Coupling and Option Price Comparisons in a Jump Diffusion Model

Vicky Henderson Nomura Centre for Quantitative Finance, OCIAM Mathematical Institute, 24-29 St Giles', Oxford. OX1 3LB. UK. henders1@maths.ox.ac.uk
David Hobson Department of Mathematical Sciences, University of Bath, Bath, BA2 7AY. UK. dgh@maths.bath.ac.uk

January 31, 2000, update: November 29, 2001

Abstract

In this paper we examine the dependence of option prices in a general jump-diffusion model on the choice of martingale pricing measure. Since the model is incomplete there are many equivalent martingale measures. Each of these measures correponds to a choice for the market price of diffusion risk and the market price of jump risk. Our main result is to show that for convex payoffs the option price is increasing in the jump-risk parameter. We apply this result to deduce general inequalities comparing the prices of contingent claims under various martingale measures which have been proposed in the literature as candidate pricing measures.

Our proofs are based on couplings of stochastic processes. If there is only one possible jump size then we are able to utilize a second coupling to extend our results to include stochastic jump intensities.

Keywords and Phrases: Coupling, Option Pricing, incomplete markets, jump diffusion models, monotonicity
JEL Classification Numbers: D52,G12, G13.
AMS (1991) Subject Classification: 60G17, 60J55, 60J75, 90A09.

1 Introduction

The aim of this paper is to prove option price monotonicity results in a jump diffusion model with both Brownian and Poisson sources of uncertainty. The market is incomplete, with many equivalent martingale measures. Each measure corresponds to a pair of choices for the market price of diffusion risk and the market price of jump risk where the pair is chosen so that the discounted price process is a martingale. Our result is that for convex payoffs the option price is monotone increasing in the market price of jump risk, and hence effectively monotone decreasing in the market price of diffusion risk.

The problem of determining how the price of an option varies with the value of the model parameters is an important question in finance, and has led to many of the 'greeks'. For example in a diffusion set-up we can ask whether option prices are monotone in volatility. This problem has been considered by El Karoui et al [11], Bergman et al [4], Hobson [20], Lyons [24], Martini [26] and Henderson [18] who show that for convex European payoffs (and some convex path-dependent options) the option price is indeed increasing with volatility. Our paper has a similar theme to these papers; we investigate whether option prices are monotonic in the value of a parameter, but in a more complex jump-diffusion model.

Jump diffusion processes were first used by Merton [27] to introduce discontinuities into the sample paths of the stock dynamics. His model, often called the normal jump diffusion model, had jumps at the event times of a Poisson process and jump size normally distributed. Options were priced in this incomplete set-up by assuming the investors' attitude to jump risk is risk neutral, or that the jump risk is unpriced.

Bardhan and Chao [1] were amongst the first authors to consider market completeness in a jump-diffusion model. They consider a model driven by a number of Brownian motions and Poisson processes with stochastic intensity, each assigned a unique predictable jump size. They assume that the number of assets is equal to the number of sources of risk and this has the effect of completing the market. However, in a subsequent paper [2] they show that in a model with random jump sizes it is not possible to complete the market by increasing the number of traded securities.

We are interested in models where the number of risky assets is smaller than the number of sources of risk. In this case the model is incomplete. Examples of such models include Kou [22], Duffie et al [8] and Prigent et al [29]. Kou [22] proposed a model where jumps occur at the times of a Poisson process and the logarithm of jump size has a double exponential distribution. The purpose of this model is to capture the leptokurtic feature of returns and the volatility smile. Duffie et al [8] describe a general class of affine jump diffusion models of which Kou's is a special case. Another way to model jumps in prices is to consider infinite activity pure jump processes such as the variance gamma model in Madan and Seneta [25], the hyperbolic model in Eberlein and Jacod [10] and the model in Carr et al [5]. Chan [6] considers a geometric Levy process for the stock price which includes many of the previous models as special cases.

Jump models have found recent applications in modelling defaults in credit models, see Duffie and Singleton [9], and are used in a paper of Elliot and Jeanblanc [13] which examines insider information. Jump models are also very important in insurance mathematics, see the overview paper of Embrechts et al [14].

This paper is most closely aligned with those of Bellamy and Jeanblanc [3] and Pham [28]. Bellamy and Jeanblanc [3] find bounds for European call prices in a jump-diffusion model. They show that if we price options as discounted expected payoffs under members of the family of equivalent martingale measures then there are upper and lower bounds on the set of feasible call option prices. The lower bound is the Black-Scholes option price obtained by assuming the jump intensity is zero and the upper bound is the trivial upper bound of the asset price itself. Moreover they show that under certain model assumptions these bounds are attained. Pham [28] extends these results to show that option prices are monotonic in the jump intensity, and that they apply to American options.

Our results generalise and extend those of these two papers in the following ways. The first contribution is to prove the price monotonicity result for convex European options in a deterministic parameter setting using a simple coupling argument. This extends the bounds result of Bellamy and Jeanblanc [3]. Our generalisation to American options gives a result equivalent to Pham [28], but without one of his assumptions (namely the r-excessive condition (C_p), page 156).

The second contribution of this paper is to obtain results for a stochastic intensity model, where the intensity is dependent on the stock price. (Both Bellamy and Jeanblanc and Pham concentrate on the case of deterministic parameters.) In order to prove that options prices are higher under one martingale measure than another we need a comparison condition on the jump intensities under the two measures. This comparison condition says that when the prices are at the same level in both models, then the intensities are ordered. If this holds, then the option prices (European and American) are also ordered. The limitation of the result is that the proof requires us to restrict the model to have a single jump size. This seems to be necessary since we give an example of a situation where there are two sorts of jumps, and the stochastic jump intensities are ordered, but the option prices are not ordered.

The key tool that we use in our comparison theorems is the coupling of stochastic processes. For a primer on coupling methods, see Lindvall [23]. Coupling has been used in finance by Henderson and Hobson [19], and Henderson [18] in a passport options context. It has also been used by Hobson [20] to investigate option prices in a level dependent stochastic volatility model when one volatility process dominates another.

The third contribution is to find relationships between the prices of options under different pricing measures. In a complete market the price of an option depends on the choice of equivalent martingale measure — we prove comparison theorems between prices derived from various popular measures from the literature.

The remainder of the paper is organised as follows. Section 2 states the model and relates it to other papers. The next section establishes the set of equivalent martingale measures in the jump diffusion model. Some popular martingale measures are identified in Section 4. These include the pricing measure used by Merton [27], the Föllmer-Schweizer minimal martingale measure [15], the martingale measure which minimises relative entropy, and the Esscher transform martingale measure. The material in these sections is fairly standard and can be found in the union of [2], [3] and [6].

In Section 5 we discuss coupling as a technique and Section 6 uses coupling ideas to prove the monotonicity result when parameters are deterministic. The next section uses a different coupling and treats the case of stochastic parameters. The results of these two sections are extended to American-style claims in Section 8. In the penultimate section we prove some comparison results using the monotonicity theorems. In particular we find some relationships between option prices under the various martingale measures mentioned in the previous paragraph. Specifically, when there is only one possible jump size, then for a European option with convex payoff the option price under the Föllmer-Schweizer minimal martingale measure is smaller than the price under the measure which minimises relative entropy, which in turn is smaller than the price under the Esscher transform martingale measure.

2 Model Description

Consider a financial market with a riskless asset and a single risky asset which are traded up to a horizon T.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a filtration $\mathcal{F}_t$ supporting a Brownian motion and a Poisson point process, and satisfying the usual conditions and assume that all parameters are previsible with respect to this filtration.

The dynamics of the riskless asset are given by

$$dR_t = R_t r(t) dt,$$

for a bounded, deterministic interest rate r so that

$$R_t = R_0 \exp\left(\int_0^t r(u)du\right).$$

Without loss of generality we assume $R_0 = 1$.

Consider the following model for the risky asset. Under the 'real-world' measure $\mathbb{P}$ the asset price process S_t is a jump-diffusion process given by

$$\frac{dS_t}{S_{t-}} = b_t dt + \sigma_t dW_t + \int_{[0,1]} \phi_t(y)\tilde{\nu}(dt, dy), \tag{1}$$

where W_t is a Brownian motion and

$$\tilde{\nu}(dt, dy) = \nu(dt, dy) - q(dt, dy)$$

is a compensated Poisson random measure on $([0, T] \times [0, 1])$ with intensity measure $q(dt, dy) = \lambda_t dtdy$. We write

$$M_t^\lambda = \int_0^t \int_{[0,1]} \nu(dt, dy) - \int_0^t \int_{[0,1]} \lambda_t dtdy,$$

where the superscript λ denotes that ν has intensity $\lambda_t dtdy$.

The interpretation that should be given to the various parameters is as follows: b is the drift of the asset under the real-world measure, and σ the volatility of the Brownian part. The rate of jumps in the price process is given by λ; associated with each jump is a label y, and a jump at time t with label y corresponds to a proportional jump in the asset price process of size $\phi_t(y)$. We assume the labels are uniformly distributed on $[0, 1]$. Both the label space and probability distribution could be made arbitrary upon a transformation of variables.

In order to ensure limited liability ($S_t \geq 0$) we assume that $\phi \geq -1$. Note that the parameters governing S can all be stochastic; for example we should write $\phi_t = \phi_t(y; \omega)$, or in the Markov case we shall be interested in later $\phi_t = \phi(y; S_{t-}, t)$ when the size of the jump at time t depends on the current asset price as well as the label y. We assume that for fixed t, ϕ is a Borel-measurable function. Since it does not make sense to consider a jump of zero size we assume $\phi_t(y) \neq 0$ for each t and y.

