

# Skorokhod embeddings, minimality and non-centred target distributions

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## Abstract

In this paper<sup>1</sup> we consider the Skorokhod embedding problem for target distributions with non-zero mean. In the zero-mean case, uniform integrability provides a natural restriction on the class of embeddings, but this is no longer suitable when the target distribution is not centred. Instead we restrict our class of stopping times to those which are minimal, and we find conditions on the stopping times which are equivalent to minimality.

We then apply these results, firstly to the problem of embedding non-centred target distributions in Brownian motion, and secondly to embedding general target laws in a diffusion.

We construct an embedding (which reduces to the Azema-Yor embedding in the zero-target mean case) which maximises the law of  $\sup_{s \leq T} B_s$  among the class of minimal embeddings of a general target distribution  $\mu$  in Brownian motion. We then construct a minimal embedding of  $\mu$  in a diffusion  $X$  which maximises the law of  $\sup_{s \leq T} h(X_s)$  for a general function  $h$ .

## 1 Introduction

The Skorokhod embedding problem was first proposed, and then solved, by Skorokhod (1965), and may be described thus:

Given a Brownian motion  $(B_t)_{t \geq 0}$  and a centred target law  $\mu$  can we find a stopping time  $T$  such that  $B_T$  has distribution  $\mu$ ?

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Skorokhod gave an explicit construction of the stopping time  $T$  in terms of independent random variables, and in doing so showed that any zero-mean probability measure may be embedded in Brownian motion. Since the problem was posed many more solutions have been given, see for example Dubins (1968), Root (1969) and Chacon and Walsh (1976), and the comprehensive survey article of Oblój (2004). With different solutions comes the question of optimal properties of the embeddings, and various optimal embeddings have been found — for example the embedding minimising the variance of  $T$  (Rost, 1976), the embedding minimising in convex order the law of the local time at zero (Vallois, 1992), or the embedding stochastically minimising the law of the maximum (Perkins, 1986).

An obvious extension of the problem is to consider more general classes of processes. Here the question of the existence of an embedding becomes more interesting. For a Markov process and an arbitrary target measure necessary and sufficient conditions are given by Rost (1971) and a construction in this general case is given by Bertoin and Le Jan (1992). In the case of diffusions on  $\mathbb{R}$  simpler necessary and sufficient conditions are given in Grandits and Falkner (2000), Pedersen and Peskir (2001) and Cox and Hobson (2004), along with some constructions.

The work we present here was motivated by the following question:

Given a diffusion  $(X_t)_{t \geq 0}$  and a target distribution  $\mu_X$  for which an embedding exists, which embedding maximises the law of  $\sup_{s \leq T} X_s$  (respectively  $\sup_{s \leq T} |X_s|$ )?

For Brownian motion, the question has been solved by Azéma and Yor (1979a) (respectively Jacka (1988)) under the condition that  $B_{t \wedge T}$  is a UI-martingale.

There are several considerations that need to be made when moving from the Brownian case to the diffusion case. Firstly, the mean-zero assumption that is made by Azéma and Yor (1979a) and Jacka (1988) is no longer natural since we are no longer necessarily dealing with a martingale. The second aspect that needs to be considered is with what restriction should we replace the UI condition? That such a condition is desirable may be seen by considering a recurrent diffusion. Here the maximisation problem can easily be seen to degenerate by considering first running the diffusion until it hits a level  $x$ , allowing it to return to its starting point and then using the reader's favourite embedding. Clearly this dominates the unmodified version of the reader's favourite embedding.

In Pedersen and Peskir (2001) an integrability condition on the maximum (specifically that  $\mathbb{E}(\sup_{t \leq T} s(X_t)) < \infty$  where  $s$  is the scale function of  $X$ ) was suggested to replace the UI condition in the Brownian case. In this work we propose using the following class of stopping times introduced by Monroe (1972) to provide us with a natural restriction on the set of admissible embeddings.

**Definition 1.** A stopping time  $T$  for the process  $X$  is *minimal* if whenever  $S \leq T$  is a stopping time such that  $X_S$  and  $X_T$  have the same distribution then  $S = T$  a.s..

The class of minimal stopping times provides us with a natural class of 'good' stopping times. In the Brownian case it includes as a subclass those embeddings for which

$\mathbb{E}[\sup_{s \leq T} B_s]$  is finite. Furthermore, there is a link to the uniformly integrable case as a consequence of the following result:

**Theorem 2.** *(Monroe, 1972, Theorem 3) Let  $T$  be an embedding in Brownian motion of a centred distribution  $\mu$ . Then  $T$  is minimal if and only if the process  $B_{t \wedge T}$  is uniformly integrable, or equivalently if  $\mathbb{E}[B_T | \mathcal{F}_S] = B_S$  for all stopping times  $S \leq T$ .*

Our first result extends this theorem to non-centred target distributions.

