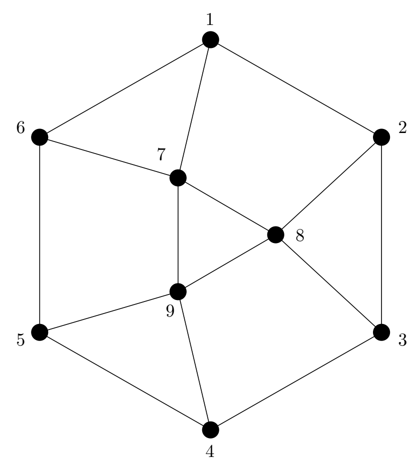
APTS Applied Stochastic Processes, Nottingham, April 2024
Exercise Sheet for Assessment

The work here is "light-touch assessment", intended to take students up to half a week to complete. Students should talk to their supervisors to find out whether or not their department requires this work as part of any formal accreditation process (APTS itself has no resources to assess or certify students). It is anticipated that departments will decide the appropriate level *of assessment locally, and may choose to drop some (or indeed all) of the parts, accordingly.*

Students are recommended to read through the relevant portion of the lecture notes before attempting each question. It may be helpful to ensure you are using a version of the notes put on the web *after* the APTS week concluded.

1 Markov chains and renewal processes

Consider a random walk X moving on the following graph. The walk starts at $X_0 = 1$ and at each step picks one of the neighbours of the current position uniformly at random and moves to it.



(a) What is the equilibrium distribution, π?

(b) What is the mean return time to state 1?

(c) Write $O = \{1, 2, 3, 4, 5, 6\}$ for the states in the outer ring and $I = \{7, 8, 9\}$ for the states in the inner ring. Consider the sequence $H_0, H_1, H_2, \ldots$ of times defined recursively by $H_0 = \inf\{n \geq 0 : X_n \in I\}$ and, for $m \geq 0$,

$$H_{m+1} = \inf\{n > H_m : X_n \in I\}.$$

Let $N(n) = \#\{m \geq 0 : H_m \leq n\}$ and consider the process $(N(n), n \geq 0)$. Explain why N is a delayed renewal process.
Hint: you may find it helpful to define a new Markov chain $(\tilde{X}_n)_{n\geq 0}$ on the state space $\{O, I\}$.

(d) Show that $H_1 - H_0$ has the same distribution as

$$1 + BG$$

where B and G are independent, $B \sim \mathrm{Ber}(1/2)$ and $G \sim \mathrm{Geom}(1/3)$ (i.e. $\mathbb{P}(G = k) = \frac{1}{3}\left(\frac{2}{3}\right)^{k-1}$ for $k \geq 1$).

(e) Now suppose that we instead start with $X_0 \sim \pi$. What is the probability mass function of H_0 in this case? (In other words, what is the special delay distribution that makes the renewal process stationary?) Deduce that H_0 has the same distribution as

$$B'G',$$

where B' and G' are a Bernoulli random variable and an independent Geometric random variable respectively, whose parameters you should determine.

2 Martingales and optional stopping

Let $S = (S_n, n \geq 0)$ be a simple *asymmetric* random walk on $\mathbb{Z}$, started at zero. That is, $S_0 = 0$ and $S_n = \xi_1 + \cdots + \xi_n$ for $n \geq 1$, where $\xi_1, \xi_2, \ldots$ are independent and identically distributed random variables with $\mathbb{P}(\xi_1 = 1) = p = 1 - \mathbb{P}(\xi_1 = -1)$, for some $p \in (0, 1/2)$.

(a) Define the function ϕ by $\phi(x) = \left(\frac{1-p}{p}\right)^x$. Show that $\phi(S_n)$ is a martingale.

(b) Let $T_x = \inf\{n \geq 0 : S_n = x\}$. Prove that for any positive integers $a, b > 0$,

$$\mathbb{P}(T_a > T_{-b}) = \frac{\phi(a) - 1}{\phi(a) - \phi(-b)}.$$

(Hint: consider the stopping time $T = T_a \wedge T_{-b} \equiv \min(T_a, T_{-b})$, which you know is almost surely finite!)

(c) Show that $S_n - (2p - 1)n$ is a martingale. Use this to prove that

$$\mathbb{E}[T] = \frac{a + b - a\phi(-b) - b\phi(a)}{(2p - 1)(\phi(a) - \phi(-b))}.$$

(You may assume that both $\mathbb{E}[n \wedge T] \to \mathbb{E}[T]$ and $\mathbb{E}[S_{n\wedge T}] \to \mathbb{E}[S_T]$ as $n \to \infty$. *In fact, these can be proved by monotone convergence and dominated convergence, respectively.*)

(d) Since $p < 1/2$ we know that the random walk S is transient, and that $S_n \to -\infty$ almost surely as $n \to \infty$. Define R to be the largest value ever reached by S, i.e., $R = \max\{S_n : n \geq 0\}$; this is a finite random variable taking values in the set $\{0, 1, 2, \ldots\}$. Using part (b), determine the distribution of R, and show that $\mathbb{P}(R = 0) = (1 - 2p)/(1 - p)$.

3 Foster–Lyapunov criteria

Let X be a Markov chain on $\mathbb{R}$ starting at X_0 and defined by

$$X_{n+1} = aX_n + W_{n+1}, \text{ for } n \geq 0,$$

where $|a| < 1$ and the random variables $W_1, W_2, \ldots$ are independent and identically distributed and independent of X_0, with a continuous density function f_W which is strictly positive on the whole of $\mathbb{R}$, and where $\mathbb{E}[|W_1|] < \infty$.

(a) Show that X is Lebesgue-irreducible. (Recall that Lebesgue measure on $\mathbb{R}$ is length measure.)

(b) Show that any set of the form $C_d = \{x : |x| \leq d\}$ is a small set.

(c) Show that X is geometrically ergodic. (Hint: use the Foster–Lyapunov criterion with $\Lambda(x) = |x| + 1$.)