

Dynamic Bradley-Terry modelling of sports tournaments

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SUMMARY

In the course of national sports tournaments, usually lasting several months, it is expected that the abilities of teams taking part in the tournament change in time. A dynamic extension of the Bradley-Terry model for paired comparison data is introduced to model the outcomes of sporting contests allowing for time-varying abilities. It is assumed that teams' home and away abilities depend on past results through exponentially weighted moving average processes. The proposed model is applied to sports data with and without tied contests, namely the 2009-2010 regular season of the National Basketball Association tournament and the 2008-2009 Italian Serie A football season.

Keywords: Bradley-Terry model; Cumulative logit model; Exponentially weighted moving average process; Paired comparisons; Sport tournaments.

1 Introduction

The analysis of sports data has always aroused great interest among statisticians. Albert *et al.* (2005) collect a number of articles that summarise various statistical aspects of interest in sports data including rating of players or teams, evaluation of sport strategies, enhancement of sport rules, illustration of statistical methods and forecasting of results.

Sports data have been investigated from different perspectives, often with the aim of forecasting the results. A first approach consists in modelling the scores of the two teams. Maher (1982) employs independent Poisson distributions for the score of each team with means that depend on the attack and defence strength of teams. Dixon and Coles (1997) propose an *ad hoc* adjustment of the Poisson distribution introducing a dependence parameter that modifies the probabilities of the results 0-0, 0-1, 1-0 and 1-1. Dixon and Coles (1997) introduce also a dynamic element in the model updating the parameter estimates including the results up to the last observation and down-weighting observations distant in time. Karlis and Ntzoufras (2003) suggest applying a bivariate Poisson distribution with a dependence parameter between the number of goals scored by the two teams and then extend the model to inflate the probabilities of draws.

McHale and Scarf (2007) model the number of shots of the two teams. They propose two different types of Archimedean copula with either Poisson or negative binomial distributions for the marginals to account for the negative dependence between shots-for and shots-against.

Extensions allowing dynamic developments of abilities of the teams are proposed by Rue and Salvesen (2000) and Crowder *et al.* (2002). Rue and Salvesen (2000) assume that the attack and defence strength parameters of each team follow a Brownian motion process. The model is estimated by employing Bayesian inference through Markov chain Monte Carlo methods. Crowder *et al.* (2002) suggest an autoregressive model for the attack and defence abilities of teams. The original model is then replaced by a derived version that is easier to handle by maximum likelihood.

A second approach to the analysis of sports data consists of modelling the difference in

scores. Clarke and Norman (1995) perform a linear regression of the difference in scores on the difference in strength of the two teams. Harville (2003) employs a similar specification, but eliminates the incentives for running up the score beyond a predetermined number of points. A dynamic specification of strength in this context is considered in Harville (1980) who proposes an autoregressive process for the strength of teams in different seasons. Also Glickman and Stern (1998) assume that the evolution of week-by-week and seasonal strength follows a first-order autoregressive process; inference is carried out in a Bayesian framework through Markov chain Monte Carlo algorithms.

Finally, sports data can be analysed by considering only the outcomes of the matches (win-draw-loss). Goddard and Asimakopoulos (2004) use an ordered probit model to determine which covariates, e.g. importance of the match, fouls, yellow and red cards, affect the result of the match. An ordered probit model is adopted also by Koning (2000) who specifies the probability of the outcome as a function of the difference of abilities of the two teams. Kuk (1995) introduces two strength parameters for each team, one denoting the strength when playing at home and the other when playing away.

Barry and Hartigan (1993) propose a dynamic extension for the ability parameters of teams; they employ a choice model assuming a prior distribution for strength of teams that changes slowly in time. Fahrmeir and Tutz (1994) consider three possible specifications for the development of abilities: a first and second order random walk and a local linear trend model. These models are estimated using empirical Bayes methods. Glickman (1999) specifies a logit model assuming a prior with normal increments for abilities of teams and proposes an approximate Bayesian algorithm for ranking purposes. Knorr-Held (2000) employs a logit model assuming random walk priors for abilities of teams. The variance of the random walk is estimated through four different predictive criteria while the abilities are estimated by means of the extended Kalman filter and smoother.

In this paper, we analyse the results of sport tournaments from the last perspective, that is modelling the outcomes of matches. Since we are interested in studying how the strengths

of the teams evolve during the season, we develop a dynamic paired comparison model. In particular, we model the evolution in time of the abilities in home and away matches of each team through an exponentially weighted moving average process.

The paper is organised as follows. Section 2 presents two motivating data sets regarding the American National Basketball Association (NBA) league and the Italian major men's football league. Section 3 describes the proposed dynamic version of the Bradley-Terry model, discusses maximum likelihood estimation and considers model validation by Brier and ranking probability scores. The methodology is applied in Section 4 to the data for the two sports. Concluding remarks and future research are summarized in Section 5.

The data and R (R Development Core Team, 2011) code written for implementing the analyses are available from www.blackwellpublishing.com/rss.

2 Description of the data and analyses with non-dynamic abilities

2.1 National Basketball Association

As first motivating example we consider the 2009-2010 regular season of the NBA league. There are thirty teams in the league playing 82 games each, 41 at home and 41 away. The total number of matches is 1,230. The schedule of the tournament includes a greater number of contests against teams in the same division and in the same conference, while competitions between teams in different conferences are less numerous. The regular season started at the end of October 2009 and ended in mid April 2010. Matches were played in 164 different days. The number of matches per day ranges from 1 to 14, the mean number is 7.5.

The description of the proposed methodology for the analysis of tournaments is simplified by the assumption of an order for the $m = 1,230$ matches among the $n = 30$ teams involved in the tournament. A convenient choice is to arrange the matches in chronological order,

with those played at the same time in alphabetic order of the home team. Let Y_i be the binary random variable which denotes the result of the i th match, $i = 1, \dots, m$, played by the home team h_i against the visiting team v_i ; with $h_i, v_i = 1, \dots, n$, $h_i \neq v_i$. We arbitrarily code $Y_i = 1$ if the home team wins and $Y_i = 0$ if the visitors win.