**Theorem 3.** *Let  $T$  be an embedding in Brownian motion of a non-centred distribution  $\mu$  with mean  $m < 0$ . Then  $T$  is minimal if and only if the process  $B_{t \wedge T}^-$  is uniformly integrable, or equivalently if  $\mathbb{E}[B_T | \mathcal{F}_S] \leq B_S$  for all stopping times  $S \leq T$ .*

It is clear that the notion of a minimality fits well with the problem of embedding in any process, such as a diffusion, and not just Brownian motion. Our approach to embeddings in diffusions can be traced back to Azéma and Yor (1979b) and will be to map the diffusion into natural scale (so that, up to a time change, it resembles Brownian motion) and use techniques developed for embedding Brownian motion. Using this method on a transient diffusion one finds that the state space and target distribution for the Brownian motion is restricted to a half-line (or sometimes a finite interval). We will show minimality to be equivalent to stopping the Brownian motion before it leaves this interval, so that a minimal stopping time is necessarily before the first explosion time of  $X$ .

When we map from the problem of embedding  $\mu_X$  in  $X$  to the Brownian motion the target law  $\mu$  we obtain for  $B$  is the image of  $\mu_X$  under the scale function. The key point is that there is no reason why this target law should have mean zero. Thus, unlike most of the other studies of Skorokhod embeddings in Brownian motion we are interested in non-centred target distributions, and non-UI stopping times. Instead, as described in Theorem 3, the class of ‘good’ stopping times satisfy slightly different integrability conditions.

Having characterised the class of minimal embeddings, we then turn to the problem of constructing optimal embeddings in Brownian motion for non-centred target distributions. In particular, amongst the class of minimal embeddings we find the stopping time which maximises the law of the maximum of the stopped process. In the centred case this embedding reduces to the classical Azema-Yor embedding. In fact most of the paper will concentrate on embedding non-centred target distributions in  $B$ , and we will only return to the diffusion case in a short final section.

The paper will proceed as follows. In Section 2 we prove some results concerning minimality of stopping times for non-centred target distributions, giving equivalent conditions to minimality in terms of the process. Next, in Section 3 we construct an extension of the Azema-Yor embedding for non-centred target distributions and show both that it is minimal, and that it retains the optimality properties of the original Azema-Yor embedding. In Section 4 we use these stopping times to construct an embedding maximising the distribution of  $\sup_{s \leq T} h(B_s)$  for a general function  $h$ . Finally in Section 5 we apply these results to the problem of embedding optimally in diffusions.

## 2 Minimal Embeddings for Non-centred Distributions

In this section we examine the properties of minimal stopping times. In particular, for embeddings in Brownian motion we aim to find equivalent conditions to minimality when the target distribution is not centred.

We begin by noting the following result from Monroe (1972) which justifies the existence of minimal stopping times:

**Proposition 4 (Monroe (1972), Proposition 2).** *For any stopping time  $T$  there exists a minimal stopping time  $S \leq T$  such that  $B_S \sim B_T$ .*

For  $\beta \in \mathbb{R}$  define  $H_\beta = \inf\{t > 0 : B_t = \beta\}$ , the first hitting time of the level  $\beta$ . The main result of this section is the following:

**Theorem 5.** *Let  $T$  be a stopping time of Brownian motion which embeds an integrable distribution  $\mu$  where  $m = \int_{\mathbb{R}} x \mu(dx) < 0$ . Then the following conditions are equivalent:*

(i)  $T$  is minimal for  $\mu$ ;

(ii) for all stopping times  $R \leq S \leq T$ ,

$$\mathbb{E}(B_S | \mathcal{F}_R) \leq B_R \quad a.s.; \quad (1)$$

(iii) for all stopping times  $S \leq T$ ,

$$\mathbb{E}(B_T | \mathcal{F}_S) \leq B_S \quad a.s.; \quad (2)$$

(iv) for all  $\gamma > 0$

$$\mathbb{E}(B_T; T > H_{-\gamma}) \leq -\gamma \mathbb{P}(T > H_{-\gamma});$$

(v) as  $\gamma \rightarrow \infty$

$$\gamma \mathbb{P}(T > H_{-\gamma}) \rightarrow 0;$$

(vi) the family  $\{B_S^-\}$  taken over stopping times  $S \leq T$  is uniformly integrable;

(vii) for all  $x > 0$

$$\mathbb{E}(B_{T \wedge H_x}) = 0.$$

In the case where  $\text{supp}(\mu) \subseteq [\alpha, \infty)$  for some  $\alpha < 0$  then the above conditions are also equivalent to the condition:

(viii)

$$\mathbb{P}(T \leq H_\alpha) = 1. \quad (3)$$

**Remark 6.** (i) Of course the Theorem may be restated in the case where  $m > 0$  by considering the process  $-B_t$ . We will use this observation extensively in Section 3.

