A Note on Irreversible Investment, Hedging and Optimal Consumption Problems †

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First version: June 3, 2005
This version: July 10, 2005

Abstract

A canonical problem in real option pricing, as described in the classic text of Dixit and Pindyck [2], is to determine the optimal time to invest at a fixed cost, to receive in return a stochastic cashflow. In this paper we are interested in this problem in an incomplete market where the cashflow is not spanned by the traded assets. We follow the formulation in Miao and Wang [21]; our contribution is to show that significant progress can be made in solving the Hamilton-Jacobi-Bellman equation and that the optimal exercise threshold can be characterized quite precisely.

1 Introduction

In this paper we consider the problem of determining the optimal timing of an irreversible investment decision in an incomplete market, for an agent who aims to maximize expected utility of consumption over an infinite horizon. The agent pays a fixed investment cost to receive rights to a stream of stochastic cashflows over an infinite horizon. The problem is incomplete since the investment cashflows are not tradeable, and so cannot be perfectly hedged. Instead, the agent has access to a traded financial asset which is correlated with these cashflows, and can be used to partially hedge risk. The agent maximizes his expected utility from consumption over an infinite horizon, subject to choice over investment timing, consumption and hedge position in the financial asset.

When fully specified this problem leads to a Hamilton-Jacobi-Bellman equation which is a non-linear second-order ordinary differential equation subject to a free-boundary condition. Our contribution in this paper is to reduce this problem to a first order ordinary differential equation subject to initial conditions. Additionally, in a special case of the parameters, the ode can be solved explicitly and the free-boundary solves a transcendental equation. We will also depict the investment boundary for a variety of parameters. The formulation of our problem was previously treated in Miao and Wang [21]. Miao and Wang reduce the HJB equation to a pair of first order equations which are then solved by approximating the solution with Chebyshev polynomials and then by adjusting the location of the free-boundary until the smooth pasting condition is approximately satisfied. Our solution method is much simpler, and we characterize the free-boundary directly.

The problem we consider is motivated by the field of real options. The standard real options problem (see Dixit and Pindyck [2] and McDonald and Siegel [16]) considers when a firm should time an irreversible, indivisible investment. In a complete market (assuming the investment cashflows are tradeable or perfectly spanned by a traded asset) and for an infinite horizon, this reduces to a perpetual American call under the Black Scholes assumptions, as studied by McKean [17] and also by Merton [20].

†Henderson is partially supported by the NSF via grant DMI 0447990. Hobson is supported by an EPSRC Advanced Fellowship.

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Recently some progress has been made to extend such investment timing models to incomplete markets. Henderson [6] solves the timing problem where a lump-sum payoff is obtained at option exercise and the agent can hedge in a traded asset. This approach has the advantage that the solution is in closed form; however, she does not consider consumption. Miao and Wang [21] and Monoyios [22] treat the lump-sum payoff with consumption. The results in their formulation are related closely to those in this paper, and although we do not treat this case here, our techniques could be used to reduce their problem to an initial value problem. Rather than consider a lump-sum payoff, in this paper we consider the opportunity to invest to receive a stochastic cashflow. This is typical of many applications including the sale of industrial products, the value of the output from a mine or the rental income from property, see Dixit and Pindyck [2].

There is a vast literature on optimal consumption and investment problems, beginning with Merton [18, 19]. This literature typically solves problems of optimal control rather than stopping problems. For instance, general utility functions were studied by Karatzas et al [12] (see also Karatzas and Shreve [13]) whilst Zariphopoulou [24] generalizes to non-linear stock dynamics in a finite time horizon model. Huang and Pages [10] show some existence and convergence results in an infinite horizon model.

After the investment option is exercised, the agent faces a consumption-investment problem with stochastic income. There is also a large literature on such problems including Duffie and Zariphopoulou [5], Duffie et al [3], Koo [15] and Henderson [7].

The problem we treat in this note is a mixed control/stopping problem. Other examples of this type of problem in mathematical finance include Davis and Zariphopoulou [1], Karatzas and Kou [11] and Karatzas and Wang [14].