Traditional paired comparison models describe the outcome probability as $\text{pr}(Y_i = 1) = F(a_{h_i} - a_{v_i})$, where F is a cumulative distribution function while a_{h_i} and a_{v_i} are the parameters representing the abilities of the home and the visiting teams in match i . This simple choice model is commonly termed the Bradley-Terry model (Bradley and Terry, 1952) or the Thurstone-Mosteller model (Thurstone, 1927; Mosteller, 1951) depending on whether F is the cumulative distribution function of a logistic or of a standard normal random variable, respectively. In the rest of the paper, we will consider the Bradley-Terry specification.

The advantage deriving from playing at home is commonly taken into account by including a common home effect parameter η for all teams (Fahrmeir and Tutz, 1994; Knorr-Held, 2000; Harville, 2003), thus leading to the model

$$\text{pr}(Y_i = 1) = \frac{\exp(\eta + a_{h_i} - a_{v_i})}{1 + \exp(\eta + a_{h_i} - a_{v_i})}. \quad (1)$$

Parameter identifiability requires one constraint in the set of abilities, such as the sum constraint $\sum_{k=1}^n a_k = 0$ or the reference team constraint $a_k = 0$ for some $k \in \{1, \dots, n\}$.

Table 1 shows the estimates of the abilities \hat{a}_k from model (1) with the sum constraint on team abilities. Teams are ranked on the basis of the number of matches won during the season out of the total 82 matches played (first column). The second column indicates which percentage of the won matches was played at home. On average circa 60% of the matches are won by the home team, therefore the advantage in playing at home seems not negligible. Columns three and five in Table 1 report the estimated abilities and the ranking according to the estimated abilities. There is a very close agreement between the ranking obtained by the estimated abilities and the number of won matches, indeed the Kendall rank correlation τ is 0.97. The estimated home effect (with standard error in brackets) is $\hat{\eta} = 0.487$ (0.067).

Table 1: 2009-2010 American National Basketball Association league. The table displays: (1) number of matches won (**won**), (2) percentage of won matches played at home (**% home**), (3) estimated abilities (**ability**), (4) quasi-standard errors (**qse**) and (5) ranks (**rank**) based on the **static** Bradley-Terry model, (6) estimated mean abilities and (7) ranks based on the **dynamic** Bradley-Terry model.

| | won | % home | static | | | dynamic | |
|------------------------|-----|--------|---------|-------|------|---------|------|
| | | | ability | qse | rank | ability | rank |
| Cleveland Cavaliers | 61 | 0.57 | 1.189 | 0.267 | 1 | 0.769 | 1 |
| Orlando Magic | 59 | 0.58 | 1.048 | 0.260 | 2 | 0.568 | 3 |
| Los Angeles Lakers | 57 | 0.60 | 0.997 | 0.254 | 3 | 0.650 | 2 |
| Dallas Mavericks | 55 | 0.51 | 0.835 | 0.249 | 4 | 0.451 | 5 |
| Phoenix Suns | 54 | 0.59 | 0.767 | 0.248 | 5 | 0.382 | 8 |
| Atlanta Hawks | 53 | 0.64 | 0.692 | 0.247 | 8 | 0.424 | 7 |
| Denver Nuggets | 53 | 0.64 | 0.744 | 0.246 | 7 | 0.482 | 4 |
| Utah Jazz | 53 | 0.60 | 0.749 | 0.248 | 6 | 0.348 | 9 |
| Boston Celtics | 50 | 0.48 | 0.493 | 0.243 | 12 | 0.447 | 6 |
| Oklahoma City Thunder | 50 | 0.54 | 0.565 | 0.242 | 10 | 0.268 | 11 |
| Portland Trail Blazers | 50 | 0.52 | 0.538 | 0.242 | 11 | 0.253 | 12 |
| San Antonio Spurs | 50 | 0.58 | 0.580 | 0.242 | 9 | 0.270 | 10 |
| Miami Heat | 47 | 0.51 | 0.307 | 0.239 | 13 | 0.070 | 14 |
| Milwaukee Bucks | 46 | 0.61 | 0.241 | 0.239 | 14 | 0.054 | 15 |
| Charlotte Bobcats | 44 | 0.70 | 0.155 | 0.239 | 15 | 0.019 | 16 |
| Houston Rockets | 42 | 0.55 | 0.140 | 0.237 | 16 | 0.080 | 13 |
| Chicago Bulls | 41 | 0.59 | 0.000 | 0.237 | 17 | -0.065 | 20 |
| Memphis Grizzlies | 40 | 0.58 | -0.011 | 0.237 | 18 | 0.019 | 17 |
| Toronto Raptors | 40 | 0.63 | -0.074 | 0.240 | 19 | 0.017 | 18 |
| New Orleans Hornets | 37 | 0.65 | -0.177 | 0.240 | 20 | -0.037 | 19 |
| Indiana Pacers | 32 | 0.72 | -0.544 | 0.246 | 21 | -0.382 | 22 |
| Los Angeles Clippers | 29 | 0.72 | -0.651 | 0.249 | 22 | -0.300 | 21 |
| New York Knicks | 29 | 0.62 | -0.749 | 0.250 | 23 | -0.415 | 23 |
| Detroit Pistons | 27 | 0.63 | -0.821 | 0.252 | 25 | -0.446 | 25 |
| Philadelphia 76ers | 27 | 0.44 | -0.830 | 0.253 | 26 | -0.457 | 26 |
| Golden State Warriors | 26 | 0.69 | -0.804 | 0.254 | 24 | -0.586 | 28 |
| Washington Wizards | 26 | 0.58 | -0.904 | 0.257 | 28 | -0.494 | 27 |
| Sacramento Kings | 25 | 0.72 | -0.888 | 0.258 | 27 | -0.436 | 24 |
| Minnesota Timberwolves | 15 | 0.67 | -1.616 | 0.302 | 29 | -0.834 | 29 |
| New Jersey Nets | 12 | 0.67 | -1.972 | 0.327 | 30 | -1.119 | 30 |

Column four of Table 1 reports the quasi-standard errors (Firth and de Menezes, 2004) of the abilities. The quasi-standard errors allow one to approximately reconstruct the uncertainty of pairwise differences $\hat{a}_k - \hat{a}_{k'}$ used for comparing teams k and k' without the need to report also the covariance between \hat{a}_k and $\hat{a}_{k'}$. For example, if it is of interest to test whether the ability of Cleveland is significantly higher than the ability of Orlando, the standard error of the difference between the estimators can be approximated using the quasi-variances simply as $(0.267^2 + 0.260^2)^{1/2} = 0.373$. In this case, the abilities of the best two teams do not appear statistically different.

