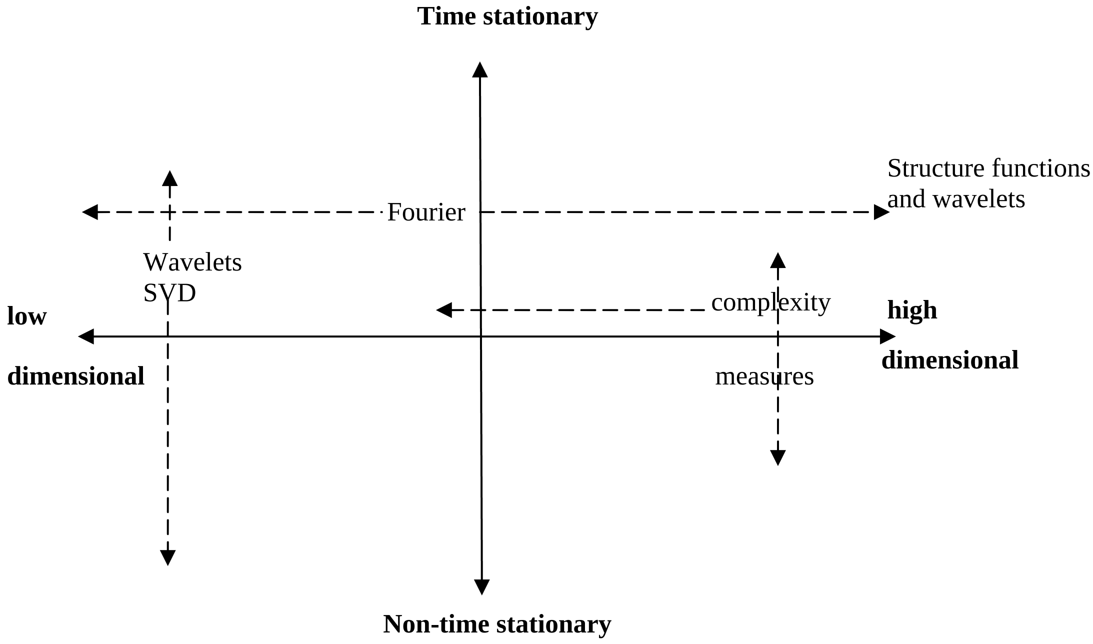
Processes



Techniques



Summary of Fourier transforms and DFT

Time series $x(t)$, its transform $S(f)$, time interval T

a) expand $x(t)$ in an orthogonal set of basis functions

$$x(t) = \sum_{m=-\infty}^{\infty} S_m e^{2\pi i f_m t}, \quad f_m = \frac{m}{T}$$

and

$$S_m = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-2\pi i f_m t}\, dt$$

NB: orthogonality $\int_{-\pi}^{\pi} e^{i(n-m)x} dx = 2\pi\delta_{mn}$

we will discuss the importance of this in a moment…

b) Parseval's theorem

$$\int_{-T/2}^{T/2} |x(t)|^2 dt = T\sum_{m=-\infty}^{\infty} |S_m|^2$$

c) Defns: since $S_m = |S_m| e^{i\phi_m}$ - in general complex

amplitude spectrum $S_{m,}\left(|S_m|^2 = S_m S_m^*\right)$ phase spectrum ϕ_m; (discrete) power spectrum $|S_m|^2 \rightarrow$ the power in mode f_m.

d) We then take the continuous limit $f_N \rightarrow f, \quad T \rightarrow \infty$ to obtain the Fourier transform pair:

$$x(t) = \int_{-\infty}^{\infty} S(f) e^{2\pi i f t}\, df$$

$$S(f) = \int_{-\infty}^{\infty} x(t) e^{-2\pi i f t}\, dt$$

NB: Notation – choosing variables *f* and *t* retains symmetry. If we worked in terms of

$\omega = 2\pi f$ in $e^{i\omega t}$ term this results in a factor $\frac{1}{2\pi}$ in front of the IFT.

e) Parseval becomes:

$$\int_{-\infty}^{\infty} |x(t)|^2\, dt = \int_{-\infty}^{\infty} |S(f)|^2\, df\ .$$

f) Defns:

$|S(f)|$ amplitude spectrum

$|S(f)|^2$ power spectral density

ie $|S(f)|^2\,df$ is the power in band $f - f + df$

again, S is complex

$$S(f) = A(f)e^{i\phi(f)}$$

amp. spectrum phase spectrum

One can construct surrogate data sets by modifying (randomising) $\phi(f)$.