In order to guarantee existence and uniqueness of a solution to (1) we need to impose some regularity conditions on the parameters. In particular in the non-stochastic case it is sufficient to assume b is integrable, σ and λ are bounded and that for each t, $\phi_t(y)$ is an increasing square-integrable function bounded below by $\delta_\phi > -1$. For further details see [3] or [6]. Essentially to guarantee existence and uniqueness of a solution to (1) it is sufficient to show that the right-hand-side is the stochastic differential of a semimartingale X_t, see Elliott [12, Theorem 13.5].

The solution to (1) is given by:

$$S_t = S_0 \mathcal{E}(b \cdot \imath + \sigma \cdot W + \phi \cdot M^\lambda)_t = S_0 \mathcal{E}(b \cdot \imath)_t \mathcal{E}(\sigma \cdot W)_t \mathcal{E}(\phi \cdot M^\lambda)_t, \tag{2}$$

where the Doléans-Dade exponentials $\mathcal{E}$ are given by:

$$\mathcal{E}(b \cdot \imath)_t = \exp\left(\int_0^t b_s ds\right),$$

$$\mathcal{E}(\eta \cdot W)_t = \exp\left(\int_0^t \eta_s dW_s - \frac{1}{2}\int_0^t \eta_s^2 ds\right),$$
$$\mathcal{E}(\theta \cdot M^\lambda)_t = \exp\left(\int_0^t \int_{[0,1]} \ln(1+\theta_u(y))\nu(du,dy) - \int_0^t \int_{[0,1]} \lambda_u \theta_u(y) du dy\right);$$

and we have used Yor's addition formula and the fact that $[W, M^\lambda] = 0$, see Elliott [12, Chapter 13].

Our model and notation are a generalisation of the model given in Bellamy and Jeanblanc [3]. If we change the label space from $[0,1]$ to $[-1,\infty)$ then we can identify the labels directly as the sizes of the (proportional) jumps in the risky asset price, and with this transformation we can recover the models of Baradhan and Chou [2], Pham [28] and Kou [22].

It is implicit in our notation that the jump rate λ is finite. It is possible to generalise this and to extend the analysis to include models where the jumps (or equivalently the right-hand-side of (1)) form a Lévy process and this is the approach taken by Chan [6]. In that case, if the Lévy measure of the jumps is not finite then the definitions of this section have to be extended. Details can be found in Elliott [12, Chapter 13], see especially Theorem 13.5.

3 Risk-neutral pricing measures

In general the jump-diffusion model described in the previous section is incomplete. Exceptions include the degenerate cases where $\phi = 0$, or where $\sigma = 0$ and there is only one possible jump size at any given instant, but see also the recent paper by Jeanblanc and Privault [21] in which a more intricate jump-diffusion model is constructed which possesses the completeness property. Since our model is incomplete there is no unique equivalent martingale measure which can be used for determining unambiguous options prices. Our goal in this section is to characterise the set of risk-neutral measures each of which corresponds to a possible pricing functional.

Henceforth we assume that σ is bounded below by a constant $\delta_\sigma > 0$ and that ϕ is non-zero. A measure $\mathbb{Q}$ is an equivalent martingale (or risk-neutral) measure if it is equivalent to the real-world probability $\mathbb{P}$ and if the discounted price process $R_t S_t$ is a martingale under $\mathbb{Q}$.

Theorem 3.1 ([2], [28], [3], [6]) *$\mathbb{Q}$ is an equivalent martingale measure if it has Radon-Nikodym density with respect to $\mathbb{P}$ of the form*

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = L_t$$

where

(3)
$$\frac{dL_t}{L_{t-}} = \psi_t dW_t + \int_{[0,1]} \gamma_u(y)\tilde{\nu}(dt,dy),$$

and ψ and γ are two previsible processes such that $1+\gamma_t > 0$, $\mathbb{E}(L_T) = 1$ and

(4)
$$b_t - r(t) + \sigma_t \psi_t + \lambda_t \int_{[0,1]} \gamma_t(y)\phi_t(y) dy = 0,$$

almost surely.

Remark 3.2 It follows that $L_t = \mathcal{E}(\psi \cdot W)_t \mathcal{E}(\gamma \cdot M^\lambda)_t$. In general L_t is a supermartingale and the assumption $\mathbb{E}(L_T) = 1$ is necessary to ensure that $\mathbb{Q}$ is a probability measure.

Remark 3.3 Let $p = 1 + \gamma$, so that $p \equiv p_t(y) > 0$. It is convenient to parameterise the space of equivalent martingale measures as $\{\mathbb{Q}^p, p \geq 0\}$. Once p is fixed, ψ is uniquely determined via (4). (The advantage of parameterising $\mathbb{Q}$ via p is that if conversely ψ is fixed, then neither existence nor uniqueness of a γ solving (4) is guaranteed. Indeed uniqueness will only follow in a limited set of special cases.)

Remark 3.4 Under the real world measure the intensity of jumps with label y at time t is λ_t; under $\mathbb{Q}^p$ the intensity of jumps with label y is $\lambda_t p_t(y)$. Thus p acts multiplicatively on the jump rate. The parameter ψ quantifies the modification to the drift of W and represents a market price of diffusion risk, and $\gamma_t(y) \equiv p_t(y) - 1$ is the risk premium associated with the jump risk.

Under $\mathbb{Q}^p$ the stochastic differential equation for S is:

$$\frac{dS_t}{S_{t-}} = r(t)dt + \sigma_t dW_t^p + \int_{[0,1]} \phi_t(y)(\nu(dt, dy) - \lambda_t p_t(y) dt\, dy) \tag{5}$$

where $W_t^p \equiv W_t - \int_0^t \psi_u du$ is a $\mathbb{Q}^p$ Brownian motion and ν now has intensity $\lambda_t p_t(y) dt\, dy$.

The solution to (5) is given by:

$$S_t = S_0 R_t \mathcal{E}(\sigma \cdot W^p)_t \mathcal{E}(\phi \cdot M^{\lambda p})_t. \tag{6}$$

4 Examples

In this section we review some of the common choices of changes of measure from the literature, and in particular determine various choices for the pair (ψ, γ) which make the discounted price process into a martingale.

Consider the standard Black-Scholes model in which there is no jump component, and the asset price is driven by a single Brownian motion B, so that $dS_t/S_t = \sigma_t dB_t + b_t dt$. In that case there is a unique equivalent martingale measure $\mathbb{Q}$ and the density of $\mathbb{Q}$ with respect to the real-world measure $\mathbb{P}$ is L_T where, with $\xi_t = -(b_t - r(t))/\sigma_t^2$,

$$\frac{dL_t}{L_t} = \xi_t \left(\frac{dS_t}{S_t} - b_t dt \right) = \xi_t \sigma_t dB_t, \qquad L_0 = 1, \tag{7}$$

or equivalently

$$L_T = \exp\left(\int_0^T \xi_t \sigma_t dB_t - \frac{1}{2} \int_0^T \xi_t^2 \sigma_t^2 dt \right). \tag{8}$$

Alternatively, we can write $S_t = S_0 \exp(Y_t)$ where

$$Y_t = \int_0^t \sigma_u dB_u + \int_0^t \left(b_u - \frac{1}{2}\sigma_u^2 \right) du.$$

With this set-up we have

$$L_T = \exp\left(\int_0^T \xi_t dY_t - \int_0^T \left(\xi_t b_t + \frac{1}{2}\sigma_t^2 \xi_t (1 - \xi_t) \right) dt \right). \tag{9}$$

4.1 Merton

In Merton's original jump-diffusion model [27] he proposes leaving the intensity rates of the discontinuous component of the price unchanged and changing the drift of the Brownian component. This is equivalent to taking $\gamma^M \equiv 0$ and $\psi_t^M = -(b_t - r(t))/\sigma_t$. The superscript M refers to Merton.

4.2 The Minimal Martingale Measure

The minimal martingale measure as introduced by Föllmer and Schweizer [15] is the martingale measure with the property that the (discounted) asset price becomes a martingale, and that martingales which are orthogonal to the asset price process continue to be martingales.

Consider a family of changes of measure L^η which we think of as a generalisation of (7) where $\sigma_t dB_t$ is replaced by the martingale part driving S. In particular

(10)
$$\frac{dL_t^\eta}{L_{t-}^\eta} = \eta_t \left(\frac{dS_t}{S_{t-}} - b_t dt \right) = \eta_t \left(\sigma_t dW_t + \int_0^1 \phi_t(y) \tilde{\nu}(dt, dy) \right).$$

Here, in the notation of (3), $\psi_t = \eta_t \sigma_t$ and $\gamma_t = \eta_t \phi_t(y)$, so that for $\mathbb{Q}^\eta$ to be a martingale measure we must have

(11)
$$b_t - r(t) + \sigma_t^2 \eta_t + \lambda_t \eta_t \int_{[0,1]} \phi_t(y)^2 dy = 0.$$

This forces $\eta_t = \eta_t^*$, where by definition

(12)
$$\eta_t^* = \frac{-(b_t - r(t))}{\sigma_t^2 + \lambda_t \int_{[0,1]} \phi_t(y)^2 dy}.$$

Hence $\psi_t^F = \eta_t^* \sigma_t$ and $\gamma^F = \eta_t^* \phi_t(y)$. It is clear from (10) that the change of measure affects the drift of the price process, and intuitively clear that any martingale orthogonal to S remains a martingale. See Chan [6] for details.