2.2 Association football

The second application concerns the 2008-2009 Italian Serie A football league. This tournament comprises $n = 20$ teams with matches played between August 2008 and May 2009. The tournament has a double round-robin structure, so each team competes twice against all the other teams in the league, once at home and once away. The total number of matches is thus $m = n(n - 1) = 380$. These matches were played on 74 different days. As in the NBA, there are days with just one match and days with up to 10 matches played. The average number of matches per day is 5.14.

The teams, ranked according to the final points order, are listed in Table 2. In the football tournament, the winning team gains 3 points while the losing team gets nothing. If the match is drawn, both teams gain 1 point. On average, 65% of the total points are gained in home matches, with percentages ranging from 45% to 79% and it is then evident that home advantage of teams should be included in the model.

In contrast to basketball, football matches can also end in a draw, hence the random variable Y_i has three categories arbitrarily coded as 2 if the home team wins, 1 in case of draw and 0 in case of victory of the visiting team. Accordingly, model (1) is extended to

Table 2: 2008-2009 Italian Serie A football league. The table displays: (1) final points (**pts**), (2) percentage of points won at home (**% home**), (3) estimated abilities (**ability**), (4) quasi-standard errors (**qse**) and (5) ranks (**rank**) based on the **static** Bradley-Terry model, (6) estimated mean abilities and (7) ranks based on the **dynamic** Bradley-Terry model.

| | pts | % home | static | | | dynamic | |
|-------------------|-----|--------|---------|-------|------|---------|------|
| | | | ability | qse | rank | ability | rank |
| 1) Internazionale | 84 | 0.56 | 1.380 | 0.348 | 1 | 0.462 | 1 |
| 2) Juventus | 74 | 0.53 | 0.928 | 0.324 | 2 | 0.303 | 2 |
| 3) Milan | 74 | 0.61 | 0.913 | 0.326 | 3 | 0.294 | 3 |
| 4) Fiorentina | 68 | 0.65 | 0.643 | 0.327 | 5 | 0.205 | 4 |
| 5) Genoa | 68 | 0.60 | 0.693 | 0.314 | 4 | 0.177 | 5 |
| 6) Roma | 63 | 0.68 | 0.422 | 0.310 | 6 | 0.108 | 6 |
| 7) Udinese | 58 | 0.66 | 0.231 | 0.306 | 7 | -0.007 | 11 |
| 8) Palermo | 57 | 0.75 | 0.124 | 0.319 | 8 | 0.071 | 7 |
| 9) Cagliari | 53 | 0.70 | -0.010 | 0.315 | 9 | 0.004 | 10 |
| 10) Lazio | 50 | 0.56 | -0.233 | 0.329 | 12 | 0.028 | 8 |
| 11) Atalanta | 47 | 0.70 | -0.224 | 0.317 | 11 | -0.009 | 12 |
| 12) Napoli | 46 | 0.76 | -0.259 | 0.307 | 13 | 0.009 | 9 |
| 13) Sampdoria | 46 | 0.70 | -0.177 | 0.295 | 10 | -0.098 | 14 |
| 14) Siena | 44 | 0.73 | -0.425 | 0.312 | 15 | -0.109 | 15 |
| 15) Catania | 43 | 0.79 | -0.409 | 0.319 | 14 | -0.070 | 13 |
| 16) Chievo | 38 | 0.45 | -0.527 | 0.306 | 16 | -0.243 | 16 |
| 17) Bologna | 37 | 0.57 | -0.627 | 0.315 | 17 | -0.252 | 17 |
| 18) Torino | 34 | 0.74 | -0.760 | 0.316 | 18 | -0.259 | 18 |
| 19) Reggina | 31 | 0.58 | -0.850 | 0.311 | 20 | -0.337 | 20 |
| 20) Lecce | 30 | 0.63 | -0.833 | 0.303 | 19 | -0.276 | 19 |

account for draws with a cumulative link specification (Agresti, 2002)

$$\text{pr}(Y_i \leq y_i) = \frac{\exp(\delta_{y_i} + \eta + a_{h_i} - a_{v_i})}{1 + \exp(\delta_{y_i} + \eta + a_{h_i} - a_{v_i})}, \quad y_i \in \{0, 1, 2\}, \quad (2)$$

where $-\infty < \delta_0 < \delta_1 < \delta_2 = \infty$ are cutpoint parameters. Parameter identifiability is achieved by imposing the “symmetrical” constraints $\delta_0 = -\delta$ and $\delta_1 = \delta$, with $\delta \geq 0$. These constraints are needed to ensure that two teams with the same ability playing on a neutral field (no home advantage) have the same probability of winning the match. If Y_i assumes only two possible values, then the cumulative logit model (2) reduces to the standard Bradley-Terry model (1), and δ becomes 0.

Column three of Table 2 shows the estimates of the abilities \hat{a}_k again with the sum constraint on team abilities. The estimates of the abilities range from -0.850 for Reggina to 1.380 for Internazionale. The ranking derived from the estimated abilities is very similar to the final points ranking, as the Kendall τ rank correlation is 0.94 . The estimated home effect parameter is $\hat{\eta} = 0.661$ (0.105) and the estimated cutpoint parameter is $\hat{\delta}_0 = -0.652$ (0.061). Column four in Table 2 reports quasi-standard errors for teams’ abilities. In this case, if, for example, it is of interest to test whether the ability of Internazionale is significantly higher than the ability of Juventus, the standard error of the difference can be computed as $(0.348^2 + 0.324^2)^{1/2} = 0.475$, so there is no evidence in this model that they are statistically different even though Internazionale ended the tournament ten points ahead of Juventus.

In the above static models, parameters a_k measure the overall abilities of the teams over a complete season. However, team abilities are expected to change during the season because of injuries to players, tiredness due to participation also in international competitions, team psychology and other factors. In the next section we develop a dynamic version of the Bradley-Terry model in which abilities are allowed to change and to depend on the past performance of the team.