(ii) Equation (2) is suggestive of the fact that when  $T$  is minimal,  $(B_{(u/1-u)\wedge T})_{0\leq u\leq 1}$  is a supermartingale. To check this we need to show also that  $\mathbb{E}B_{t\wedge T}^- < \infty$  for all  $t$ . We show this more generally, for a stopping time  $S \leq T$ . Using (2),

$$\mathbb{E}(B_T; B_T \leq 0) \leq \mathbb{E}(B_T; B_S \leq 0) \leq \mathbb{E}(B_S; B_S \leq 0),$$

so that  $\mathbb{E}B_S^- < \mathbb{E}B_T^- < \infty$  and the process is indeed a supermartingale.

It follows from comparing parts (ii) and (iii) of Theorem 5 that if  $S \leq T$  is a stopping time and  $T$  is minimal, then  $S$  is minimal provided  $\mathbb{E}B_S \leq 0$  and  $\mathbb{E}|B_S| < \infty$ . The first condition is a trivial consequence of (1) on taking  $R = 0$ , the second condition then follows from Remark 6(ii) on noting that, since  $\mathbb{E}B_S \leq 0$ , we have  $\mathbb{E}B_S^+ \leq \mathbb{E}B_S^- < \infty$ .

Consequently we have the following corollary of Theorem 5:

**Corollary 7.** *If  $T$  is minimal and  $S \leq T$  for a stopping time  $S$  then  $S$  is minimal for  $\mathcal{L}(B_S)$ .*

For ease of exposition we divide the proof of Theorem 5 into a series of smaller results. The first is a key result which shows that the strongest of the stopping time conditions is sufficient for minimality. Throughout this section it is to be understood that  $\mu$  is a distribution with negative mean and  $T$  a stopping time embedding  $\mu$ .

**Lemma 8.** *Suppose that for all stopping times  $R, S$  with  $R \leq S \leq T$  we have*

$$\mathbb{E}(B_S | \mathcal{F}_R) \leq B_R \quad a.s.. \tag{4}$$

*Then  $T$  is minimal.*

*Proof.* Let  $R \leq T$  be a stopping time such that  $R$  embeds  $\mu$  (so that  $\mathbb{E}|B_R| = \mathbb{E}|B_T| < \infty$ ). For  $a \in \mathbb{R}$ ,

$$\begin{aligned} \sup_{A \in \mathcal{F}_T} \mathbb{E}(a - B_T; A) &= \mathbb{E}(a - B_T; B_T \leq a) \\ &= \mathbb{E}(a - B_R; B_R \leq a) \\ &\leq \mathbb{E}(a - B_T; B_R \leq a) \\ &\leq \sup_{A \in \mathcal{F}_T} \mathbb{E}(a - B_T; A) \end{aligned} \tag{5}$$

where we use (4) to deduce (5). However since we have equality in the first and last expressions, we must also have equality throughout and so

$$\{B_T < a\} \subseteq \{B_R \leq a\} \subseteq \{B_T \leq a\}.$$

Since this holds for all  $a \in \mathbb{R}$  we must have  $B_T = B_R$  a.s..

It follows that for  $S$  with  $R \leq S \leq T$ ,  $B_R = \mathbb{E}[B_R | \mathcal{F}_S] = \mathbb{E}[B_T | \mathcal{F}_S] \leq B_S$ , and since  $\mathbb{E}[B_S | \mathcal{F}_R] \leq B_R$  we must have  $B_S = B_R = B_T$ . Thus  $B$  is constant on the interval  $[R, T]$  and  $R = T$ . Hence  $T$  is minimal. □

**Lemma 9.** *If  $T$  is a stopping time such that  $B_T \sim \mu$  and  $\gamma\mathbb{P}(T > H_{-\gamma}) \rightarrow 0$  as  $\gamma \rightarrow \infty$  then  $\mathbb{E}|B_S| < \infty$  and  $\mathbb{E}B_S \leq 0$  for all stopping times  $S \leq T$ .*

*Proof.* We show that, for  $S \leq T$ ,  $\mathbb{E}B_S^- < \infty$  and  $\mathbb{E}B_S^+ \leq \mathbb{E}B_S^-$  from which the result follows. Suppose  $\gamma > 0$ . Since  $B_{t \wedge H_{-\gamma}}$  is a supermartingale,

$$\mathbb{E}(B_{T \wedge H_{-\gamma}}; B_S < 0, S < H_{-\gamma}) \leq \mathbb{E}(B_{S \wedge H_{-\gamma}}; B_S < 0, S < H_{-\gamma}).$$