Recall that we need $\gamma_t^F \equiv \eta_t^* \phi_t(y) > -1$, so that in the typical circumstance of $b_t > r(t)$ we need an upper bound on the possible jump sizes for the minimal martingale measure to be well defined. This is an example of the fact that in general the minimal martingale measure may be a signed measure.

4.3 The Minimal Entropy Martingale Measure

Given the 'real-world' measure $\mathbb{P}$ the relative entropy $I_{\mathbb{P}}(\mathbb{Q})$ of the probability measure $\mathbb{Q}$ with respect to $\mathbb{P}$ is given by

$$I_{\mathbb{P}}(\mathbb{Q}) = \mathbb{E} \left(\frac{d\mathbb{Q}}{d\mathbb{P}} \ln \frac{d\mathbb{Q}}{d\mathbb{P}} \right) = \mathbb{E}^{\mathbb{Q}} \left(\ln \frac{d\mathbb{Q}}{d\mathbb{P}} \right).$$

In the diffusion setting the minimal entropy martingale measure and the Föllmer-Schweizer minimal martingale measure are identical, though in our setting with jumps they differ. A further motivation for the use of the minimal entropy measure is the fact that there are strong connections between the use of this measure and pricing under exponential utility. See Delbaen et al [7], Rouge and El Karoui [30] and Frittelli [16].

Suppose that all the parameters σ, b, ϕ and λ are deterministic and define

$$\frac{dL_t}{L_{t-}} = \alpha_t\sigma_t dW_t + \int_{[0,1]} \beta_t(y)\tilde{\nu}(dt, dy), \qquad L_0 = 1,$$

and suppose L_t is the Radon-Nikodym derivative of a measure $\mathbb{Q}^{\alpha,\beta}$ with respect to $\mathbb{P}$ on $\mathcal{F}_t$. Then

$$L_T = \exp\left(\int_0^T \alpha_t\sigma_t dW_t - \frac{1}{2}\alpha_t^2\sigma_t^2 dt\right)\exp\left(\int_0^T\int_{[0,1]} \ln(1+\beta_t(y))\nu(dt,dy) - \int_0^T\int_{[0,1]} \lambda_t\beta_t(y)dt\,dy\right).$$

Under $\mathbb{Q}^{\alpha,\beta}$, the process W has drift $\alpha_t\sigma_t$ and the Poisson process ν has intensity $\lambda_t(1+\beta_t(y))dt\,dy$. Then

$$\mathbb{E}^{\mathbb{Q}^{\alpha,\beta}} \ln L_T = \mathbb{E}^{\mathbb{Q}^{\alpha,\beta}} \int_0^T \mathcal{G}_t(\alpha_t, \beta_t)dt \tag{13}$$

where

$$\mathcal{G}_t(\alpha_t,\beta_t) = \frac{1}{2}\alpha_t^2\sigma_t^2 + \lambda_t\int_{[0,1]}[(1+\beta_t(y))\ln(1+\beta_t(y)) - \beta_t(y)]dy. \tag{14}$$

We seek to minimise this expression over choices of α and β which make the discounted price process into a martingale.

For fixed t and ω consider minimising $\mathcal{G}_t(\alpha_t,\beta_t)$ subject to the martingale condition $\mathcal{M}_t(\alpha_t,\beta_t) = 0$, where

$$\mathcal{M}_t(\alpha_t,\beta_t) = b_t - r(t) + \alpha_t\sigma_t^2 + \lambda_t\int_{[0,1]}\beta_t(y)\phi_t(y)dy. \tag{15}$$

If we define the Lagrangian $\mathcal{G}_t(\alpha_t,\beta_t) - \xi_t\mathcal{M}_t(\alpha_t,\beta_t)$ then it is easy to show that the minimising parameter choices are $\alpha_t = \xi_t$ and $\beta_t(y) = e^{\xi_t\phi_t(y)} - 1$. We find that $\psi_t^{RE} = \alpha_t^*\sigma_t$ and $\gamma_t^{RE}(y) = (e^{\alpha_t^*\phi_t(y)} - 1)$ where α_t^* solves

$$b_t - r(t) + \sigma_t^2\alpha_t^* + \lambda_t\int_{[0,1]}(e^{\alpha_t^*\phi_t(y)} - 1)\phi_t(y)dy = 0. \tag{16}$$

Note in particular that under our assumption that the parameters are deterministic, the solution α^* is again deterministic. Write $\beta_t^* = e^{\alpha_t^*\phi_t(y)} - 1$ and $\mathbb{Q}^* = \mathbb{Q}^{\alpha_t^*,\beta_t^*}$ and note that β_t^* and $\mathcal{G}_t(\alpha_t^*,\beta_t^*)$ are also deterministic.

Hence we find that for parameter pairs (α,β) satisfying the martingale condition we have $\mathcal{G}_t(\alpha_t,\beta_t) \geq \mathcal{G}_t(\alpha_t^*,\beta_t^*) \equiv \mathbb{E}^{\mathbb{Q}^*}(\mathcal{G}_t(\alpha_t^*,\beta_t^*))$. Finally, therefore, for any (α,β) satisfying (16)

$$\mathbb{E}^{\mathbb{Q}^{\alpha,\beta}}\mathcal{G}_t(\alpha_t,\beta_t) \geq \mathcal{G}_t(\alpha_t^*,\beta_t^*) \equiv \mathbb{E}^{\mathbb{Q}^{\alpha^*,\beta^*}}(\mathcal{G}_t(\alpha_t^*,\beta_t^*))$$

and (13) is minimised by the choices (α^*,β^*).

Remark 4.1 Instead of choosing the measure $\mathbb{Q}$ to minimise entropy we could instead attempt to minimize $J_q(\mathbb{Q})$ where, for $q > 1$

$$J_q(\mathbb{Q}) = \mathbb{E}\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)^q.$$

We can then modify the above calculations to deduce that the optimal change of measure parameters are $\psi_t^q = \alpha_t^{q,*}\sigma_t$ and $\gamma_t^q(y) = (1 + (q-1)\alpha_t^{q,*}\phi_t(y))^{1/(q-1)} - 1$ where $\alpha^{q,*}$ solves

$$b_t - r(t) + \sigma_t^2 \alpha_t^{q,*} + \lambda_t \int_{[0,1]} ((1 + (q-1)\alpha_t^{q,*}\phi_t(y))^{1/(q-1)} - 1)\phi_t(y)dy = 0. \tag{17}$$

If we take $p = 2$ we recover the Föllmer-Schweizer minimal martingale measure. Further, the limits $p \uparrow \infty$ and $p \downarrow 1$ correspond to the Merton measure and the relative entropy measure respectively.

Remark 4.2 When the parameters of the price process are stochastic the story is more complicated. In that case, although the definitions of the α and β which minimise $\mathcal{G}_t$ subject to (16) are unchanged, they themselves are now non-deterministic. Hence, although $\mathcal{G}_t(\alpha_t, \beta_t) \geq \mathcal{G}_t(\alpha_t^*, \beta_t^*)$ it does not follow that $\mathbb{E}^{\mathbb{Q}^{\alpha,\beta}} \mathcal{G}_t(\alpha_t, \beta_t)$ is greater than $\mathbb{E}^{\mathbb{Q}^*} \mathcal{G}_t(\alpha_t^*, \beta_t^*)$ since the expectations are taken with respect to different measures. In essence we can define a martingale measure using (α^*, β^*) but the definitions of these quantities are local in t, and there is no guarantee that they lead to the measure which minimisizes entropy, which is a condition that is global in t.

4.4 Esscher Transforms

Suppose that $S_t = S_0 \exp(Y_t)$ where Y_t is a time-homogeneous Lévy process. Since Y_t has stationary independent increments we have

$$\mathbb{E}(\exp(\theta Y_t)) = \exp(t\Theta(\theta)),$$

where Θ is the Lévy exponent. The Esscher transform (Gerber and Shiu [17]) of the Lévy process Y is the process whose law $\mathbb{Q}^\theta$ is given by

$$L_t = \left.\frac{d\mathbb{Q}^\theta}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \exp\left\{\theta Y_t - t\Theta(\theta)\right\}.$$

Now consider our more general inhomogeneous case. Again we write $S_t = S_0 \exp(Y_t)$ where

$$Y_t = \int_0^t \sigma_u dW_u + \int_0^t \int_{[0,1]} \ln(1 + \phi_u(y))\nu(du, dy) + \int_0^t \left(b_u - \frac{1}{2}\sigma_u^2 - \lambda_u \int_{[0,1]} \phi_u(y)dy \right) du.$$

We consider generalised Esscher transforms of the form

$$L_t = \left.\frac{d\mathbb{Q}^\theta}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \exp\left\{ \int_0^t \theta_u dY_u - \int_0^t \Theta_u(\theta_u)du \right\} \tag{18}$$

where

$$\Theta_u(\theta) = \frac{1}{2}\theta^2\sigma_u^2 + \theta \left(b_u - \frac{1}{2}\sigma_u^2 - \lambda_u \int_{[0,1]} \phi_u(y)dy \right) + \lambda_u \int_{[0,1]} [(1 + \phi_u(y))^\theta - 1]dy.$$

We can think of this change of measure as a generalisation of (9). It follows that

$$\frac{dL_t}{L_{t-}} = \theta_t \sigma_t dW_t + \int_{[0,1]} [(1 + \phi_t(y))^{\theta_t} - 1]\tilde{\nu}(dt, dy). \tag{19}$$

and L is a martingale under $\mathbb{P}$ as required.