3 Dynamic Bradley-Terry model

3.1 The model

We model the match results with the following dynamic Bradley-Terry model

$$\text{pr}(Y_i \leq y_i | Y_{i-1} = y_{i-1}, \dots, Y_1 = y_1) = \frac{\exp \{ \delta_{y_i} + a_{h_i}(t_i) - a_{v_i}(t_i) \}}{1 + \exp \{ \delta_{y_i} + a_{h_i}(t_i) - a_{v_i}(t_i) \}}, \quad (3)$$

where $a_{h_i}(t_i)$ describes the ability of the home team h_i in match i played against the visiting team v_i at time t_i . We specify an evolution of the team ability in home matches which depends only on past matches played at home, while the ability when playing away depends only on past matches played as visitors. First, consider the ability in home matches and let $t_i^{(-1)}$ be the time of the match previous to match i in which h_i was the home team. The ability of the home team $a_{h_i}(t_i)$ is assumed to evolve in time following the exponentially weighted moving average (EWMA) process

$$a_{h_i}(t_i) = \lambda_1 \mu_{h_i}(t_i) + (1 - \lambda_1) a_{h_i}(t_i^{(-1)}), \quad (4)$$

for some home-specific smoothing parameter $\lambda_1 \in [0, 1]$. The term $\mu_{h_i}(t_i)$ denotes the mean home ability of team h_i based only on the result of the nearest previous match played at home by h_i

$$\mu_{h_i}(t_i) = \beta_1 r_{h_i}(t_i^{(-1)}), \quad (5)$$

with β_1 being a home-specific parameter and $r_{h_i}(t_i^{(-1)})$ a variable measuring the result of team h_i in the match played at time $t_i^{(-1)}$. In the NBA application, we specify $r_{h_i}(t_i^{(-1)})$ as the binary variable equal to 1 if team h_i won its previous home match and 0 if it was defeated. Thus, if j denotes the nearest match previous to match i , which was played by h_i at home, *i.e.* $t_i^{(-1)} = t_j$, then $r_{h_i}(t_i^{(-1)}) = y_j$. Instead, in the Serie A application, we specify $r_{h_i}(t_i^{(-1)})$ as the number of points earned by team h_i at time $t_i^{(-1)}$ of its previous home match: 3 points

in case of victory, 1 for a draw and 0 for a loss.

The ability model (4) must be complemented by an initial condition. We assume that all teams start with the same home ability equal to $\beta_1 \bar{r}_h$, where \bar{r}_h is an average of variables $r_{h_i}(t)$ over the *previous* season. In the analysis of NBA 2009-2010, \bar{r}_h is 0.608, the frequency of victories at home during the NBA 2008-2009 regular season. In the analysis of the Serie A 2008-2009 data, \bar{r}_h is 1.676 points, the average number of points gained by home teams during the Serie A 2007-2008 season.

Suppose that the home team h_i has played K matches at home before the match played at time t_i . Then, by iterated back-substitution, the model based on the pair of equations (4)-(5) can be reformulated as

$$a_{h_i}(t_i) = \beta_1 \left\{ \lambda_1 \sum_{k=0}^{K-1} (1 - \lambda_1)^k r_{h_i}(t_i^{(-k-1)}) + (1 - \lambda_1)^K \bar{r}_h \right\} = \beta_1 x_{h_i}(t_i; \lambda_1), \quad (6)$$

with $t_i^{(-r)}$ denoting the time of the r th previous match played at home by team h_i . Thus, the ability $a_{h_i}(t_i)$ is a function of the entire past history of home matches, $r_{h_i}(t_i^{(-1)})$, $r_{h_i}(t_i^{(-2)})$, \dots , $r_{h_i}(t_i^{(-K)})$. The derived covariate $x_{h_i}(t_i; \lambda_1)$ is a weighted mean of these past results with weights $\lambda_1, \lambda_1(1 - \lambda_1), \dots, \lambda_1(1 - \lambda_1)^K$, geometrically decreasing to zero. The smoothing parameter λ_1 specifies the persistence of the dependency on previous home matches. In the limiting case $\lambda_1 = 1$, the home team's ability depends only on the previous home match, $a_{h_i}(t_i) = \beta_1 r_{h_i}(t_i^{(-1)})$. On the contrary, if $\lambda_1 = 0$ the home ability is constant in time and equal for all teams, $a_{h_i}(t_i) = \beta_1 \bar{r}_h = \eta$. Values of $\lambda_1 \in (0, 1)$ specify different levels of smoothing. In particular, home abilities smoothed in time are obtained when λ_1 approaches zero.

Similarly, the ability of the visiting team is modelled by a second EWMA process

$$a_{v_i}(t_i) = \lambda_2 \mu_{v_i}(t_i) + (1 - \lambda_2) a_{v_i}(t_i^{(-1)}),$$

where $\lambda_2 \in [0, 1]$ is the visitor-specific smoothing parameter and $\mu_{v_i}(t_i) = \beta_2 r_{v_i}(t_i^{(-1)})$ for a

visitor-specific coefficient β_2 . The starting values for r are computed similarly to those for the home abilities. In the NBA 2009-2010 data \bar{r}_a is set equal to 0.392, the frequency of visitors' victories during season 2008-2009, while in the Serie A 2008-2009 data $\bar{r}_a = 1.029$ points, the average number of points gained by visitors in the Serie A 2007-2008 tournament.

Thus, in the proposed dynamic Bradley-Terry model, the EWMA specification is used to account for the serial association among match results of the same team, with suitable differences between home and away matches.

3.2 Likelihood inference

EWMA processes are routinely used in time series forecasting (Holt, 2004) and in statistical quality control charts (Montgomery, 2005). In these contexts, the smoothing parameter is often chosen by trials or by using ad hoc methods based on previous experience. However, many have argued that automatic, data driven choices of the smoothing parameter would be preferable. For example, in classical time series, the smoothing parameter “is often estimated by minimizing the sum of squared one-step-ahead forecasts” (Chatfield, 2000, p. 105).

Here, we follow the recommendation to identify the smoothing parameters using available observations and consider maximum profile likelihood estimation of the two smoothing parameters λ_1 and λ_2 .