Although we concentrate here on the solution for the investment threshold, the option value can be obtained with a certainty equivalence or utility indifference argument via the value function (see Hodges and Neuberger [9], the survey of Henderson and Hobson [8] or, in the current context, Miao and Wang [21]). With the techniques of this paper to help determine the investment threshold, it becomes much easier to solve for the value function over the region up to this known threshold than to solve the shooting problem when the boundary is unknown.

2 The Model

We consider an agent who aims to maximize expected utility from consumption over an infinite horizon. The agent is able to invest in a complete frictionless financial market, which consists of a riskless bond (paying constant rate of interest $r > 0$) and a risky asset with price process P satisfying

$$\frac{dP_t}{P_t} = rdt + \chi_t(dB_t + \eta dt).$$

In principle the financial market could contain many risky assets, but the problem can easily be reduced to the univariate case. To facilitate calculations we take the Sharpe ratio η to be a constant. We assume that this agent has initial wealth w, and that his self-financing wealth process satisfies

$$dW_t = (rW_t - C_t)dt + \pi_t \left(\frac{dP_t}{P_t} - rdt \right) \tag{1}$$

where C is the consumption process, and π is the investment strategy, expressed as a cash amount. This problem is the classical Merton problem. In the case of exponential utility $U(C) = -e^{-\beta t}e^{-\gamma C}/\gamma$ for β a discount rate, and γ the risk aversion parameter, the problem is easily solved, see Merton [19]. In particular we find that if $V_0(w)$ is the expected utility of the agent under optimal behavior, then

$$V_0(w) = -\frac{1}{\gamma r} \exp \left(-\gamma r w + \frac{D_0}{r} \right)$$

where $D_0 = r - \beta - \eta^2/2$.

Now suppose that the agent also possesses a perpetual real option: the right to invest (irreversibly) at a time of their choosing and at a cost K, to receive in return a stochastic cashflow stream Y_s over an infinite horizon. We model Y_s as a diffusion process:

$$dY_s = \alpha(Y_s)ds + \sigma(Y_s)dZ_s \tag{2}$$

where Z is correlated to B with correlation $-1 < \rho < 1$. Later we will take Y to be a drifting Brownian motion. Note Y is a stream of cashflows from a real investment and as such, is not a traded variable. The agent is thus exposed to unhedgeable idiosyncratic risk and faces an incomplete market. Note if $|\rho| = 1$, the agent's position in the financial asset P would fully hedge his exposure to Y and the market would be complete.

The agent's problem is to maximize with respect to investment time τ, consumption strategy C and investment strategy π, the discounted expected utility from consumption, i.e. to maximize

$$\mathbb{E}\left[\int_0^\infty U(s, C_s)ds\right] = \mathbb{E}\left[-\frac{1}{\gamma}\int_0^\infty e^{-\beta s}e^{-\gamma C_s}ds\right]$$

subject to a transversality condition, and subject to modified wealth dynamics

$$dW_t = (rW_t - C_t)dt + \pi_t\left(\frac{dP_t}{P_t} - rdt\right) + \left(Y_t I_{\{t\geq\tau\}} - K\delta(t-\tau)\right)dt \tag{3}$$

where δ is the Dirac function. Let $V(w, y)$ be the maximal expected utility for this problem, given that the real option has not yet been exercised. By the assumption of a constant Sharpe ratio for the market asset, and by the diffusion assumption on Y, this value function is a function of initial wealth and the current value of Y alone.

Note this is the original formulation in Miao and Wang [21]. There is an alternative formulation (see Henderson [6] for the case without consumption and also Miao and Wang [21] and Monoyios [22] for the case with consumption) in which the agent receives a one-off payment of magnitude Y_t so that the modified wealth dynamics become

$$dW_t = (rW_t - C_t)dt + \pi_t\left(\frac{dP_t}{P_t} - rdt\right) + (Y_t - K)\delta(t-\tau)dt$$

We will not discuss this case in detail, but identical methods apply. The only difference is that the boundary condition is slightly modified.