The martingale measure Esscher transform is the measure $\mathbb{Q}^{\theta^*}$, where θ^* is chosen to make the discounted price process into a martingale. Comparing (19) with (3) we find $\psi_t^E \equiv \theta_t^* \sigma_t$ and $\gamma_t^E(y) = (1+\phi_t(y))^{\theta_t^*} - 1$ where θ^* is the unique solution to

$$b_t - r(t) + \sigma_t^2 \theta_t^* + \lambda_t \int_{[0,1]} [(1+\phi_t(y))^{\theta_t^*} - 1]\phi_t(y)dy = 0. \tag{20}$$

Remark 4.3 In defining the Esscher transform we wrote $S_t = S_0 \exp(Y_t)$ and considered exponential transforms of Y. A related approach is to consider $dS_t/S_{t-} = d\tilde{Y}_t$ for a Lévy process $\tilde{Y}$ and to consider changes of measure of the form

$$\tilde{L}_t = \left.\frac{d\tilde{\mathbb{Q}}^\theta}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \exp\left\{ \int_0^t \theta_u d\tilde{Y}_u - \int_0^t \tilde{\Theta}_u(\theta_u)du \right\}. \tag{21}$$

If S has no jumps then these formulations are identical, but otherwise the jumps manifest themselves in Y and $\tilde{Y}$ in different ways. Chan [6] has shown than if we consider the martingale Esscher transform derived from (21) then we recover exactly the minimal relative entropy measure derived in Section 4.3, at least in the case of deterministic parameters.

5 Coupling

A standard text on coupling is Lindvall [23].

By a coupling of the models $(\Omega', \mathcal{F}', \mathbb{P}', X')$ and $(\Omega'', \mathcal{F}'', \mathbb{P}'', X'')$ we mean a model $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}}, (\hat{X}', \hat{X}''))$ such that

$$X' \stackrel{\mathcal{D}}{=} \hat{X}', \qquad\qquad X'' \stackrel{\mathcal{D}}{=} \hat{X}''.$$

In words, a coupling is a model which supports two processes each of which is an identical stochastic copy of one of the original processes.

We will be interested in two couplings of Poisson random measures. As a motivational example we consider these couplings in terms of Poisson processes.

Let N' and N'' be Poisson processes with constant intensities $\lambda' \le \lambda''$ respectively and let $\delta = \lambda'' - \lambda' \ge 0$. These processes are possibly defined on different probability spaces. Now for the coupling. On a suitable probability triple $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$ define a pair of independent Poisson processes $(\hat{N}^{\lambda'}, \hat{N}^{\delta})$ with rates λ' and δ respectively. Define $\hat{N}^{\lambda''}$ by $\hat{N}_t^{\lambda''} = \hat{N}_t^{\lambda'} + \hat{N}_t^{\delta}$. Then

$$N' \stackrel{\mathcal{D}}{=} \hat{N}^{\lambda'}, \qquad\qquad N'' \stackrel{\mathcal{D}}{=} \hat{N}^{\lambda''}.$$

Further it is clear from the coupling that $\hat{N}^{\lambda''} \ge \hat{N}^{\lambda'}$, so that for example for any function G_t

$$\begin{aligned} \mathbb{P}'(N_t' \ge G_t, \forall 0 \le t \le T) &= \hat{\mathbb{P}}(\hat{N}_t^{\lambda'} \ge G_t, \forall 0 \le t \le T) \\ &\le \hat{\mathbb{P}}(\hat{N}_t^{\lambda''} \ge G_t, \forall 0 \le t \le T) \\ &= \mathbb{P}''(N_t'' \ge G_t, \forall 0 \le t \le T) \end{aligned} \tag{22}$$

This coupling can easily be extended to deterministic rate functions with the property $\lambda_t' \le \lambda_t''$ uniformly in t.

There is a second coupling of Poisson processes which is also extremely useful, particularly when the rate processes are stochastic. Again suppose that N' and N'' are Poisson processes with constant intensities $\lambda' \leq \lambda''$. Suppose $\hat{N}$ is a Poisson process of unit rate. Set $A'_t = \lambda' t$ and $A''_t = \lambda'' t$ and define

$$\hat{N}_t^{\lambda'} \equiv \hat{N}_{A'_t}, \qquad \hat{N}_t^{\lambda''} \equiv \hat{N}_{A''_t}$$

If

$$N' \stackrel{\mathcal{D}}{=} \hat{N}^{\lambda'}, \qquad N'' \stackrel{\mathcal{D}}{=} \hat{N}^{\lambda''},$$

then we have a coupling of N' and N'' which again can be used to deduce inequalities of the form (22). This coupling can easily be extended to deterministic rate functions with the comparison property

$$\int_0^t \lambda'_u du \leq \int_0^t \lambda''_u du \qquad \forall\, t.$$

6 Monotonicity of Option Prices: Deterministic Case

Suppose that all the parameters b, σ, λ and $\phi = \phi(t, y)$ are deterministic functions of time. In this section we prove a comparison theorem which includes the results of Bellamy and Jeanblanc [3] and Pham [28]. In the next section we use different methods to extend our results to cover some non-deterministic situations.

If the exogenous parameters b, σ, λ and ϕ are deterministic functions of time then there are solutions to (4) for which ψ and p are stochastic. However we will restrict attention to the case where $p_t(y) = p(t, y)$ is (chosen to be) deterministic also. It then follows that ψ is deterministic.

Suppose $p(t, y)$ and $\psi(t)$ have been chosen to satisfy (4) and let $\mathbb{Q}^p$ be the corresponding equivalent risk-neutral measure. Then $W_t^p = W_t - \int_0^t \psi(u)du$ is a $\mathbb{Q}^p$-Brownian motion, and

$$M_t^{\lambda p} = \int_0^t \int_{[0,1]} \nu(du, dy) - \int_0^t \int_{[0,1]} p(u, y)\lambda(u) du\, dy$$

is a compensated Poisson random measure under $\mathbb{Q}^p$. Since $p(t, y) - 1 = \gamma(t, y)$ and $\psi(u)$ solve (4) we can assume that $\int_{[0,1]} p(t, y)\phi(t, y)dy < \infty$. If we assume further that this integral is uniformly bounded then we have that $\mathcal{E}(\phi \cdot M^{\lambda p})$ is well defined.

Theorem 6.1 Price monotonicity result (deterministic case)
For (European) options with convex payoff, h, the option price is increasing in the deterministic parameter p.

Remark 6.2 In the current set-up we imagine a single model and various candidate equivalent martingale measures, and this theorem compares prices under these different measures. However, another fruitful way to think about this result is as a comparison of expected payoffs under different models for the underlying.

Proof of Theorem 6.1: Suppose $p''(u, y) \geq p'(u, y)$ uniformly in u and y and set $\lambda''(u, y) := \lambda(u, y)p''(u, y) \geq \lambda(u, y)p'(u, y) =: \lambda'(u, y)$. Let $\delta(u, y) = \lambda''(u, y) - \lambda'(u, y)$.

Write $\mathbb{E}'$ as shorthand for $\mathbb{E}^{\mathbb{Q}^{p'}}$ and similarly $\mathbb{E}''$. Our goal is to show that $\mathbb{E}'' h(S''_T) \geq \mathbb{E}' h(S'_T)$ where S' (respectively S'') denotes the solution to (6) under the parameter value p' (repectively p'').

We have

$$\mathbb{E}'h(S'_T) = \mathbb{E}'h(S_0 R_T \mathcal{E}(\sigma \cdot W')_T \mathcal{E}(\phi \cdot M')_T) \tag{23}$$

where $W' \equiv W^{p'}$ and $M' \equiv M^{\lambda'}$ are martingales under $\mathbb{E}'$, and similarly

$$\mathbb{E}''h(S''_T) = \mathbb{E}''h(S_0 R_T \mathcal{E}(\sigma \cdot W'')_T \mathcal{E}(\phi \cdot M'')_T). \tag{24}$$

Now consider a new probability space with probability measure $\hat{\mathbb{P}}$ supporting a Brownian motion $\hat{W}$ and a pair of independent Poisson random measures $\hat{\nu}'$ and $\hat{\nu}^\delta$ on $([0,T] \times [0,1])$ with rates $\lambda'(u,y)dudy$ and $\delta(u,y)dudy$. Let $\hat{M}'$ and $\hat{M}^\delta$ denote the corresponding compensated processes.