We may rewrite the term on the left of the equation as

$$\mathbb{E}(B_T; B_S < 0, T < H_{-\gamma}) - \gamma\mathbb{P}(B_S < 0, S < H_{-\gamma} < T),$$

and by hypothesis  $\gamma\mathbb{P}(B_S < 0, S < H_{-\gamma} < T) \leq \gamma\mathbb{P}(H_{-\gamma} < T) \rightarrow 0$  as  $\gamma \rightarrow \infty$ . Further, by dominated convergence,  $\mathbb{E}(B_T; B_S < 0, T \leq H_{-\gamma}) \rightarrow \mathbb{E}(B_T; B_S < 0)$  and it follows that

$$\mathbb{E}(B_S; B_S < 0) = \lim_{\gamma \rightarrow \infty} \mathbb{E}(B_S; B_S < 0, S < H_{-\gamma}) \geq \mathbb{E}(B_T; B_S < 0).$$

Hence  $\mathbb{E}B_S^- \leq -\mathbb{E}(B_T; B_S < 0) \leq \mathbb{E}B_T^- < \infty$ .

Again using the fact that  $B_{t \wedge H_{-\gamma}}$  is a supermartingale,  $0 \geq \mathbb{E}(B_S \wedge H_{-\gamma}) = \mathbb{E}(B_S; S < H_{-\gamma}) - \gamma\mathbb{P}(H_{-\gamma} \leq S)$  so that

$$\mathbb{E}(B_S^+; S < H_{-\gamma}) \leq \mathbb{E}(B_S^-; S < H_{-\gamma}) + \gamma\mathbb{P}(H_{-\gamma} \leq S).$$

By monotone convergence the term on the left increases to  $\mathbb{E}B_S^+$ , while by monotone convergence and the hypothesis of the Lemma the right hand side converges to  $\mathbb{E}B_S^-$ . Consequently  $\mathbb{E}B_S^+ \leq \mathbb{E}B_S^- < \infty$ .  $\square$

The previous two lemmas are crucial in determining sufficient conditions for minimality. Now we consider necessary conditions.

**Lemma 10.** *If  $T$  is minimal then, for all  $\gamma \leq 0$ ,  $\mathbb{E}(B_T - B_{T \wedge H_\gamma}) \leq 0$ .*

*Proof.* For  $\gamma \leq 0$  let  $f(\gamma) = \mathbb{E}(B_T - B_{T \wedge H_\gamma}) = \mathbb{E}(B_T - \gamma; T > H_\gamma)$ . Note that  $f(0) = m < 0$ .

It is easy to see that for  $0 \geq \gamma \geq \gamma'$

$$\mathbb{P}(T \in (H_\gamma, H_{\gamma'})) = \mathbb{P}(\inf_{s \leq T} B_s \in (\gamma', \gamma)) \rightarrow 0$$

as  $\gamma \downarrow \gamma'$  or  $\gamma' \uparrow \gamma$ . Since we may write

$$f(\gamma) - f(\gamma') = \mathbb{E}(B_T; H_\gamma < T < H_{\gamma'}) + (\gamma' - \gamma)\mathbb{P}(T \geq H_{\gamma'}) - \gamma\mathbb{P}(H_\gamma < T < H_{\gamma'})$$

it follows from the dominated convergence theorem that  $f$  is continuous. As a corollary if  $f(\gamma_0) > 0$  for some  $\gamma_0 < 0$ , then there exists  $\gamma_1 \in (\gamma_0, 0)$  such that  $f(\gamma_1) = 0$ .

Given this  $\gamma_1$ , and conditional on  $T > H_{\gamma_1}$ , let  $T'' = T - H_{\gamma_1}$ ,  $W_t = B_{H_{\gamma_1} + t} - \gamma_1$ , and  $\mu'' = \mathcal{L}(W_{T''})$ . Suppose that  $T''$  is not minimal, so there exists  $S'' \leq T''$  with law  $\mu''$ . If we define

$$S = \begin{cases} T & \text{on } T \leq H_{\gamma_1} \\ H_{\gamma_1} + S'' & \text{on } T > H_{\gamma_1} \end{cases}$$

then  $S$  embeds  $\mu$  and  $S \leq T$  but  $S \neq T$ , contradicting the minimality of  $T$ . Hence  $T''$  is minimal. But then by Theorem 2,  $W_{t \wedge T''}$  is uniformly integrable and so, for  $\gamma < \gamma_1$

$$\mathbb{E}(W_{T''} - (\gamma - \gamma_1); T'' > H_{\gamma - \gamma_1}^W) = 0$$

or equivalently

$$f(\gamma) = \mathbb{E}(B_T - \gamma; T > H_\gamma) = 0.$$

Hence  $f(\gamma) \leq 0$  for all  $\gamma \in (-\infty, 0]$ .  $\square$

**Lemma 11.** *Suppose  $B_T \sim \mu$  and  $m < 0$ . If  $\mathbb{E}[B_{T \wedge H_x}] = 0$  for all  $x > 0$  then we have  $\mathbb{E}[B_T | \mathcal{F}_S] \leq B_S$  for all stopping times  $S \leq T$ .*