In order to solve the utility maximization problem it is necessary to first solve the problem under the assumption that the agent has already started to receive the cashflow Y (and has already paid K). Problems of utility maximization and investment with stochastic income are considered by Duffie and Jackson [4] and Svensson and Werner [23]. Following Miao and Wang [21], suppose that $\alpha(Y_s) = \alpha$ and $\sigma(Y_s) = \sigma$ so that Y is a drifting Brownian motion. This has the interpretation that cashflows may be negative, representing losses. Set

$$V_1(w, y) = \sup_{C,\pi}\mathbb{E}\left[-\frac{1}{\gamma}\int_0^\infty e^{-\beta s}e^{-\gamma C_s}ds\right] \tag{4}$$

subject to $Y_0 = y$ and

$$dW_t = (rW_t - C_t)dt + \pi_t\left(\frac{dP_t}{P_t} - rdt\right) + Y_t dt$$

with $W_0 = w$. The set of admissible wealth/consumption processes is also subject to a transversality condition:

$$\lim_{T\to\infty}\mathbb{E}[e^{-\beta T}e^{-\gamma r W_T - \gamma Y_T}] = 0$$

Then

$$V_1(w, y) = -\frac{1}{\gamma r}\exp\left(-\gamma r w - \gamma y + \frac{D_1}{r}\right)$$

where $D_1 = D_0 - \gamma(\alpha - \sigma\rho\eta) + \sigma^2\gamma^2(1-\rho^2)/2$.

Now we can solve the full problem:

$$V(w,y) = \sup_{C,\pi,\tau} \mathbb{E}\left[-\frac{1}{\gamma}\int_0^\infty e^{-\beta s}e^{-\gamma C_s}ds\right]$$

subject to wealth dynamics (3). Using the value function V_1 above this can be rewritten as

$$V(w,y) = \sup_{\tau} \sup_{\{C_s,\pi_s:0\leq s\leq\tau\}} \mathbb{E}\left[-\frac{1}{\gamma}\int_0^\tau e^{-\beta s}e^{-\gamma C_s}ds + e^{-\beta\tau}V_1(W_\tau,Y_\tau)\middle| W_0 = w, Y_0 = y\right].$$

It is straightforward to write down a Hamilton-Jacobi-Bellman equation for this problem (for details see Miao and Wang [21]). If we guess that the solution takes the form $V(w,y) = V_0(w+g(y))$ then g has the interpretation as the certainly equivalent value of the real option to invest. Miao and Wang reduce the problem to solving (finding g and y^*) the free-boundary problem

$$rg(y) = (\alpha - \rho\eta\sigma)g'(y) + \frac{\sigma^2}{2}g''(y) - \frac{\gamma r\sigma^2}{2}(1-\rho^2)g'(y)^2 \tag{5}$$

subject to

$$\lim_{y\to-\infty} g(y) = 0 \tag{6}$$

$$g'(y^*) = 1/r \tag{7}$$

$$g(y^*) = \frac{y^*-D}{r} \tag{8}$$

where D is a constant

$$D = D(r,\alpha,\eta,\sigma,\gamma,\rho,K) = \frac{D_1 - D_0}{\gamma r} + rK = -\frac{(\alpha-\eta\sigma\rho)}{r} + \frac{\gamma\sigma^2(1-\rho^2)}{2r} + rK.$$

Here (6) is a condition which says that as the initial value of Y becomes smaller and smaller, the value of the option to receive the cashstream Y_s becomes zero. The other conditions (7) and (8) are smooth pasting and value matching respectively, and are derived from the functional form (4) of V_1.