Note that the $\mathbb{Q}^{p'}$ law of W', the $\mathbb{Q}^{p''}$ law of W'' and the $\hat{\mathbb{P}}$ law of $\hat{W}$ are identical (each are Brownian motion), so for any function f, $\mathbb{E}'f(W'_T) = \mathbb{E}''f(W''_T) = \hat{\mathbb{E}}f(\hat{W}_T)$ and moreover

$$\mathbb{E}'h(S_0 R_T \mathcal{E}(\sigma \cdot W')_T \mathcal{E}(\phi \cdot M')_T) = \hat{\mathbb{E}}h(S_0 R_T \mathcal{E}(\sigma \cdot \hat{W})_T \mathcal{E}(\phi \cdot \hat{M}')_T). \tag{25}$$

Now define $\hat{\nu}''$ to be the sum of the Poisson random measures $\hat{\nu}'$ and $\hat{\nu}^\delta$. By the independence of these last two quantities, $\hat{\nu}''$ is again a Poisson random measure on $([0,T] \times [0,1])$ with rate $\lambda''(u,y)dudy$, and let $\hat{M}''$ be the corresponding compensated process. Then the law of $\hat{M}''$ under $\hat{\mathbb{P}}$ is the same as the law of M'' under $\mathbb{P}''$. Let $\hat{S}'_t = S_0 R_t \mathcal{E}(\sigma \cdot \hat{W})_t \mathcal{E}(\phi \cdot \hat{M}')_t$ and similarly for $\hat{S}''_t$.

Now

$$\begin{aligned} \mathbb{E}''h(S''_T) &= \mathbb{E}''h(S_0 R_T \mathcal{E}(\sigma \cdot W'')_T \mathcal{E}(\phi \cdot M'')_T) \\ &= \hat{\mathbb{E}}h(S_0 R_T \mathcal{E}(\sigma \cdot \hat{W})_T \mathcal{E}(\phi \cdot \hat{M}'')_T) \\ &= \hat{\mathbb{E}}h(\hat{S}''_T) \end{aligned} \tag{26}$$

since the laws of $W^{p''}$ and $M^{p''}$ under $\mathbb{E}''$ are identical to those of $\hat{W}$ and $\hat{M}''$ under $\hat{\mathbb{E}}$. But

$$\begin{aligned} &\hat{\mathbb{E}}h(S_0 R_T \mathcal{E}(\sigma \cdot \hat{W})_T \mathcal{E}(\phi \cdot \hat{M}'')_T) = \\ &\qquad \hat{\mathbb{E}}h(S_0 R_T \mathcal{E}(\sigma \cdot \hat{W})_T \mathcal{E}(\phi \cdot \hat{M}')_T \mathcal{E}(\phi \cdot \hat{M}^\delta)_T) \end{aligned} \tag{27}$$

where we have used the independence of the constituent parts of $\hat{\nu}''$. In particular, since $\hat{\nu}'$ and $\hat{\nu}^\delta$ are independent, $[\phi \cdot \hat{M}', \phi \cdot \hat{M}^\delta] = 0$ and by Yor's addition formula $\mathcal{E}(\phi \cdot (\hat{M}' + \hat{M}^\delta))_t = \mathcal{E}(\phi \cdot \hat{M}')_t \mathcal{E}(\phi \cdot \hat{M}^\delta)_t$.

Conditioning on $\hat{W}$ and $\hat{M}'$, using a conditional Jensen's inequality and the fact that $\hat{\mathbb{E}}\mathcal{E}(\phi \cdot \hat{M}^\delta)_T = 1$, we have

$$\begin{aligned} &\hat{\mathbb{E}}h(S_0 R_T \mathcal{E}(\sigma \cdot \hat{W})_T \mathcal{E}(\phi \cdot \hat{M}')_T \mathcal{E}(\phi \cdot \hat{M}^\delta)_T) \\ &\quad \geq \hat{\mathbb{E}}h(S_0 R_T \mathcal{E}(\sigma \cdot \hat{W})_T \mathcal{E}(\phi \cdot \hat{M}')_T \hat{\mathbb{E}}(\mathcal{E}(\phi \cdot \hat{M}^\delta)_T | \hat{W}, \hat{M}')) \\ &\quad = \hat{\mathbb{E}}h(S_0 R_T \mathcal{E}(\sigma \cdot \hat{W})_T \mathcal{E}(\phi \cdot \hat{M}')_T). \end{aligned} \tag{28}$$

Finally, combining (26), (27), (28), and (25) we have the result:

$$\begin{aligned} \mathbb{E}''h(S''_T) &= \mathbb{E}''h(S_0 R_T \mathcal{E}(\phi \cdot M'')_T \mathcal{E}(\sigma \cdot W'')_T) \\ &\geq \mathbb{E}'h(S_0 R_T \mathcal{E}(\phi \cdot M')_T \mathcal{E}(\sigma \cdot W')_T) \\ &= \mathbb{E}'h(S'_T) \end{aligned} \tag{29}$$

□

Remark 6.3 This proof fails if the parameters ϕ, σ or λ are stochastic. For example, if $\phi = \phi(S_{t-})$ then

$$\begin{aligned} \mathbb{E}'' h(S''_T) &= \hat{\mathbb{E}} h(S_0 R_T \mathcal{E}(\phi(\hat{S}'') \cdot \hat{M}'')_T \mathcal{E}(\sigma \cdot \hat{W})_T) \\ &\geq \hat{\mathbb{E}} h(S_0 R_T \mathcal{E}(\phi(\hat{S}'') \cdot \hat{M}')_T \mathcal{E}(\sigma \cdot \hat{W})_T) \end{aligned}$$

However this last term cannot be identified with $\hat{\mathbb{E}} h(\hat{S}'_T)$ since

$$\hat{S}'_T = S_0 R_T \mathcal{E}(\phi(\hat{S}') \cdot \hat{M}')_T \mathcal{E}(\sigma \cdot \hat{W})_T)_T)$$

and in general the terms $\mathcal{E}(\phi(\hat{S}'') \cdot \hat{M}')_T$ and $\mathcal{E}(\phi(\hat{S}') \cdot \hat{M}')_T$ are not the same.

7 Monotonicity of Option Prices: non-deterministic case

The results of the previous section, and the price comparison results of Bellamy and Jeanblanc [3] and Pham [28], rely on the crucial assumption that both the parameters of the price process and of the equivalent martingale measure are deterministic. In this section we make partial progress in extending this result to allow for state dependent jump rates.

This generalisation comes at a cost however: our proof only works in the case where there is at most one possible (proportional) jump size, so that the size of any jump is independent of both time and the label, and in particular $\phi_t(y) \equiv \phi > -1$. In this case there is no point in considering $p_t(\cdot)$ as a function of the label, so we drop any dependency of p on y.

We consider the Markovian case. Assume λ and b are functions of time and the current value of the asset price, so that for example $\lambda = \lambda(S_{t-}, t)$, but suppose that $\sigma = \sigma(t)$ is deterministic. The space of equivalent martingale measures is parameterised by p where p and ψ solve (4). Again, this equation does not force p and ψ to be functions of time and asset price alone, but we will only consider p of the form $p = p(S_{t-}, t)$, and given our other assumptions this forces $\psi = \psi(S_{t-}, t)$ also. Hence we find that p and ψ solve

(30) $$b(S_{t-}, t) - r(t) + \sigma(t)\psi(S_{t-}, t) + (p(S_{t-}, t) - 1)\lambda(S_{t-}, t)\phi = 0,$$

For the purposes of this section we make some further technical assumptions. Recall that λ and σ are bounded and assume that p is bounded also. Assume further that λ and p satisfy a global Lipschitz condition. Hence we have $|\lambda| \leq K_\lambda$ and $|\lambda(x_1, t) - \lambda(x_2, t)| \leq C_\lambda |x_1 - x_2|$, with similar expressions for the other parameters.

Our main result is

Theorem 7.1 Price monotonicity result: non-deterministic case.
Assume that there is only one possible (proportional) jump size ϕ, and that the functions b and λ depend on t and S_{t-} alone, and σ is deterministic. If we consider equivalent martingale measures parameterised by $p = p(t, S_{t-})$, then for a (European) options with convex payoff, h, the option price is increasing in p.

Remark 7.2 Consider two alternative pricing measures, $\mathbb{Q}' \equiv \mathbb{Q}^{p'}$ and $\mathbb{Q}'' \equiv \mathbb{Q}^{p''}$. Suppose

(31) $$p''(x,t) \geq p'(x,t) \geq 0 \quad \forall\, x,\ t \geq 0$$

If (S'_t, S''_t) are a pair of realisations from the different models then we are not requiring that $p''(S''_{t-}, t) \geq p'(S'_{t-}, t)$ uniformly in S'_{t-}, S''_{t-} and t, but rather the much weaker condition that, if the price levels are identical in both models so that $S''_{t-} = x = S'_{t-}$, then $p''(x,t) \geq p'(x,t)$.