*Proof.* Fix  $x > 0$  and define  $T_x = T \wedge H_x$ . We begin by showing that  $T_x$  is minimal for the centred probability distribution  $\mathcal{L}(B_{T_x})$ . Fix  $R \leq T_x$ . The stopped process  $B_{(t/1-t) \wedge T_x}$  is a submartingale so  $\mathbb{E}[B_{T_x} | \mathcal{F}_R] \geq B_R$  and  $\mathbb{E}[B_R] \geq 0$ . Thus  $0 = \mathbb{E}[B_{T_x}] \geq \mathbb{E}[B_R] \geq 0$  and  $B_R = \mathbb{E}[B_{T_x} | \mathcal{F}_R]$ . Since  $B_{t \wedge T_x}$  is UI, by Theorem 2 we have that  $T_x$  is minimal.

Now fix  $S \leq T$  and define  $S_x = S \wedge H_x$ . We show  $\mathbb{E}[|B_S|] \leq \mathbb{E}[|B_T|] < \infty$ . We have

$$\begin{aligned} \mathbb{E}[|B_S| I_{\{S \leq H_x\}}] &= \mathbb{E}[B_{S_x} I_{\{S \leq H_x\}} I_{\{B_S \geq 0\}}] - \mathbb{E}[B_{S_x} I_{\{S \leq H_x\}} I_{\{B_S < 0\}}] \\ &= \mathbb{E}[B_{T_x} I_{\{S \leq H_x\}} I_{\{B_S \geq 0\}}] - \mathbb{E}[B_{T_x} I_{\{S \leq H_x\}} I_{\{B_S < 0\}}] \\ &\leq \mathbb{E}[|B_{T_x}| I_{\{T \leq H_x\}}] \end{aligned}$$

Then, by two applications of Monotone Convergence  $\mathbb{E}[|B_S|] \leq \mathbb{E}[|B_T|] < \infty$ .

To complete the proof we show that for  $A \in \mathcal{F}_S$ ,  $\mathbb{E}[B_S I_A] \geq \mathbb{E}[B_T I_A]$ . Let  $A_x = A \cap \{S \leq H_x\}$ . Then  $\mathbb{E}[B_S I_A] = \lim_x \mathbb{E}[B_{S_x} I_{A_x}] = \lim_x \mathbb{E}[B_{T_x} I_{A_x}]$  by Monotone convergence and the minimality of  $T_x$  respectively. Further,

$$\mathbb{E}[B_{T_x} I_{A_x}] \geq \mathbb{E}[B_{T_x} I_A I_{\{S \leq T \leq H_x\}}] \rightarrow \mathbb{E}[B_T I_A].$$

$\square$

We now turn to the proof of the main result:

*Proof of Theorem 5.* We begin by showing the equivalence of conditions (ii) – (v). It is clear that (ii)  $\implies$  (iii)  $\implies$  (iv)  $\implies$  (v). Hence it only remains to show that (v)  $\implies$  (ii). Suppose (v) holds and choose stopping times  $R \leq S \leq T$  and  $A \in \mathcal{F}_R$ . Set  $A_\gamma = A \cap \{R < H_{-\gamma}\}$ . Since  $B_{t \wedge H_{-\gamma}}$  is a supermartingale

$$\mathbb{E}(B_{S \wedge H_{-\gamma}}; A_\gamma) \leq \mathbb{E}(B_{R \wedge H_{-\gamma}}; A_\gamma). \quad (6)$$

By Lemma 9  $\mathbb{E}|B_R| < \infty$  and by dominated convergence the right hand side converges to  $\mathbb{E}(B_R; A)$  as  $\gamma \rightarrow \infty$ . For the term on the left we consider

$$\mathbb{E}(B_{S \wedge H_{-\gamma}}; A_\gamma) = \mathbb{E}(B_S; A \cap (S < H_{-\gamma})) - \gamma \mathbb{P}(A \cap (R < H_{-\gamma} < S)).$$

Again by Lemma 9 and dominated convergence the first term on the right converges to  $\mathbb{E}(B_S; A)$  while the other term converges to 0 by (v). Hence on letting  $\gamma \rightarrow \infty$  in (6) we have

$$\mathbb{E}(B_S; A) \leq \mathbb{E}(B_R; A)$$

and we have shown (ii).

We have already shown that minimality is equivalent to these conditions: (ii)  $\implies$  (i) is Lemma 8, while (i)  $\implies$  (iv) is Lemma 10.