Let $\boldsymbol{\gamma} = (\beta_1, \beta_2, \delta)^T$ be the vector of parameter of interests and let $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)^T$ be the vector of nuisance smoothing parameters. Under the chosen order for the match results, the likelihood function for $\boldsymbol{\theta} = (\boldsymbol{\gamma}^T, \boldsymbol{\lambda}^T)^T$ is written as

$$\mathcal{L}(\boldsymbol{\theta}; \mathbf{y}) = \text{pr}(Y_1 = y_1; \boldsymbol{\theta}) \prod_{i=2}^n \text{pr}(Y_i = y_i | Y_{i-1} = y_{i-1}, \dots, Y_1 = y_1; \boldsymbol{\theta}).$$

Given the home smoothing parameter λ_1 , the home ability can be written as $a_{h_i}(t_i) = \beta_1 x_{h_i}(t_i; \lambda_1)$, where $x_{h_i}(t_i; \lambda_1)$ is the weighted average of past home results $r_{h_i}(t_i^{(-k)})$ with weights $\lambda_1(1 - \lambda_1)^{k-1}$, see formula (6). In parallel, given the visitors' smoothing parameter λ_2 ,

the visitors ability is $a_{v_i}(t_i) = \beta_2 x_{v_i}(t_i; \lambda_2)$ where $x_{v_i}(t_i; \lambda_2)$ has a specification analogous to the home case. Accordingly, the conditional probability for the result of match i is expressed by the cumulative logit model

$$\text{pr}(Y_i \leq y_i | Y_{i-1} = y_{i-1}, \dots, Y_1 = y_1; \boldsymbol{\theta}) = \frac{\exp\{\delta_{y_i} + \beta_1 x_{h_i}(t_i; \lambda_1) - \beta_2 x_{v_i}(t_i; \lambda_2)\}}{1 + \exp\{\delta_{y_i} + \beta_1 x_{h_i}(t_i; \lambda_1) - \beta_2 x_{v_i}(t_i; \lambda_2)\}}.$$

Thus, if the smoothing parameters are known, the likelihood function for the interest parameter $\boldsymbol{\gamma}$ corresponds to that of a standard logistic regression model if draws are not allowed (*e.g.* basketball) or that of a cumulative logistic regression model with constrained cutpoints in case of draws (*e.g.* football). The simplicity of computation of $\widehat{\boldsymbol{\gamma}}_\lambda$, the estimates of $\boldsymbol{\gamma}$ given $\boldsymbol{\lambda}$, suggests a two-step maximization of the likelihood. First, the smoothing parameter vector $\boldsymbol{\lambda}$ is estimated by maximizing the profile likelihood $\mathcal{L}(\widehat{\boldsymbol{\gamma}}_\lambda, \boldsymbol{\lambda}; \mathbf{y})$, then $\boldsymbol{\gamma}$ is estimated as $\widehat{\boldsymbol{\gamma}}_{\widehat{\boldsymbol{\lambda}}}$. This approach is employed in the applications illustrated in Section 4.

3.3 Model validation

Model validation can be based upon comparison of the fitted probabilities from the proposed model with fitted probabilities from the unstructured model (2). The proposed model aims to capture the evolution in time of all teams abilities with only four parameters (five in case of draws), while the unstructured model has n free parameters ($n + 1$ when draws are allowed), with n being the number of teams. Clearly, the unstructured model is expected to fit the data better, and thus it may be viewed as a benchmark. The closer the fitted probabilities of the proposed model are to those of the unstructured model, the better is the fit. In order to summarize the fitted probabilities we consider the Brier score (Brier, 1950) which is defined for match i as

$$\text{BS}_i = \sum_{q=0}^{Q-1} \left\{ \text{pr}(Y_i = q | Y_{i-1} = y_{i-1}, \dots, Y_1 = y_1; \widehat{\boldsymbol{\theta}}) - 1(y_i = q) \right\}^2, \quad i = 1, \dots, m,$$

where $1(y_i = q)$ is the indicator function of the event $\{y_i = q\}$, $Q = 2$ for sports without draws and $Q = 3$ when draws are allowed, and $\hat{\theta}$ is the maximum likelihood estimate of θ based on the results of all the played matches, *i.e.* y_1, \dots, y_m . When the fit is perfect, giving probability 1 to the observed outcome, the Brier score is equal to zero, while a completely erroneous fit produces a Brier score equal to 2.

Some authors have suggested that in the case of more than two categories it is better to employ an index which accounts for the whole distribution of probabilities, such as the rank probability score (Czado *et al.*, 2009)

$$\text{RPS}_i = \sum_{q=0}^{Q-1} \left\{ \text{pr}(Y_i \leq q | Y_{i-1} = y_{i-1}, \dots, Y_1 = y_1; \hat{\theta}) - 1(y_i \leq q) \right\}^2.$$

In the analysis of sport tournaments the real interest usually lies in forecasting future results. Hence, it may be more relevant to evaluate the fitted model from a predictive point of view. In this case, we quantify the BS_i and RPS_i using the maximum likelihood estimate $\hat{\theta}_{(i-1)}$ computed only with matches played before the forecasted match i , that is only with results y_1, \dots, y_{i-1} .

4 Applications

4.1 Application to the NBA tournament

We fit the proposed model to the NBA 2009-2010 regular season. Figure 1 shows the profile log-likelihood for the smoothing parameters λ_1 and λ_2 . This is maximized at $\hat{\lambda}_1 = 0.043$ and $\hat{\lambda}_2 = 0.025$. These two values are close to zero thus supporting the effect also of remote match results on the estimation of the present ability. The limiting model with one common home advantage parameter and no evolution in time of abilities corresponds to $\lambda_1 = \lambda_2 = 0$. This pair of values for the smoothing parameters is strongly *not* supported by the data. In fact, the maximized profile log-likelihood is -752.22 while the profile log-likelihood for

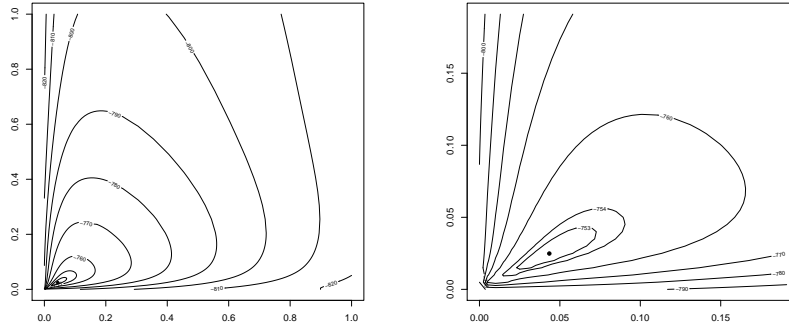


Figure 1: NBA 2009-2010. Contour plot of the profile log-likelihood for the smoothing parameters λ_1 and λ_2 . Left panel: contour plot of the profile log-likelihood for all the range of values of λ_1 and λ_2 . Right panel: zoom of the area of highest likelihood. The point marks the maximum likelihood estimate.