Some important theorems:

g) Convolution

Defn: convolution $g(t) * h(t) = \int_{-\infty}^{\infty} g(\tau)h(t-\tau)\,d\tau$

then FT $\int_{-\infty}^{\infty} g(t) * h(t) e^{-2\pi ift}\,dt = G(f).H(f)$

product of FT of g and h
(convolution theorem)

h) Cross correlation

Defn: cross correlation $g(t) \times h(t) = \int_{-\infty}^{\infty} g^*(\tau)h(t+\tau)\,d\tau$

not the same as convolution. However:

$$\text{FT} \quad \int_{-\infty}^{\infty} g(t) \times h(t) e^{-2\pi ift}\,dt = G^*(f).H(f)$$

NB: g^* - complex conjugate

i) Auto correlation – put $h = g = x$

$$x(t) \times x(t) = \int_{-\infty}^{\infty} x^*(\tau)x(t+\tau)\,d\tau = R(t)$$

then

$$\int_{-\infty}^{\infty} [x(t) \times x(t)] e^{-2\pi ift}\,dt = S^*(f)S(f) = |S(f)|^2$$

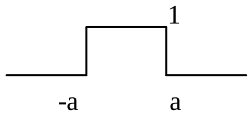
$$\text{or } R(t) = \int_{-\infty}^{\infty} S^2(f) e^{2\pi i f t} \, df$$

Wiener-Khintchine theorem
[Will relate to "statistical correlation" and covariance in a moment.]

j) <u>Uncertainty principle/resolution</u>

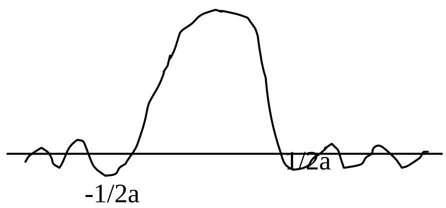
Consider a well known example (diffraction)

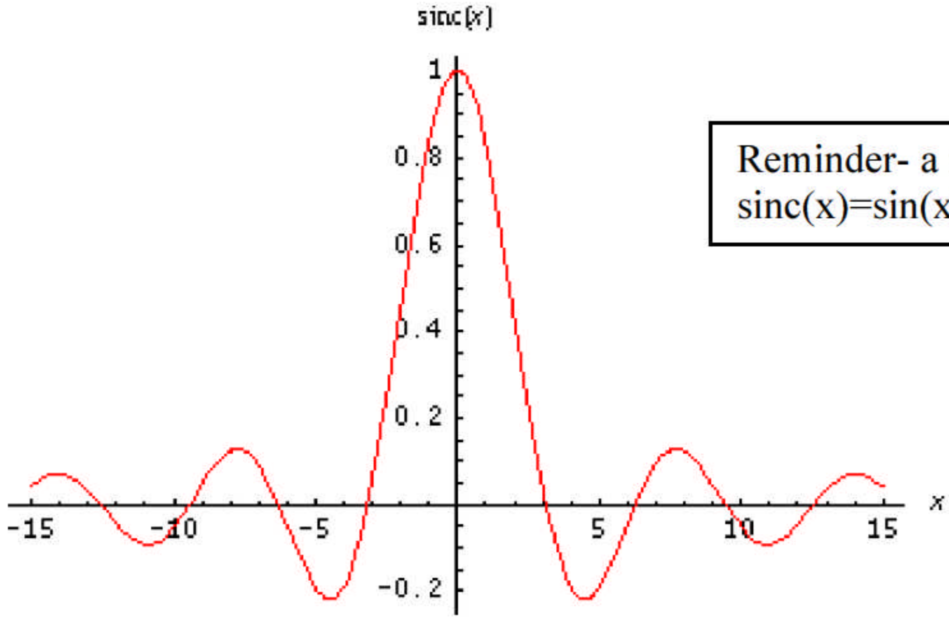
$x(t)$ is a "top hat" function



cf diffraction at
slits in optics

$S(f)$ is a sinc function





Reminder- a nice plot of
sinc(x)=sin(x)/x

proof: $$S(f) = \int_{-a}^{a} e^{-2\pi i f t} \, dt = \left[-\frac{e^{-2\pi i f t}}{2\pi i f} \right]_{-a}^{a} = \frac{e^{2\pi i a f} - e^{-2\pi i a f}}{2\pi i f}$$

$$= \frac{\sin 2\pi a f}{\pi f}$$

1st zero at $2\pi a f = \pi$ or

$$f = 1/2a$$

thus, "width" of function has the property $\Delta t \Delta f \sim 1$.