Observe that the discount rate β does not enter into the solution (Miao and Wang take $\beta = r$, but this is not necessary). The set of parameters can be reduced to

$$\Gamma = \gamma(1-\rho^2) \qquad \xi = \alpha - \rho\eta\sigma \qquad \Sigma = \sigma^2,$$

(respectively the effective risk aversion parameter, the effective drift on the stochastic cashflow and the squared volatility), together with K and r. The last new parameter is purely a relabeling, but it allows us to distinguish between the economic parameters and the parameters of the differential equation.

Note that the problem with the investment opportunities in the market is no more difficult to solve than the problem faced by the agent with no access to the market, and with only the perpetual option to exercise. Note also that different dynamics for Y would not affect the form of the ode for the value function in the continuation region, but they would affect the boundary conditions. The choice of drifting Brownian motion for Y ensures that it is possible to get a closed form expression for V_1.

We assume we are in an incomplete market so that $\rho^2 < 1$. The case where there are no financial market investment opportunities is easily deduced from the general case by taking zero correlation ($\rho = 0$) and a zero Sharpe ratio on the market asset ($\eta = 0$).

The key equation (5) is a non-linear, second-order ordinary differential equation, subject a condition at the free-boundary. Miao and Wang [21] solve (5)—(8) firstly by reducing (5) to a pair of first order equations which are solved by approximating the solution with Chebyshev polynomials, and secondly by guessing the location of the free-boundary and then adjusting the location of this boundary until the smooth pasting condition is approximately satisfied.

In contrast, the contribution of this paper is to show that the problem of finding the free-boundary can be reduced to solving a first order differential equation subject to initial conditions. For certain special parameter values this ode can be solved explicitly and the free-boundary can be determined as the solution to a transcendental equation, but even in the general case it is much simpler to solve a first-order (non-linear) initial value problem than to solve a second order (non-linear) free-boundary problem. If the critical exercise value is the object of interest then we are done, but even if we wish to find the value function, it is much easier to solve for g over the region $(-\infty, y^*]$ when y^* is known, than to solve the shooting problem when y^* is unknown.

Theorem 1 *(i) Suppose $\xi = 0$, for example if $\alpha = \rho = 0$. Then the critical value y^* is the unique solution for $y \geq D$ of the equation $\Lambda(y) = 0$ where*

$$\begin{aligned}\Lambda(y) &= e^{2\Gamma y - 2\Gamma D} - 1 - 2y\Gamma + 2D\Gamma - \frac{\Sigma\Gamma^2}{r} \\ &= \exp\left(2\Gamma y - 2\Gamma r K - \Gamma^2\Sigma/r\right) - 2y\Gamma + 2\Gamma r K - 1\end{aligned}$$

(ii) If $\xi \neq 0$ and if $F(f)$ is the (positive increasing) solution to

$$\frac{dF}{df} = \frac{\Sigma f}{\Sigma\Gamma f^2 - 2\xi f + 2rF} \qquad F(0) = 0 \tag{9}$$

then the critical investment threshold is given by

$$y^* = F(1) + D. \tag{10}$$

Proof: The key idea is to set $G = dg/dy$, and to write G as a function of g. With g as the independent variable we have $d^2g/dy^2 = (dg/dy)(dG/dg) = GG_g$. Equation (5) becomes

$$rg = \xi G(g) + \frac{\Sigma}{2} G(g) G_g - \frac{\Gamma r \Sigma}{2} G(g)^2 \tag{11}$$

subject to

$$\begin{aligned} G(0) &= 0 \\ G(g^*) &= \left.\frac{dg}{dy}\right|_{g=g^*} = \frac{1}{r} \end{aligned}$$

Here g^* is the value function at the critical value, and the value matching condition is not necessary for the determination of g^*, but is required to convert this value into the critical threshold y^*.