Now consider (vi). If (2) holds then  $B_S^- \leq \mathbb{E}(B_T | \mathcal{F}_S)^- \leq \mathbb{E}(B_T^- | \mathcal{F}_S)$  and uniform integrability follows. Conversely, (vi) implies  $\sup_{S \leq T} \mathbb{E}(B_S^-; B_S^- \geq \gamma) \rightarrow 0$  as  $\gamma \rightarrow \infty$ . Taking  $S = H_{-\gamma} \wedge T$  yields (v).

The implication (vii) implies (iii) is the content of Lemma 11. We now show (vi) implies (vii). Fix  $x > 0$ . Since  $B_S^-$  is uniformly integrable for all  $S$ , we have that  $B_{t \wedge H_x \wedge T}^-$  is uniformly integrable. Since  $B_{t \wedge H_x \wedge T}$  is also bounded above, it follows that  $B_{t \wedge H_x \wedge T}$  is UI. Hence  $\mathbb{E}[B_{H_x \wedge T}] = \lim_t \mathbb{E}[B_{t \wedge H_x \wedge T}] = 0$ .

We have shown equivalence between (i) – (vii). We are left with showing that if  $\mu$  has support bounded below then (viii) is also equivalent. So assume that the target distribution  $\mu$  has support contained in  $[\alpha, \infty)$  for some  $\alpha < 0$  and that  $T$  is an embedding of  $\mu$ . Then  $B_{t \wedge H_\alpha}$  is a continuous supermartingale, bounded below and therefore if  $S \leq T \leq H_\alpha$ ,

$$\mathbb{E}(B_T | \mathcal{F}_S) \leq B_S.$$

The reverse implication follows from considering the stopping time  $H_{\alpha-\epsilon} = \inf\{t \geq 0 : B_t \leq \alpha - \epsilon\}$ , for then if  $A = \{\omega : H_{\alpha-\epsilon} < T\}$  and  $S = H_{\alpha-\epsilon} \wedge T$ ,

$$(\alpha - \epsilon)\mathbb{P}(A) = \mathbb{E}(B_{H_{\alpha-\epsilon}}; A) = \mathbb{E}(B_S; A) \geq \mathbb{E}(B_T; A) \geq \alpha\mathbb{P}(A)$$

which is only possible if  $\mathbb{P}(A) = 0$ . □

We close this section with a discussion of the case where the mean of the target distribution is well defined, but infinite. In this case the notion of minimality still makes perfect sense, but many of the conditions outlined in Theorem 5 can be shown to be no longer equivalent to minimality.

Suppose that  $\mu$  only places mass on  $\mathbb{R}^-$  and that that  $\mu((-\infty, x)) > (1 + |x|)^{-1}$  for  $x < 0$ . Let  $\phi : \mathbb{R}^- \mapsto \mathbb{R}^-$  solve

$$\mu((-\infty, x)) = \frac{1}{1 + |\phi^{-1}(x)|}.$$

Such a  $\phi$  is an increasing function with  $\phi(x) < x$ .

Define  $J = \inf\{B_u; u \leq H_1\}$ . By construction  $\phi(J) \sim \mu$  and we can construct an embedding of  $\mu$  by setting  $T = \inf\{u > H_1 : B_u = \phi(J)\}$ . Further, at the stopping time  $T$  the Brownian motion is at a current minimum. Hence  $T$  is minimal.

Now consider the alternative conditions in Theorem 5. In reverse order, (vii) and (vi) both clearly fail, whereas (v) may or may not hold depending on the target law  $\mu$ , and (iv) holds trivially. (However (iv) holds for all embeddings of  $\mu$  so that it is not sufficient



for minimality.) The condition in (iii) also holds, since on stopping  $B$  is at a minimum value. However the choice  $R = 0$  and  $S = H_1$  shows that (ii) fails and is not necessary for minimality. Hence, in the case where  $m = \int_{\mathbb{R}} x\mu(dx)$  is well defined but equal to  $-\infty$ , the only condition from the list in Theorem 5 which could possibly be necessary and sufficient for minimality is (iii).

### 3 An extension of the Azéma-Yor embedding to the non-centred case

Let  $\mu$  be a target distribution on  $\mathbb{R}$ , and let  $B_t$  be a Brownian motion with  $B_0 = 0$ . Define  $M_T = \sup_{s \leq T} B_s$  and  $J_T = \inf_{s \leq T} B_s$ . In this section we are interested in finding an embedding to solve the following problem:

Given a Brownian motion  $(B_t)_{t \geq 0}$  and an integrable (but possibly not centred) target distribution  $\mu$  with mean  $m$ , find a minimal stopping time  $T$  such that  $T$  embeds  $\mu$  and

$$\mathbb{P}(M_T \geq x)$$

is maximised over all minimal stopping times  $T$  embedding  $\mu$ , uniformly in  $x$ .