Finite window in *t* thus implies spectral leakage in *f* – to come later.

k) Finally….(an aside)

A relationship with the <u>moments</u> of a function
(relates to cumulants)

Since $$S(f) = \int_{-\infty}^{\infty} x(t) e^{-2\pi i f t}\, dt$$

$$\frac{dS}{df} = \int_{-\infty}^{\infty} (-2\pi i t) x(t) e^{-2\pi i f t}\, dt$$

so, $$\frac{dS}{df}(f = 0) = -2\pi i \int_{-\infty}^{\infty} t x(t) dt$$

and $$\frac{1}{(-2\pi i)^P} \frac{d^P}{df^P} S(f = 0) = \int_{-\infty}^{\infty} t^P x(t) dt = m_p$$

the *p*th moment of $x(t)$

hence, if $x(t)$ is PDF then $S(f)$ is the characteristic function – useful in handing PDFs.

Discrete Fourier Theory

What you actually calculate for real time series

a) $x(t)$ is sampled every Δt over interval T

$$x(t) \to x_k \quad k = 0, 1 \dots N-1 \quad t \to k\Delta t$$

Discrete Fourier Transform (DFT) are

$$x_k = \frac{1}{N\Delta t}\sum_{m=0}^{N-1} S_m e^{2\pi imk/N}$$
$$S_m = \Delta t \sum_{k=0}^{N-1} x_k e^{-2\pi ikm/N}$$

NB: usually $\Delta t = 1$ - given explicitly here to be clear about units, normalisation.

b) Now S_m is associated with frequency (mode) $f_m = \frac{m}{N\Delta t}$
and x_k is associated with time $t_k = k\Delta t$.
Thus "resolution" Δt corresponds to $\Delta f = \frac{1}{N\Delta t}$, "uncertainty principle" becomes $\Delta f \Delta t \sim 1/N$.

c) Cyclic behaviour

Note, indices can be written as

$$k \to k + p \qquad m \to m + p$$

or sums over
$$k = p \dots N + p - 1$$
$$m = p \dots N + p - 1$$

This implies periodicity over *N*, that is periodicity in time over *T*.

General implications of the DFT pair - I
(which also hold for continuous limit – not done here).

Write the DFT pair again:

$$x_k = \frac{1}{T}\sum_{m=0}^{N-1} S_m e^{2\pi i f_m t_k}$$

- this is just an expansion of $x(t)$ in an orthogonal basis $e^{2\pi i f_m t_k}$, but the series is truncated at $m = N - 1$, ie, summing over finite number of modes f_m.

- this expresses the fact that $x(t)$ is represented by a linear superposition of a finite set of modes - a linear process.

General implications – II – co-ordinate rotations

Write

$$A_{mk} = e^{2\pi imk/N} \qquad \text{a matrix}$$
$$A^*_{mk} = e^{-2\pi imk/N} \qquad \text{complex conjugate}$$
$$= \tilde{A} \qquad \text{complex conjugate transpose}$$

since A is symmetric

Setting normalizations $1/N, \Delta t = 1$ etc for now;
Then

$$x_k = A_{km} S_m \qquad \text{a rotation/projection}$$

The DFT simply projects the vector x_k into a (useful) co-ordinate system to give vector S_m.

A_{mk} is a rotation matrix.

Follows that $S_m = A^{-1}_{mk} x_k$ - inverse rotation.

We can treat all transforms in this way.

Desirable property – from the DFT – defn. we have the following:

$$S_m = \tilde{A}_{mk} x_k = \tilde{A}_{mk}\left(A_{kp} S_p\right) \text{ - from DFT}$$

thus $\tilde{A}_{mk} A_{kp} = \delta_{mp}$ - A is orthogonal

$\rightarrow$ so inverse FT exists.