Equation (11) becomes

$$\frac{dG}{dg} = \frac{2r}{\Sigma}\frac{g}{G} - \frac{2}{\Sigma}\xi + \Gamma r G \tag{12}$$

If $\xi = 0$ this has general solution

$$G(g) = \left(Ae^{2\Gamma r g} - \frac{1}{\Sigma\Gamma^2 r} - \frac{2g}{\Sigma\Gamma}\right)^{1/2}$$

(this can be checked by differentiation) and using the initial value condition $G(0) = 0$ we obtain

$$G(g) = \left(\frac{e^{2\Gamma r g} - 1 - 2g\Gamma r}{\Sigma\Gamma^2 r}\right)^{1/2}. \tag{13}$$

Note that G is an increasing function on $[0, \infty)$, which grows without bound. Substituting for the value $G(g^*) = 1/r$, squaring and cross multiplying gives that g^* solves

$$\frac{\Sigma\Gamma^2}{r} = e^{2\Gamma r g^*} - 1 - 2g^*\Gamma r$$

Note that this equation has two solutions, but that we want the positive solution corresponding to the positive square root. Finally, returning to the original variables, and using the fact that $g(y^*) = (y^* - D)/r$, we get that y^* solves

$$\frac{\Sigma\Gamma^2}{r} = e^{2\Gamma(y^*-D)} - 1 - 2\Gamma y^* + 2\Gamma D$$

and the first part of the theorem follows.

In the general case, writing H for the inverse function to G we have that H solves

$$\frac{dH}{dh} = \frac{\Sigma h}{\Sigma\Gamma r h^2 - 2\xi h + 2rH} \qquad\qquad H(0) = 0$$

and the boundary conditions $G(g^*) = 1/r$ and $g^* = g(y^*) = (y^* - D)/r$ become $H(1/r) = g^*$ and $y^* = rH(1/r) + D$.

Finally, these expressions can be simplified by setting $f = rh$ and $F(f) = rH(f/r)$. Then

$$y^* = F(1) + D$$

where F solves (9).

□

As a final transformation, let

$$E(f) = F(f) - \frac{\xi}{r}f + \frac{\Gamma\Sigma}{2r} + rK.$$

Then $y^* = E(1)$ where

$$\text{(14)} \qquad \frac{dE}{df} = \frac{f}{(2r/\Sigma)(E(f) - rK) - \Gamma(1 - f^2)} - \frac{\xi}{r} \qquad\qquad E(0) = \frac{\Gamma\Sigma}{2r} + rK.$$

It is clear from (14) that $E(f) \geq rK$. The advantage of this representation is that it is easy to determine the dependence of y^* on each of the parameters Γ, Σ and ξ.

3 Numerical Implementation

The problem of determining the optimal exercise threshold has been reduced to solving a first order linear differential equation. Note that we need to choose the positive increasing solution to (9), and that care needs to be taken at $f = F = 0$. By considering solutions of the form $F(f) = \theta f$ for small f we find there are two candidate values of θ. We want the increasing solution and so we take the positive value θ_+, and initialise the ode at $F(\epsilon) = \theta_+\epsilon$. In the numerical solutions we took $\epsilon = 0.0001$. Further, some care is needed in choosing an appropriate numerical ode solver. We found that the *Matlab* function ode23s which is designed to cope with stiff equations, was particularly efficient. With these provisos, constructing a numerical solution to (9) is straightforward.

4 Discussion

In this section we give a brief discussion of the conclusions that can be drawn from the analysis of the previous sections. There is a much more extensive discussion of the financial implications in Miao and Wang [21] and so we content ourselves with a small number of observations. The three key parameters in (12) are Γ, ξ and Σ.

In terms of these parameters, the critical threshold is increasing in Γ and Σ and decreasing in ξ. (This is easy to see from the representation (14). Note that both $E(0)$ and dE/df are increasing in each of Γ and Σ.) These relationships are portrayed in Figure 1 for representative parameter values. Miao and Wang make similar observations on the impacts of these parameters. However, they rely on expansions of the solution in powers of Σ rather than proving the result.