Before proving the theorem we introduce some further notation and prove a crucial lemma. Suppose that $\mathbb{Q}^p$ is any pricing measure and that under $\mathbb{Q}^p$

$$\frac{dS_t}{S_{t-}} = r_t dt + \sigma_t dW_t + \phi(dN_t^{\lambda p} - p(t, S_{t-})\lambda(t, S_{t-})dt) \tag{32}$$

where W is a $\mathbb{Q}^p$-Brownian motion, and $N^{\lambda p}$ is a Poisson process with rate λp. Then

$$S_t = S_0 R_t \mathcal{E}(\sigma \cdot W)_t (1+\phi)^{N_t^{\lambda p}} e^{-\phi A_t^{\lambda p}} \tag{33}$$

where $A_t^{\lambda p} = \int_0^t \lambda(S_{u-}, u) p(S_{u-}, u) du$. It is profitable to think of $N^{\lambda p}$ as a time change of a Poisson process N^1 of unit rate, so that $N_t^{\lambda p} = N^1(A_t^{\lambda p})$. In particular (33) becomes

$$S_t = S_0 R_t \mathcal{E}(\sigma \cdot W)_t (1+\phi)^{N^1(A_t^{\lambda p})} e^{-\phi A_t^{\lambda p}}.$$

We wish to use a coupling to obtain a price comparison result. To this end suppose that $\hat{\mathbb{P}}$ is a probability measure and that $\hat{W}$ is a $\hat{\mathbb{P}}$-Brownian motion and $\hat{N}^1$ is a unit rate Poisson process. The discount factor R_T and the exponential martingale $\mathcal{E}(\sigma \cdot \hat{W})_t$ can be defined as before. Fix ω and let $\hat{A}$ and $\hat{S}$ solve the coupled equations

$$\hat{A}_t \equiv \hat{A}_t^{\lambda p} = \int_0^t \lambda(\hat{S}_{u-}, u) p(\hat{S}_{u-}, u) du \tag{34}$$

$$\hat{S}_t = S_0 R_t \mathcal{E}(\sigma \cdot \hat{W})_t (1+\phi)^{\hat{N}^1(\hat{A}_t)} e^{-\phi \hat{A}_t}. \tag{35}$$

Now suppose we are given p' and p'' and let the pairs $(\hat{A}', \hat{S}')$ and $(\hat{A}'', \hat{S}'')$ solve (34) and (35) with p replaced by p' or p'' respectively. Note that we use the same $\hat{W}$ and $\hat{N}^1$ in the construction of each pair; only the form of p varies.

Our first result is that if the functions p are ordered then so are the time changes A.

Lemma 7.3 *Suppose $p''(x,t) \geq p'(x,t)$ for each pair (x,t) and that p'' and the parameters σ, λ, r satisfy the smoothness, growth and boundedness conditions given above. Then, except on a null set, for each ω we have $\hat{A}''_t(\omega) \geq \hat{A}'_t(\omega)$.*

Proof:
Fixing ω is equivalent to fixing the realisation of the Brownian motion $\hat{W}$ and the unit rate Poisson process $\hat{N}^1$. It is clear that $\hat{A}'_T$ and $\hat{A}''_T$ are both bounded by $K_\lambda K_p T$. Hence, except on a set of measure 0, we can assume for our ω that each of $R_t, \mathcal{E}(\sigma \cdot \hat{W})_t, \hat{N}^1(\hat{A}'_t)$ and $\hat{N}^1(\hat{A}''_t)$ is bounded on $[0,T]$.

Let $Y_u = \hat{A}'_u - \hat{A}''_u$; we show $Y_u \leq 0$. Let T_1 be the time of the first jump of N^1 and suppose $t \leq T$ is such that $\hat{A}'_t \vee \hat{A}''_t < T_1$. Observe that Y is finite variation so that for

$$\begin{aligned} Y_t^+ &= \int_0^t I_{\{Y_u > 0\}} dY_u \\ &= \int_0^t I_{\{Y_u>0\}} \Big(\lambda(\hat{S}'_{u-}, u) p'(\hat{S}'_{u-}, u) - \lambda(\hat{S}''_{u-}, u) p''(\hat{S}''_{u-}, u) \Big) du \end{aligned}$$

$$
\begin{aligned}
&\leq \int_0^t I_{\{Y_u>0\}} \Big(\lambda(\hat{S}'_{u-},u)p''(\hat{S}'_{u-},u) - \lambda(\hat{S}''_{u-},u)p''(\hat{S}''_{u-},u)\Big)\,du \\
&= \int_0^t I_{\{Y_u>0\}} \Big(\lambda(\hat{S}'_{u-},u)p''(\hat{S}'_{u-},u) - \lambda(\hat{S}''_{u-},u)p''(\hat{S}'_{u-},u)\Big)\,du \\
&\quad + \int_0^t I_{\{Y_u>0\}} \Big(\lambda(\hat{S}''_{u-},u)p''(\hat{S}'_{u-},u) - \lambda(\hat{S}''_{u-},u)p''(\hat{S}''_{u-},u)\Big)\,du \\
&\leq (C_\lambda K_p + C_p K_\lambda)\int_0^t I_{\{Y_u>0\}}|\hat{S}'_{u-} - \hat{S}''_{u-}|du
\end{aligned}
$$

Since there have been no jumps by time t we have that

$$
\begin{aligned}
|\hat{S}'_{u-} - \hat{S}''_{u-}| &= S_0 R_t \mathcal{E}(\sigma\cdot\hat{W})_t |e^{-\phi\hat{A}'_t} - e^{-\phi\hat{A}''_t}| \\
&\leq K|A'_u - A''_u|.
\end{aligned}
$$

for some constant K. We deduce

$$
\begin{aligned}
Y_t^+ &\leq K(C_\lambda K_p + C_p K_\lambda)\int_0^t (A'_u - A''_u)^+ du \\
&= K(C_\lambda K_p + C_p K_\lambda)\int_0^t Y_u^+ du
\end{aligned}
$$

Then, by Gronwall's Lemma [31, 5.11.11] we must have $Y_t^+ = 0$.

Thus, $\hat{A}''_t \geq \hat{A}'_t$ for all t with $\hat{A}''_t < T_1$, and since $\hat{A}''$ and $\hat{A}'$ are both continuous and increasing processes this holds for all $t \leq t'(1) = (\hat{A}')^{-1}(T_1)$. We extend this result to all $t \leq T$ by induction on the number of jumps of the Poisson process.

Let T_k be the time of the k^{th} jump of $\hat{N}^1$, and define $t'(k) = (\hat{A}')^{-1}(T_k)$. As the inductive hypothesis suppose $\hat{A}''_{t'(k)} \geq \hat{A}'_{t'(k)}$. Then $Y_{t'(k)} \leq 0$ and we prove $Y_s \leq 0$ for $t'(k) \leq s \leq t'(k+1)$.

If $\hat{A}''_{t'(k)} \geq T_{k+1}$ then $\hat{A}'_s \leq \hat{A}'_{t'(k+1)} = T_{k+1} \leq \hat{A}''_{t'(k)} \leq \hat{A}''_s$ and there is nothing to prove. Otherwise, for t such that $\hat{A}''_t \vee \hat{A}'_t \leq t'(k+1)$

$$
Y_t^+ = \int_{t(k)}^t I_{\{Y_u>0\}}(\lambda(\hat{S}'_{u-},u)p'(\hat{S}'_{u-},u) - \lambda(\hat{S}''_{u-},u)p''(\hat{S}''_{u-},u))du
$$

The argument proceeds exactly as before except for an additional factor $(1+\phi)^k$ in the modulus of $|\hat{S}'_{u-} - \hat{S}''_{u-}|$. Again we deduce from Gronwall's Lemma that $Y_t^+ = 0$, and by extension $\hat{A}''_t \geq \hat{A}'_t$ for $t \leq t'(k+1) \wedge T$.

For fixed ω the number of jumps of the Poisson process is finite and we deduce $\hat{A}''_t \geq \hat{A}'_t$ for all t as desired. □

We now prove the theorem.

Proof of Theorem 7.1:

If $\mathbb{Q}'$ and $\mathbb{Q}''$ are the martingale measures associated with p' and p'' then the goal is to show

$$
\mathbb{E}' h(S'_T) \leq \mathbb{E}'' h(S''_T). \tag{36}
$$

Note that, with W' a $\mathbb{Q}'$-Brownian motion and N' a Poisson process with rate $\lambda p'$

$$
\begin{aligned}
\mathbb{E}' h(S'_T) &= \mathbb{E}' h(S_0 R_T \mathcal{E}(\sigma\cdot W')_T (1+\phi)^{N'_T} e^{-\phi A'_T}) \\
&= \hat{\mathbb{E}} h(S_0 R_T \mathcal{E}(\sigma\cdot\hat{W})_T (1+\phi)^{\hat{N}^1(\hat{A}'_T)} e^{-\phi\hat{A}'_T}).
\end{aligned}
$$

Similarly,

(37) $$\mathbb{E}'' h(S''_T) = \hat{\mathbb{E}} h(S_0 R_T \mathcal{E}(\sigma \cdot \hat{W})_T (1+\phi)^{\hat{N}^1(\hat{A}''_T)} e^{-\phi \hat{A}''_T})$$

Using the independent increments property of $\hat{N}^1$ and Lemma 7.3

(38) $$\hat{N}^1(A''_T) = (\hat{N}^1(A''_T) - \hat{N}^1(A'_T)) + \hat{N}^1(A'_T))$$

and we can rewrite the right-hand side of (37) as

$$\hat{\mathbb{E}} h(S_0 R_T \mathcal{E}(\sigma \cdot \hat{W})_T (1+\phi)^{\hat{N}^1(\hat{A}'_T)} e^{-\phi \hat{A}'_T} (1+\phi)^{(\hat{N}^1(A''_T) - \hat{N}^1(A'_T))} e^{-\phi(\hat{A}''_T - \hat{A}'_T)})$$

Conditioning on the value of $\mathcal{E}(\sigma \cdot \hat{W})_T$ and $\hat{N}^1(\hat{A}'_T)$, and using a conditional form of Jensen's inequality we deduce that

$$\mathbb{E}'' h(S''_T) \geq \hat{\mathbb{E}} h(S_0 R_T \mathcal{E}(\sigma \cdot \hat{W})_T (1+\phi)^{\hat{N}^1(\hat{A}'_T)} e^{-\phi \hat{A}'_T})$$

and the result follows. □

Remark 7.4 Again it is interesting to consider where this proof breaks down if there is more than one possible jump size. In this case for each possible jump size $\phi(y)$ we would have to consider cumulative intensities $\hat{A}'(y)$ and $\hat{A}''(y)$ and we would have to prove a condition of the type $\hat{A}''_t(y) \geq \hat{A}'_t(y)$ uniformly in y. If there is only one possible jump size then we know that if $\hat{A}''_t = \hat{A}'_t$ then $\hat{S}''_t = \hat{S}'_t$, and hence we deduce $\hat{A}'' \geq \hat{A}'$. If there is more than one possible jump size, then if for a single y we have $\hat{A}''(y) = \hat{A}'(y)$ it does not follow that $\hat{S}''_t = \hat{S}'_t$.