$\lambda_1 = \lambda_2 = 0$ is much smaller, at -830.56 . The maximum likelihood estimates for the home and away coefficients computed at the estimated smoothing parameters are $\hat{\beta}_1 = 5.503$ and $\hat{\beta}_2 = 7.379$, with estimated standard errors 0.465 and 0.699, respectively. Both estimates appear strongly significant.

To validate the fitted model, we compute the fitted Brier scores for each of the 1,230 matches both with the proposed model, which includes only 4 parameters, and with the unstructured model involving 31 parameters. There is a high positive association between the Brier scores of the two models, in fact their correlation is 0.784. The mean of the Brier scores for the fitted model is 0.424, while for the unstructured model it is 0.379, thus the latter is 10.6% smaller in mean. This result was expected given the larger number of parameters of the unstructured model.

However, it is more interesting to consider the appropriateness of the proposed model by evaluation of its predictive performance. For this purpose, we fit the model to the data coming from half of the competition days, *i.e.* 82 days, then predict the results of the matches taking place in the following competition day, the 83rd, and compare them with the observed results by the Brier score. Then, the model is refitted including also the matches in day 83 and used to forecast results in day 84, and so on until the last day of matches (day 164).

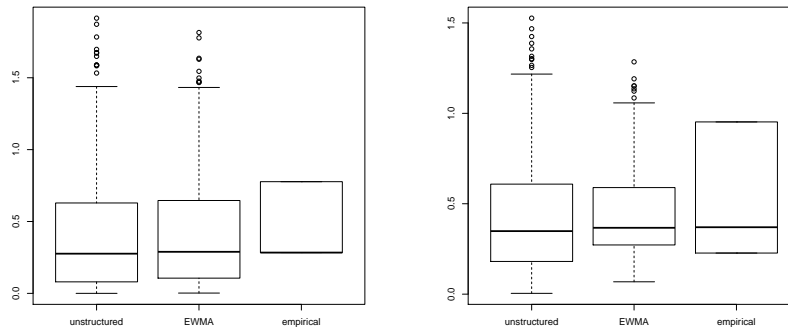


Figure 2: Boxplots of the predictive Brier scores for the NBA data (left panel) and boxplots of rank probability scores for the football data (right panel) computed for the unstructured model (**unstructured**), the EWMA model (**EWMA**) and the forecasts based on empirical proportions (**empirical**).

Following this scheme we compute predictions for a total of 638 matches. Predictions from the unstructured model are similarly computed. These predictions are also compared with those obtained simply using the empirical proportions of wins and losses of the home team computed on the first half of the competition days. These fixed empirical proportions are employed to forecast all the results of the second half of the tournament. The left panel in Figure 2 shows the boxplots of the Brier scores computed for each forecasted match using the unstructured model, the EWMA model and the empirical proportions of wins and losses. Visual inspection of the boxplots shows that the Brier scores of our model are very similar to those of the unstructured model while the Brier scores of the empirical proportions are noticeably worse. Specifically, the mean predictive Brier score for the proposed and the unstructured model are very close: 0.421 and 0.409, respectively. The correlation between the two sets of Brier scores is 0.879. The overall conclusion is that the proposed model is very competitive with the unstructured one from a predictive point of view. The mean Brier score of empirical proportions is 0.497, so it is 15% larger than the one of the proposed model. Furthermore, both the unstructured and the proposed models present Brier scores which are lower than the Brier score of empirical proportions in 66% of the forecasts.

Figure 3 shows the smoothed abilities for nine teams during the complete regular season.

The smoothed abilities are computed with parameters estimated from the complete tournament. For each team there is a home and a visiting ability that are plotted in the same graph. The time scale is the sequence of matches; at the end of the regular season each team has played 41 matches at home and 41 away. Cleveland Cavaliers ended the season with the highest number of victories, 61, and, as expected, their ability both in home and away matches increases noticeably from the starting mean ability common to all teams. It seems