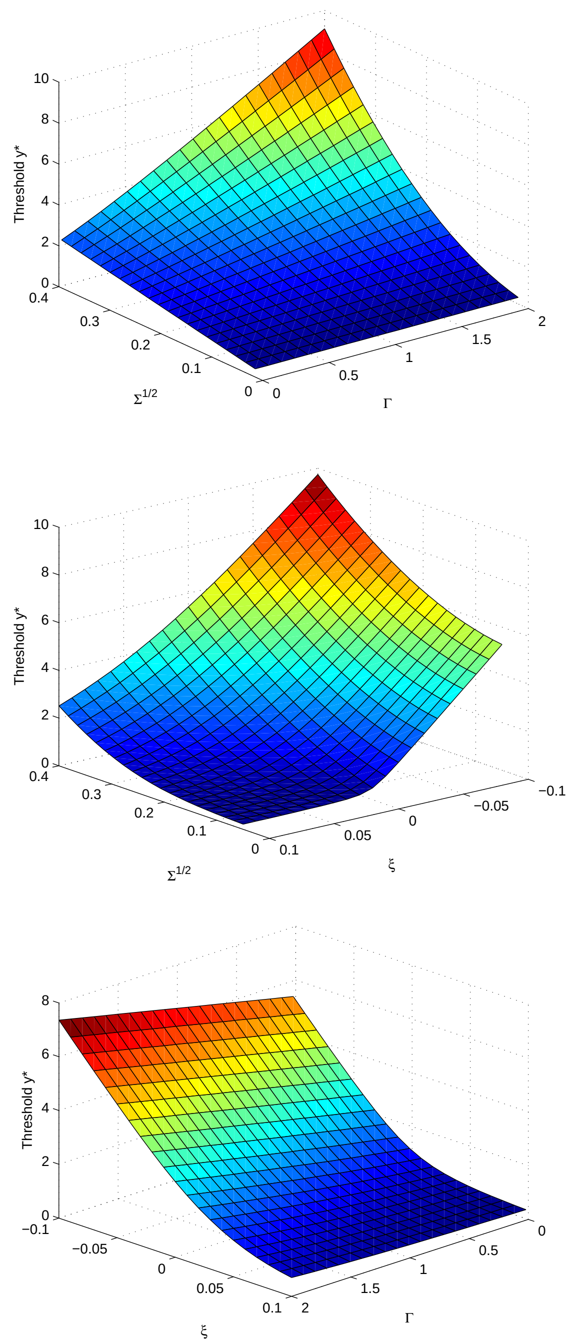

Figure 1: Investment thresholds, y^* as a function of other model parameters. Case (i) Theorem 1: The top panel gives y^* as a function of $\Sigma^{1/2}$ and Γ in the special case $\xi = 0$. Case (ii) Theorem 1: The middle and lower panels give y^* obtained from solving the ode (9) and using (10). The middle panel holds fixed $\Gamma = 1$ and varies $\Sigma^{1/2}$ and ξ. The lower panel fixes $\Sigma^{1/2} = 0.2$ and varies ξ and Γ. In all panels, other parameters are $r = 0.02, K = 10$. Care should be taken to observe the direction in which the various parameters are increasing or decreasing, since in order to make the graphs as clear as possible this varies from picture to picture.

Now we turn to the dependence of the critical threshold on the original (economic) parameters α, η, ρ, σ and γ. In most cases these parameters are present in the definitions of at most one of the key parameters, and in those cases the dependence of y^* on these parameters can be deduced easily. However the comparative statics of the optimal threshold on both σ and ρ are more complicated.

The correlation ρ enters the definitions of both Γ and ξ. As ρ increases, Γ decreases and, provided the Sharpe ratio η is positive, ξ decreases. Thus y^* increases in this case. However, if η is negative the effect of a change in correlation is mixed and the critical threshold can go up or down depending on the other parameter values.

Similarly, the volatility σ enters the definitions of ξ and Σ. Thus, holding the other economic parameters fixed, an increase in volatility causes a direct increase in Σ, and, if $\rho\eta > 0$, a decrease in ξ. In this case y^* increases. However, if correlation is negative or the Sharpe ratio of the market is negative (but not both), then the effect of an increase in volatility can be to either increase or decrease the critical threshold.

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