Finally, in this section, we provide a counter-example to show that if there is more than one jump size then an ordering on the factors p is not sufficient to guarantee an ordering on the prices of options.

Example 7.5 Suppose we have a model with two potential jump components: 'unit' jumps with $\phi_1 \equiv 1$ and 'large' jumps with and $\phi = \theta \gg 1$. (Suppose further that $S_0 = 1$, interest rates are zero and that $\sigma \equiv 0$ so that there is no Brownian component. We remove this last restriction later.)

We consider an option with maturity $T = 1$ and payoff $h(x) = (x-3)^+$.

Suppose the rate of large jumps is θ^{-1} and the rate of unit jumps is c, but that both these jump types can only occur when the price level is in the range $\ln S_{t-} \in J_\epsilon := [-\epsilon, \epsilon]$. Here ϵ is small and certainly $\epsilon < \ln 2$.

Then,

$$S_t = 2^{N^1_{t\wedge\tau}} (1+\theta)^{N^2_{t\wedge\tau}} e^{-(c+1)(t\wedge\tau)}$$

where τ is the time of the first jump, or the time when $\ln S$ first leaves J_ϵ and N^1 and N^2 are Poisson processes governing the times of the first jumps of type 'unit' and 'large' respectively.

At the time of the first jump of either type the process stops. If there are no jumps the process $\ln S$ leaves J_ϵ at $t = (1+c)^{-1}\epsilon$, and again the process is constant from then on.

The option pays out a positive amount if and only if there is a large jump before time 1. If there is a large jump at τ then the option pays out $(1+\theta)e^{-(c+1)\tau} - 3$ and since $\ln S_\tau \in J_\epsilon$ and ϵ is small this is approximately $\theta - 2$.

The probability that there is a large jump is the probability there is any jump multiplied by the probability that conditional on a jump, the jump is large. Until $\ln S$ leaves J_ϵ the rate of jumps is $c+\theta^{-1}$ and since in the absence of jumps $\ln S$ leaves J_ϵ at time $(1+c)^{-1}\epsilon$ the probability there is a jump is

$$1-\exp\left(-(c+\theta^{-1})\frac{\epsilon}{1+c}\right)$$

which is approximately $\epsilon c/(1+c)$. Given there is a jump the probability it is large jump is

$$\frac{\theta^{-1}}{c+\theta^{-1}}\sim\frac{1}{c\theta}$$

Finally we deduce that the expected value of the option is approximately

$$(\theta-2)\frac{\epsilon c}{1+c}\frac{1}{c\theta}\sim\frac{\epsilon}{1+c}.$$

This is a decreasing function of c.

When there is more than one possible jump size, increasing the rates of jumps affects the dynamics of S_t. In this case increasing the rate of unit jumps means that the drift of S is stronger and reduces the length of period for which $\ln S\in J_\epsilon$ and reduces the overall probability of large jumps. This decreases the value of far-out-of-the-money call options.

In fact it is not possible to have the volatility parameter σ equal to zero and remain within the framework of this paper, not least because if $\sigma\equiv 0$ there then is only one value of c for which the model of this example is a martingale. However if the volatility co-efficient is a small positive constant (small even in comparison with ϵ) then the spirit of the above example still holds, each c corresponds to a different martingale measure, and with careful analysis it can be shown that option prices are not increasing in c.

8 American Options

It is straightforward to extend the results of the previous sections to a American options and we provide a sketch of these extensions here. Note that in the deterministic case these results can also be found in Pham [28]. The prices of American-style options under the pricing measure $\mathbb{Q}^p$ are given by

$$\sup_{\tau\le T} R_\tau^{-1}\mathbb{E}^{\mathbb{Q}^p}h(S_\tau).$$

First assume that we are in the setting of Section 6 so there are jumps of various sizes, but parameters are deterministic. We borrow notation from Section 6 and assume $p''_t(y)\ge p'_t(y)$ uniformly in t and y.

Then, for any stopping time τ

$$\hat{\mathbb{E}}h(\hat{S}''_\tau)=\hat{\mathbb{E}}h(S_0R_\tau\mathcal{E}(\sigma\cdot\hat{W})_\tau\mathcal{E}(\phi\cdot\hat{M}')_\tau\mathcal{E}(\phi\cdot\hat{M}^\delta)_\tau)$$

and conditioning on $\hat{W}$ and $\hat{M}'$ and using the martingale property of $\mathcal{E}(\phi\cdot\hat{M}^\delta)_t$ we have by Jensen's inequality

$$\begin{aligned}\hat{\mathbb{E}}h(\hat{S}''_\tau) &\ge \hat{\mathbb{E}}h(S_0R_\tau\mathcal{E}(\sigma\cdot\hat{W})_\tau\mathcal{E}(\phi\cdot\hat{M}')_\tau)\\ &= \hat{\mathbb{E}}h(\hat{S}'_\tau)\end{aligned}$$

It follows that

$$\sup_{\tau \le T} R_\tau^{-1} \hat{\mathbb{E}} h(S''_\tau) \ge \sup_{\tau \le T} R_\tau^{-1} \hat{\mathbb{E}} h(S'_\tau)$$

and hence Theorem 6.1 holds for American options also.

Now suppose the assumptions of Section 7 hold so that there is a unique jump size, but some of the parameters may be stochastic. Since Lemma 7.3 holds for all t it also holds at any stopping time τ. The extension of Theorem 7.1 is straightforward; we simply use the conditional Jensen's inequality at the stopping time τ and note that given $\hat{N}^1(A'_\tau)$ the increment $\hat{N}^1(A''_\tau) - \hat{N}^1(A'_\tau)$ is a Poisson random variable.

9 Comparison Results

To begin with we suppose that all parameters are deterministic so that we are in the setting of Theorem 6.1. Suppose further that $b(t) > r(t)$.

Theorem 9.1 *Suppose we are considering an option with convex payoff h.*
(a) *Suppose all jumps are positive; $\phi_u(y) > 0$ uniformly in u and y. Then the price under the Merton measure is greater than the price under each of the Föllmer-Schweizer (if defined), relative entropy and Esscher transform martingale measures.*
(b) *Suppose all jumps are negative; $\phi_u(y) < 0$ uniformly in u and y. Then the price under the Merton measure is smaller then the price under each of the Föllmer-Schweizer, relative entropy and Esscher transform martingale measures.*

Proof.
Following the results of Section 7 it is sufficient to prove corresponding results about the orderings of the parameters $\gamma^M, \gamma^F, \gamma^{RE}, \gamma^E$.

Recall that $\gamma^M \equiv 0$ and $\gamma_t^F(y) \equiv \eta_t^* \phi_t(y)$ where η_t^* is given by (12). By our assumption on b and r, η^* is negative, so if $\phi_t(y)$ is uniformly positive (respectively negative) then $\gamma_t^F(y)$ is uniformly negative (respectively positive). The comparisons between the Merton and Föllmer-Schweizer prices follow.

Define

$$H_t(\alpha) = b(t) - r(t) + \sigma^2(t)\alpha + \lambda(t) \int_{[0,1]} [e^{\alpha\phi(t,y)} - 1]\phi(t,y)dy \tag{39}$$

so that the minimal entropy martingale measure is parameterised by α^* where $H_t(\alpha_t^*) = 0$. (This equation has a unique solution since

$$\frac{\partial H_t}{\partial \alpha} = \sigma^2(t) + \lambda(t) \int_{[0,1]} e^{\alpha\phi(t,y)} \phi(t,y)^2 dy$$

is positive for all α.) Further $H_t(0) > 0$ so we must have $\alpha_t^* < 0$. Since $\gamma_t^{RE}(y) = e^{\alpha_t^* \phi(t,y)} - 1$ we have that $\gamma_t^{RE}(y)$ is positive if and only if $\phi(t,y)$ is negative.

The argument for the Esscher transform martingale measure is very similar except that it is based on

$$H_t^E(\theta) = b(t) - r(t) + \sigma^2(t)\theta + \lambda(t) \int_{[0,1]} [(1+\phi(t,y))^\theta - 1]\phi(t,y)dy \tag{40}$$

□

Remark 9.2 Bellamy and Jeanblanc [3] show that the set of prices for a European call option taken under the various equivalent martingale measures is bounded below by the corresponding Black-Scholes price when the jump rates are set to zero and bounded above by the price of the underlying itself. This result can be seen as an alternative application of Theorem 6.1, the lowest possible price corresponds to the smallest possible jump rates, and the highest possible price arises in the limit as the rate of jumps grows without bound.

Now suppose that there is only one possible jump size.