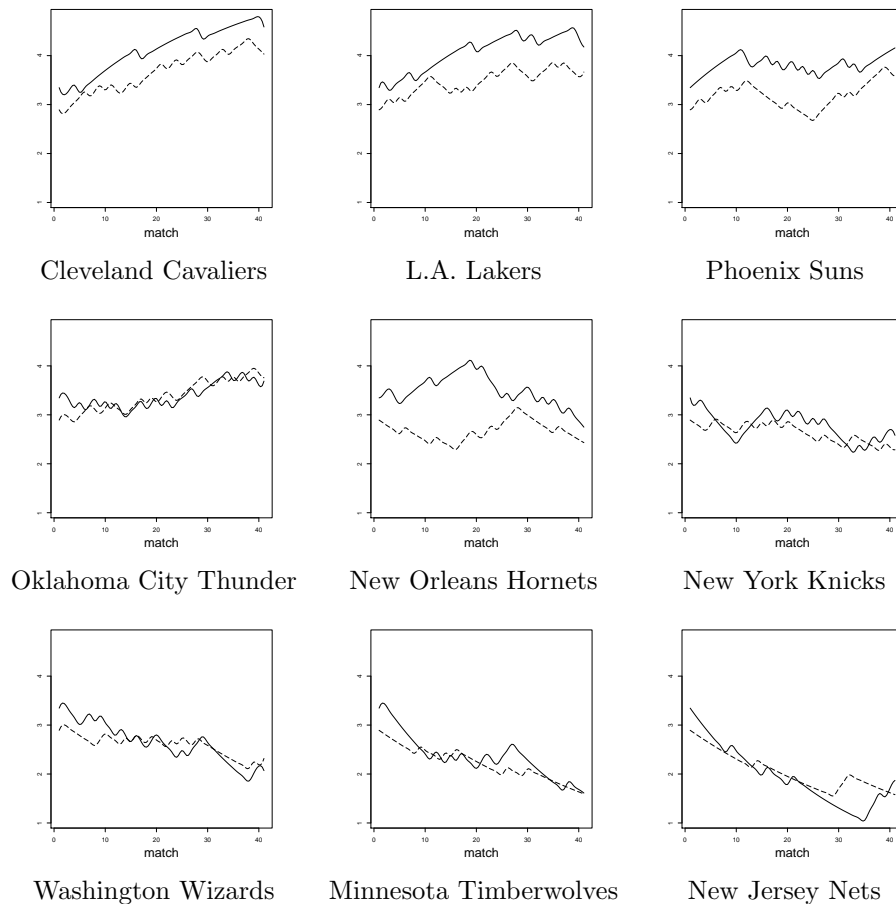


Figure 3: NBA 2009-2010. Smoothed home and visiting abilities for nine different teams. Each plot displays the time series of smoothed home (solid line) and visiting (dashed line) estimated abilities.

that Cleveland benefits from an important home-advantage effect, indeed it won 85% of the matches that it played at home and 63% of the matches played away. Also New Orleans Hornets show an important home effect especially in the first half of the season. This team

won 16 matches at home out of the first 20 matches while it succeeded only in 5 matches away out of the first 20. However, in the second part of the season it showed no particular advantage in playing at home since it won 8 matches out of the last 21 both at home and away. New Jersey Nets was the team that performed worst during the season. They won a total of 12 matches during the whole tournament and performed poorly both at home and away. The increase in the visiting ability after match 30 is due to their winning three away matches in a row.

Finally, column six in Table 1 reports the ranking derived from the proposed model based on the average of the team abilities in each of the 82 days in which the team played. The reported abilities are computed so as to sum to zero, by analogy with the unstructured model. The Kendall rank correlation between the rankings of the unstructured and the proposed model is 0.89. In this case we cannot report quasi-standard errors since the abilities are not individual parameters as in the static Bradley-Terry model. Suppose we are interested in testing whether the ability of Cleveland Cavaliers is significantly higher than the ability of Orlando Magic. The ability of the former team is 0.769 while the ability of the latter is 0.568, so their difference amounts to 0.201. This difference appears statistically significant since its standard error is 0.018. This result is different from the result given by the unstructured model because the proposed model has fewer parameters and thus the ability estimates are more precise.

4.2 Application to the Serie A tournament

The second data set concerns the 2008-2009 Serie A football tournament which allows also for ties. The profile log-likelihood is maximized at $\hat{\lambda}_1 = 0.108$ and $\hat{\lambda}_2 = 0.078$, again a pair of values quite close to zero. The support for this model, in contrast to the static model resulting when $\lambda_1 = \lambda_2 = 0$, is given by the maximized profile log-likelihood which is equal to -383.27 , while it takes the value -393.68 in the model with smoothing parameters equal to 0. The maximum likelihood estimates for the coefficients computed at the estimated values

of the smoothing parameters are $\hat{\beta}_1 = 0.925$, $\hat{\beta}_2 = 1.031$, with standard error 0.173 and 0.294, respectively, while the estimated cutpoint parameter is $\hat{\delta}_0 = -0.579$ with standard error 0.054.

Since in football data there are three possible results, it is more appropriate to employ the rank probability score in order to validate the model. The mean rank probability score for the whole tournament, which comprises $m = 380$ matches, is 0.416 for the proposed model and 0.369 for the unstructured model, hence the proposed model has a mean rank probability score 11.2% higher than that of the unstructured model. The correlation between rank probability scores for the two models is 0.715.

As for the NBA, it is however more interesting to consider the predictive performance of the model. In order to forecast the results we employ the same scheme used for basketball data. The model is estimated using the matches in the first half of the competition days (37 days) and the results of matches occurred in the 38th day of competitions are predicted. Then we updated the model with the matches performed in day 38 and predict the following ones. The procedure is repeated until the end of the tournament with a total of 198 matches predicted. The mean rank probability scores for the unstructured and the EWMA model are essentially equal: 0.450 and 0.451, respectively. The correlation of the RPS_i in the two models is 0.762. The right panel of Figure 2 shows the boxplots of the rank probability scores of the two models and the scores computed using the empirical proportions of home wins, draws and losses computed in the first 37 days. It is evident that the medians of the RPS_i for the unstructured and the EWMA models are equal, however the EWMA model presents a shorter box, thus with respect to the unstructured model it assigns higher probability to results actually observed in fewer cases, but also it less often assigns higher probabilities to results that do not occur. The mean RPS_i of empirical proportions is 0.468, which is 4% larger than the value for the EWMA model. Furthermore, the rank probability scores of the empirical proportions are higher than the scores computed for the proposed model in 58% of the forecast matches.

Figure 4 shows the estimated abilities for nine of the teams which competed in the 2008-2009 Serie A league. Internazionale won the tournament ending the season with 84 points, while Juventus and Milan got the second position gaining 74 points each. The plots show that the home performance of Internazionale is more stable during the season, in particular they never lost a home match, while Juventus and Milan lost two home matches each. In general, teams tend to perform better at home than away. This is particularly evident for Siena which won 9 matches and drew 5 matches out of the 19 home matches, while in the other 19 away matches they won only 3 times and drew 3 matches.