Alternatively, if A is a rotation

$$S_m = A^{-1}_{mk} x_k \qquad \text{so } A^{-1} = \tilde{A}$$
$$= A^{-1}_{mk}\left(A_{kp} S_p\right) \qquad \text{- } A \text{ is orthogonal}$$

d) Orthogonality of the DFT
For the Fourier transform:

$$A_{mk} = e^{2\pi imk/N}$$

so

$$A_{mp}\tilde{A}_{qp} = \sum_{p=0}^{N-1} e^{2\pi imp/N} e^{-2\pi iqp/N}$$
$$= \sum_{p=0}^{N-1} e^{2\pi ip(m-q)/N} = \delta_{mq}$$

The discrete version of the orthogonality condition seen earlier

e) Parseval's theorem - follows from orthogonality.

Consider inner product

$$
\begin{aligned}
x_k x_k &= \left(A_{kp} S_p \right)\left(A_{kq} S_q \right) \\
&= \left(A_{kp} S_p \right)\left(\tilde{A}_{qk} S_q \right) \text{ since } \tilde{A} = A^{*T} \\
&= \tilde{A}_{qk} A_{kp}\ S_p S_q \\
&= \delta_{pq} S_p S_q = S_p S_p
\end{aligned}
$$

ie:

$$\sum_{k=0}^{N-1} x_k^2 = \sum_{p=0}^{N-1} S_p^2$$

or putting back normalisation!

$$\Delta t \sum_{k=0}^{N-1} \left| x_k \right|^2 = \frac{1}{N\Delta t} \sum_{k=0}^{N-1} \left| S_k \right|^2$$

ie: Parseval's theorem states that the length of the vector x_k is unchanged under rotation $\rightarrow$ to the S_m.

$\Rightarrow$ will hold for any transform that is orthogonal.

f) Convolution – of time series g_k, h_k

$$g_k * h_k = \Delta t \sum_{u=0}^{N-1} g_u h_{k-u}$$

and DFT gives convolution theorem:

$$\Delta t \sum_{k=0}^{N-1} \left(g_k * h_k \right) e^{-2\pi ikm/N} = G_m H_m$$

where $G_m = \Delta t \sum_{k=0}^{N-1} g_k e^{-2\pi ikm/N}$ - same for H_m

again, this is cyclic – write $k \rightarrow k + p$

sum is over $p \ldots N + p - 1$.

NB: notation – often refer to "lag" τ

$$\left(g * h \right)_\tau = \Delta t \sum_{k-0}^{N-1} g_k h_{\tau - k} \text{ and } \Delta\tau \equiv \Delta t.$$

g) Cross correlation

$$C_\tau = \sum_{k=0}^{N-1} g_k h_{k+\tau}$$

and again we have

$$\Delta\tau \sum_{\tau=0}^{N-1} C_\tau e^{-2\pi i m\tau/N} \qquad = G_m^* H_m$$

(note $\Delta\tau = \Delta t$ here).

h) Auto correlation

$$R_\tau = \sum_{k=0}^{N-1} x_k x_{k+\tau}$$

and

$$S_m^* S_m = \left| S_m \right|^2 = \Delta\tau \sum_{\tau=0}^{N-1} R_\tau e^{-2\pi i m\tau/N}$$

- Wiener-Khintchine.

NB: Normalisation – often normalised to the "zero lag" ($\tau = 0$ term).

$$\bar{R}_\tau = \sum_{k=0}^{N-1} x_k x_{k+\tau} - \sum_{k=0}^{N-1} x_k x_k$$

Statistical correlation and covariance

Other things are called "correlation", ie: **correlation coeff:**

$$Q_{xy} = \sum_i^N \left(x_i - \bar{x} \right)\left(y_i - \bar{y} \right)$$

this is just the $\tau = 0$ value C_0 of the cross correlation for an iid zero mean process

ie: $Q_{xx} = R_0$

covariance

$$\bar{Q}_{xy} = \sum_i^N \frac{\left(x_i - \bar{x} \right)\left(y_i - \bar{y} \right)}{N}$$

since the correlation (statistical) is often normalised to $\left\langle x_i^2 \right\rangle$ and $\left\langle y_i^2 \right\rangle$ we have

$$\frac{\bar{Q}_{xy}}{\sigma_x \sigma_y} = Q_{xy} \quad \text{where} \quad \sigma_x = \left\langle x^2 \right\rangle - \left\langle x \right\rangle^2$$