We call an embedding with this property the max-max embedding, and denote it by  $T_{max}$ .

Without some condition on the class of admissible stopping times the problem is clearly degenerate — any stopping time may be improved upon by waiting for the first return of the process to 0 after hitting level  $x$  and then using the original embedding. For this improved embedding  $\mathbb{P}(M_T \geq x) = 1$ . Further, since no almost surely finite stopping time can satisfy  $\mathbb{P}(M_T \geq x) = 1$  for all  $x > 0$ , there can be no solution to the problem above in the class of all embeddings. As a consequence some restriction on the class of admissible stopping times is necessary for us to have a well defined problem.

Various conditions have been proposed in the literature to restrict the class of stopping times. In the case where  $m = 0$ , the condition on  $T$  that  $B_{t \wedge T}$  is a UI martingale was suggested by Dubins and Gilat (1978), and in this case the maximal embedding is the Azema-Yor embedding. When  $m = 0$  Monroe (1972) tells us that minimality and uniform integrability are equivalent conditions, so the Azema-Yor stopping time is the max-max embedding. For the case where  $m > 0$ , Pedersen and Peskir (2001) showed that  $\mathbb{E}M_T < \infty$  is another suitable condition, with the optimal embedding being based on that of Azema and Yor. We argue that the class of minimal embeddings is the appropriate class for the problem under consideration since minimality is a natural and meaningful condition, which makes sense for all  $m$  (and which, for  $m > 0$ , includes as a subclass those embeddings with  $\mathbb{E}(M_T) < \infty$ .)

We now describe the construction of the candidate max-max stopping time. This construction is an extension of the classical Azéma and Yor (1979a) stopping time. However we derive the stopping rule in a slightly non-standard way via a Chacon and Walsh (1976)

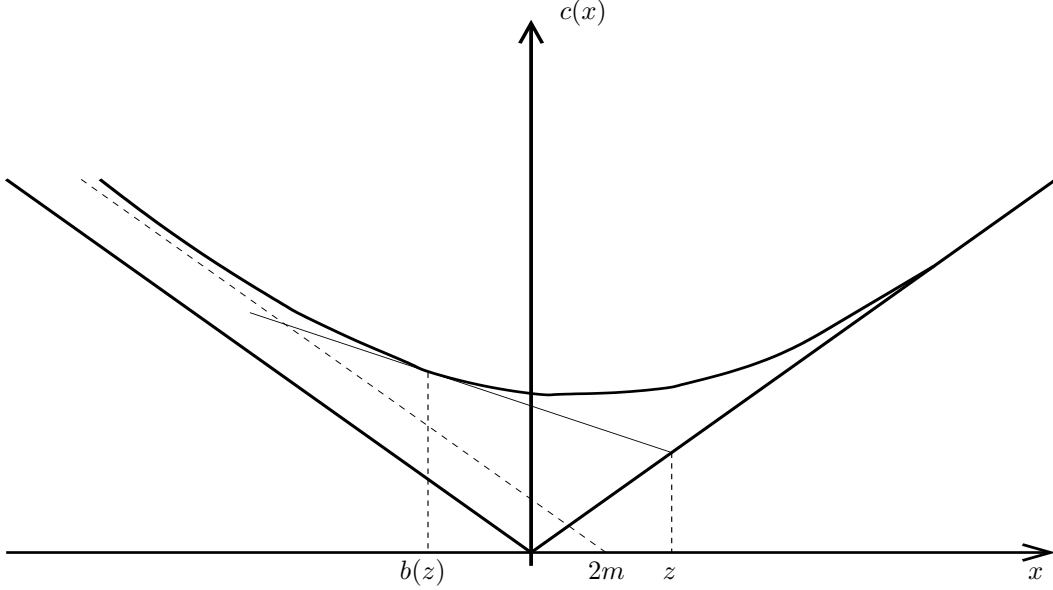


Figure 1:  $c(x)$  for a  $\mu$  with support bounded above, and positive non-zero mean  $m$ . Also shown is an intuitive idea of  $b(z)$ .

style argument. This interpretation of the Azéma-Yor construction (in the centred case) is due to Meilijson (1983).