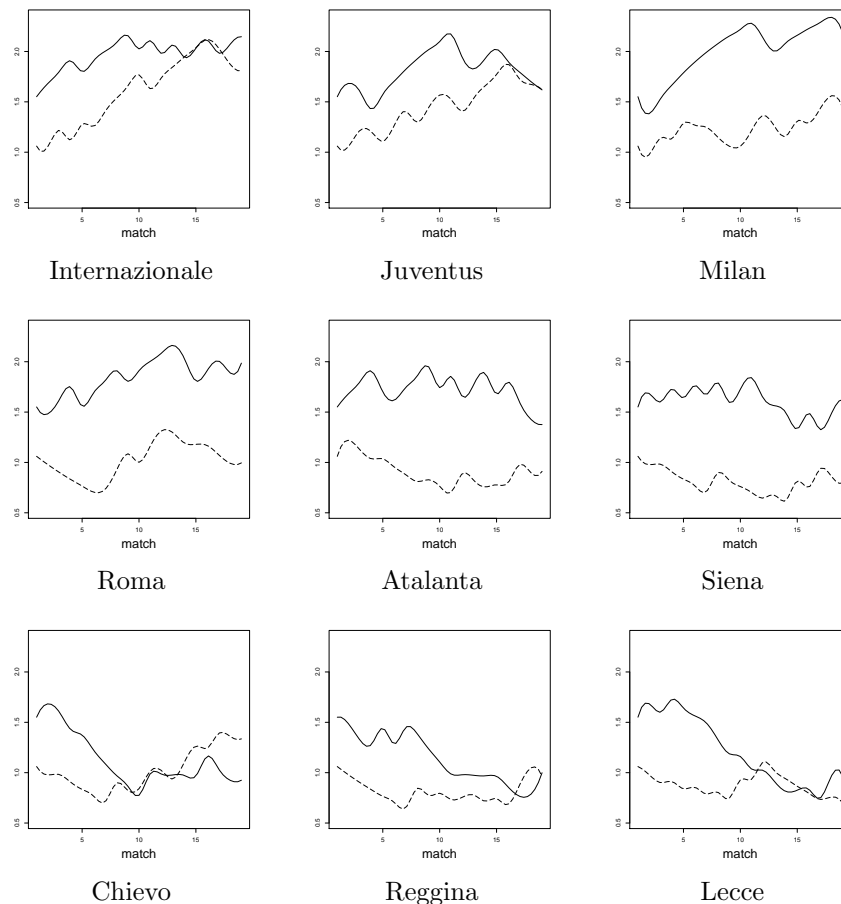


Figure 4: Football 2008-2009. Smoothed home and visiting abilities for nine different teams. Each plot displays the time series of smoothed home (solid line) and visitor (dashed line) estimated abilities.

Column six in Table 2 reports the mean abilities of football teams resulting from the

EWMA model. The Kendall rank correlation between the rankings derived from the unstructured and the proposed models is 0.86. In this case, the standard error needed to test whether the teams Internazionale and Juventus have the same ability is 0.037. Since the estimated ability of Internazionale is 0.462 and the estimated ability of Juventus is 0.303, yielding a difference of 0.159, the two teams appear to have statistically different abilities whereas the unstructured model led to the opposite conclusion.

5 Conclusions

We have described a dynamic paired comparison model for the results of matches in sport tournaments. The model specification describes the temporal evolution of teams' abilities by separate EWMA processes for the home and away results. The two applications to basketball and football tournaments show that the proposed model seems to capture the relevant aspects of the evolution in time of team abilities thus providing sensible forecasts if compared with the unstructured model that fits one ability parameter for each team.

The proposed model uses only information about the final result of previous matches. It seems reasonable that using more detailed information about previous matches may result in more accurate data fitting and improved forecasts. Inclusion of additional information about previous matches is easily handled under the model framework described in this paper, for example one could substitute $r_{h_i}(t_i^{(-1)})$ and $r_{v_i}(t_i^{(-1)})$ with vectors $\mathbf{r}_{h_i}(t_i^{(-1)})$ and $\mathbf{r}_{v_i}(t_i^{(-1)})$ and thus consider vector home and visitor parameters β_1 and β_2 . With a possible large number of covariates from previous matches, it could be sensible to consider some form of shrinkage to avoid overfitting, for example by a lasso penalty (Tibshirani, 1996) on parameters β_1 and β_2 .

The proposed model requires starting values for the covariates. We considered equal values for all teams, \bar{r}_h and \bar{r}_a , based on the results in the previous season. It is possible to consider team-specific starting values, as for example the proportions of wins and losses at

home for each team in the previous season. The use of team-specific starting values for the NBA data leads to somewhat different estimates of the model parameters, but the fitting and predictive qualities of the resulting model, as evaluated by the Brier score, are almost identical to those of the model with all equal starting values. This exercise seems not possible for the football data since the teams in the Serie A league change season-by-season: at the end of the regular season the last three teams in the league are demoted to the lower league and the three best teams of the lower league are promoted to the Serie A. Hence, the teams in the league are not the same in different seasons and it seems inappropriate to use team-specific values computed for teams in different leagues.

The issue of whether team-specific home advantages should be included in paired comparison models was considered by many authors with contrasting conclusions. Knorr-Held (1997) does not find much evidence of home advantage heterogeneity among teams in the *Bundesliga*. Neither do the results in Harville and Smith (1994) show much difference in home field advantages among college basketball teams. Analyses for some other contexts do, however, support heterogeneity in home advantages, see Clarke and Norman (1995), Kuk (1995) and Glickman and Stern (1998) for different analyses of the English Premier Football League. The model proposed in this paper can be seen as a convenient way to induce home abilities which vary between teams and in time depending on past performances of teams.

We considered sports with matches that may end with two or three different results. There are other competitions where more than three levels of results are possible. An example is volleyball where points are assigned as follows: if a match ends 3 – 0 or 3 – 1 the winning team gains 3 points and the losing team remains empty handed, while if the match ends 3 – 2, the winning team gains 2 points and the losing team is awarded 1 point. As suggested by a referee, the analysis of volleyball matches would require a categorical variable Y_i with four ordered levels. There is no special difficulty in extending our modelling framework to this case. More generally, assume that Y_i is a categorical variable that may assume Q different categories where $Q - 1$ denotes the best result achievable for the home team and 0 denotes

the best result for the visiting team. The cumulative logit model (2) is easily extended to handle Q levels

$$\text{pr}(Y_i \leq y_i) = \frac{\exp\{\delta_{y_i} + a_{h_i}(t_i) - a_{v_i}(t_i)\}}{1 + \exp\{\delta_{y_i} + a_{h_i}(t_i) - a_{v_i}(t_i)\}}, \quad y_i \in \{0, 1, \dots, Q-1\}, \quad (7)$$

where $-\infty < \delta_0 < \dots < \delta_{Q-1} = \infty$ are cutpoint parameters. To preserve model identifiability, the symmetrical constraint now becomes $\delta_q = -\delta_{Q-2-q}$, $q = 0, \dots, Q-1$, and $\delta_{Q/2-1} = 0$ when Q is even.

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