Theorem 9.3 *The price of an option with convex payoff h under the Föllmer-Schweizer minimal martingale measure is lower than the price under the minimal relative entropy martingale measure which in turn is lower than the price under the Esscher transform martingale measure.*

Proof.
By hypothesis there is exactly one possible jump size so we can suppress the jump-label y. We know that each of η_t^*, α_t^* and θ_t^* must be negative and it is easy to check that for any non-zero ϕ and negative x

$$(1+\phi)^x > e^{\phi x} > \phi x + 1.$$

Hence for negative α and positive ϕ

$$H_t(\alpha) > b(t) - r(t) + \sigma^2(t)\alpha + \lambda(t)\alpha\phi^2$$

and in particular $H_t(\eta_t^*) > 0$. Hence if ϕ is positive we have $\alpha_t^* < \eta_t^*$. Conversely, if ϕ is negative we have $\alpha_t^* > \eta_t^*$ so that $\phi(\eta_t^* - \alpha_t^*) > 0$. Further, from (12) and (16) we have

$$\begin{aligned} 0 &= b(t) - r(t) + \sigma^2(t)\eta_t^* + \lambda(t)\phi\gamma_t^F \\ 0 &= b(t) - r(t) + \sigma^2(t)\alpha_t^* + \lambda(t)\phi\gamma_t^{RE} \end{aligned}$$

Subtracting and multiplying by ϕ_t we find

$$\lambda(t)\phi^2(\gamma_t^{RE} - \gamma_t^F) = \sigma^2(t)\phi(\eta_t^* - \alpha_t^*) > 0.$$

Hence, $\gamma_t^{RE} > \gamma_t^F$ and the first part of the theorem follows as an application of Theorem 6.1.

Conversely

$$H_t(\alpha) < b(t) - r(t) + \sigma^2(t)\alpha + \lambda(t)(1+\phi)^\alpha\phi$$

and in particular $H_t(\theta_t^*) < 0$. From this we can proceed as above to deduce that $\gamma_t^E > \gamma_t^{RE}$, and the inequality between the Esscher and relative entropy martingale measures follows. □

Suppose that $\sigma = \sigma(t)$ is deterministic, and that there is only one possible jump size ϕ. The intensity process $\lambda = \lambda(S_{t-}, t)$ and drift $b = b(S_{t-}, t) > r(t)$ may be stochastic. Hence we are in the setting of Theorem 7.1. Since the parameters are stochastic, the construction in Section 4.3 does not necessarily lead to the minimal relative entropy measure. However, if we call the measure $\mathbb{Q}^*$ the locally minimal entropy martingale measure then we have the following result the proof of which is a direct copy of the proof of Theorem 9.3, except that it appeals to Theorem 7.1 at the final step.

Theorem 9.4 *Suppose that $\sigma = \sigma(t)$ is deterministic and that there is only one possible jump size ϕ. The price of an option with convex payoff h under the Föllmer-Schweizer minimal martingale measure is lower than the price under the locally minimal entropy martingale measure which in turn is lower than the price under the Esscher transform martingale measure.*

Remark 9.5 These result gives us some relationships between the various martingale measure prices in a simple model. It is an open question as to whether these relationships will hold more generally. It seems intuitively natural that the Föllmer-Schweizer minimal martingale measure should give the lowest price since this measure leaves risks which are orthogonal to the asset price process unpriced.

10 Conclusion

In this article we considered option pricing in a jump-diffusion model for the underlying. In general such a model is incomplete and there are no unique preference independent prices for options. Equivalently the set of equivalent martingales measures is not a singleton.

Several authors have characterised the set of equivalent martingale measures. This set is parameterised by two processes which correspond to the market price of jump risk and the market price of diffusion risk. Our results show that if we consider two martingale measures, and if the jump intensity under the first measure is higher than the jump intensity under the second measure, then the price of an option whose payoff is a convex function of the underlying is higher under the first model than under the second. In the case of deterministic parameter values this result is derived using pde methods by Pham [28]. Our methods, which involve coupling of Poisson processes, allow us to make some progress towards considering stochastic parameters. In particular we give an example to show that option prices are not monotonically increasing in the jump intensity if there is more than one possible jump size, and the jump intensities are stochastic.

In the situation where there is only one possible jump size there is more to be said. In this case it is possible to prove a comparison between the options prices under various candidate martingale measures which have been proposed in the literature. In particular, again for options whose payoff is a convex function of the underlying, we have that the price under the Föllmer-Schweizer minimal martingale measure is smaller than the price under the minimal relative entropy measure which in turn is smaller than the price under the Esscher transform martingale measure. We conjecture that the Föllmer-Schweizer minimal martingale measure continues to give the lowest price amongst these three common pricing measures even if there is more than one possible jump size.

References

[1] Bardhan I. and X. Chao; Pricing Options on securities with discontinuous returns, *Stochastic Processes Appl.*, 48, 123-137, 1993.

[2] Bardhan I. and X. Chao; On martingale measures when asset returns have unpredictable jumps, *Stochastic Processes Appl.*, 63, 35-54, 1996.

[3] Bellamy N. and M. Jeanblanc; Incompleteness of markets driven by a mixed diffusion, *Finance and Stochastics*, Vol 4, 2, February 2000.

[4] Bergman Y.Z., Grundy B.D and Wiener Z.; General Properties of options prices, Journal of Finance, Vol 51, p1573-1610, 1996.

[5] Carr P., Geman H., Madan D.B. and Yor M.; The fine structure of asset returns: an empirical investigation, Preprint, July 2000.

[6] Chan T.; Pricing contingent claims on stocks driven by Levy processes, *Annals of Applied Probability*, 9,2, p504-528, 1999.

[7] Delbaen F., Grandits P., Rheinlander T., Samperi D, Schweizer M. and Stricker C.; Exponential Hedging and Entropic Penalties, To appear in *Mathematical Finance*.

[8] Duffie D., Pan J. and Singleton K.; Transform Analysis and option pricing for affine jump diffusions, *Econometrica*, 68, p1343-1376, 2000.

[9] Duffie D. and K.J. Singleton; Modelling Term Structure Models of default bonds, *Review of Financial Studies*, 12, p687-720, 1999.

[10] Eberlein E. and J. Jacod; On the range of option prices, *Finance and Stochastics* 1(2), 131-140, 1997.

[11] El Karoui N, Jeanblanc Picque M. and S.E. Shreve; Robustness of the Black and Scholes formula, *Mathematical Finance*, Vol 8, No.2, p93-126, 1998.

[12] Elliot R.J.; Stochastic Calculus and Applications, Springer-Verlag, New York, 1982.

[13] Elliot R.J. and Jeanblanc M.; Incomplete markets with jumps and informed agents, *Math Methods Oper Res*, 50, p475-492, 1999.

[14] Embrechts P., Frey R. and H. Furrer; Stochastic Processes in Insurance and Finance, In: Handbook of Statistics, Vol 19, 'Stochastic Processes: Theory and Methods', Elsevier Science, Amsterdam, p365-412, 2001.

[15] Föllmer H. and Schweizer, M; Hedging of contingent claims under incomplete information. In *Applied Stochastic Analysis* (Eds M.H.A. Davis and R.J. Elliott), Gordon and Breach, New York. 1990.

[16] Frittelli M; The Minimal Entropy Martingale measure and the valuation problem in incomplete markets, *Mathematical Finance*, 10, p39-52, 2000.

[17] Gerber H.U. and E.S.W. Shui, Option pricing by Esscher transforms. With discussion. *Transactions of the Society of Actuaries*, XLVI, p99-191. 1994.

[18] Henderson V.; Price comparison results and super-replication: an application to Passport options, *Applied Stochastic models in Business and Industry*, 16, 4, p297-310, 2000.

[19] Henderson V. and D.G. Hobson; Local time, coupling and the Passport option, *Finance and Stochastics*, Vol 4, 1, p69-80, January 2000.

[20] Hobson D.G.; Volatility mis-specification, option pricing and super-replication via coupling, *Annals of Applied Probability*, 8, No.1, p193-205, 1998.

[21] Jeanblanc, M. and Privault, N.; A complete market model with Poisson and Brownian components, Prépublication du Département de Mathèmatiques, Université de La Rochelle. 1999.

[22] Kou S.; A jump diffusion model for option pricing, preprint, Columbia University, August 2001.

[23] Lindvall T.; Lectures on the Coupling method, Wiley, New York, 1992.

[24] Lyons, T.; Uncertain volatility and the risk-free synthesis of derivatives, *Applied Mathematical Finance*, 2, p117-133, 1995.

[25] Madan D.B. and Seneta E.; The variance gamma model for share market returns, *Journal of Business*, 63, p511-524, 1990.

[26] Martini, C.; Propagation de la convexite, *These, Universite d'Evry*, 1996.

[27] Merton R, C.; Option pricing when underlying stock returns are discontinuous, *Journal of Financial Economics*, 3, p125-144, 1976.

[28] Pham H.; Optimal stopping, free boundary and American option in a jump diffusion model, *Appl. Math. Optim.*, 35, p145-164, 1997.

[29] Prigent, J. Renault, O. and Scaillet, O; Option pricing with discrete rebalancing; Preprint 2001.

[30] Rouge R. and El Karoui N.; Pricing via utility maximisation and entropy, *Mathematical Finance*, 10, 2, p259-276, 2000.

[31] Rogers, L.C.G. and D. Williams, Diffusions, Markov Processes and Martingales, Volume 2. Wiley, 1987.