In the following we treat the cases  $m > 0$ ,  $m = 0$  and  $m < 0$  in one go, since the basic idea is identical. Define the convex function:

$$c(x) := \mathbb{E}^\mu |X - x| + |m|. \quad (7)$$

We note that  $c$  is related to the potential of  $\mu$ , that it is Lebesgue-almost everywhere differentiable with left-derivative  $c'_-(x) = 1 - 2\mu([x, \infty))$  and  $c(x) - |x| \rightarrow |m| \mp m$  as  $x \rightarrow \pm\infty$ . For  $\theta \in [-1, 1]$  define

$$u(\theta) := \inf\{y \in \mathbb{R} : c(y) + \theta(x - y) \leq c(x), \forall x \in \mathbb{R}\},$$

$$z_+(\theta) := \frac{c(u(\theta)) - \theta u(\theta)}{1 - \theta},$$

and (note that  $z_+^{-1}$  is well defined) for  $x \geq 0$

$$b(x) := u(z_+^{-1}(x)).$$

The intuition behind these definitions is as follows. Let  $\theta$  denote a gradient and consider the unique tangent to  $c$  with gradient  $\theta$ . Then  $z_+(\theta)$  is the  $x$ -coordinate of the point where this tangent crosses the line  $y = x$ . Similarly,  $b(x)$  is the  $x$ -coordinate of the point for which the tangent to  $c$  at that point passes through  $(x, x)$ . The mathematical definitions above, and the fact that  $u$  is left-continuous, ensure that  $b$  is well defined. Note that when  $m = 0$ ,

a few lines of calculus are sufficient to check that  $b^{-1}$  is precisely the barycentre function which arises in the classical Azema-Yor construction.

**Theorem 12.** *Let  $T$  be a stopping time of  $(B_t)_{t \geq 0}$  which embeds  $\mu$  and is minimal. Then for  $x \geq 0$*

$$\mathbb{P}(M_T \geq x) \leq \frac{1}{2} \inf_{\lambda < x} \left( \frac{c(\lambda) - \lambda}{x - \lambda} \right). \quad (8)$$

Define the stopping time  $T_{max}$  via

$$T_{max} := \inf\{t > 0 : B_t \leq b(M_t)\}. \quad (9)$$

Then  $T_{max}$  embeds  $\mu$ , is minimal, and attains equality in (8) for all  $x \geq 0$ .

**Remark 13.** (i) By the comments before the theorem relating  $b$  to the barycentre function, when  $m = 0$  the above theorem is a restatement of the classical Azema-Yor result, which by Theorem 2 can be stated in terms of minimal, rather than uniformly integrable embeddings. For  $m > 0$ ,  $b(x) = -\infty$  for  $x < m$ , and consequently  $T_{max} \geq H_m$ . Moreover, for  $x > m$ ,  $b^{-1}$  is a shifted version of the barycentre function. Consequently, when  $m > 0$  the embedding  $T_{max}$  may be thought of as ‘wait until the process hits  $m$  then use the Azema-Yor embedding,’ and the conclusion of Theorem 12 is similar to Proposition 3.1 of Pedersen and Peskir (2001), except that  $T_{max}$  is shown to be optimal amongst the larger class of minimal stopping times rather than the class of embeddings for which  $M_T \in L^1$ . However, the truly original part of the theorem is in the case  $m < 0$ . In that case the embedding ‘wait until the process hits  $m$  and then use the Azema-Yor embedding’ does not achieve equality in (8).

(ii) Note that

$$\frac{c(\lambda) - \lambda}{x - \lambda} = 1 - \frac{x - c(\lambda)}{x - \lambda}. \quad (10)$$

We can relate the right-hand-side of (10) to the slope of a line joining  $(x, x)$  with  $(\lambda, c(\lambda))$ . In taking the infimum over  $\lambda$  we get a tangent to  $c$  and a value for the slope in  $[-1, 1]$ . Thus the bound on the right-hand-side of (8) lies in  $[0, 1]$ .

(iii)  $T_{max}$  has the property that it maximises the law of  $M_T$  over minimal stopping times which embed  $\mu$ . If we want to minimise the law of the minimum, or equivalently we wish to maximise the law of  $-J_T$ , then we can deduce the form of the optimal stopping time by reflecting the problem about 0, or in other words by considering  $-B$ . Let  $T_{min}$  be the embedding which arises in this way, so that amongst the class of minimal stopping times which embed  $\mu$ , the stopping time  $T_{min}$  maximises

$$\mathbb{P}(-J_T \geq x)$$

simultaneously for all  $x \geq 0$ .

We now turn to the proof of Theorem 12.

*Proof.* The following inequality for  $x > 0$ ,  $\lambda < x$  may be verified on a case by case basis:

$$\mathbf{1}_{\{M_T \geq x\}} \leq \frac{1}{x - \lambda} \left[ B_{T \wedge H_x} + \frac{|B_T - \lambda| - (B_T + \lambda)}{2} \right]. \quad (11)$$

In particular, on  $\{M_T < x\}$ , (11) reduces to

$$0 \leq \begin{cases} 0 & \lambda \geq B_T \\ \frac{B_T - \lambda}{x - \lambda} & \lambda < B_T, \end{cases} \quad (12)$$

and on  $\{M_T \geq x\}$  we get

$$1 \leq \begin{cases} \frac{x - B_T}{x - \lambda} & \lambda > B_T \\ 1 & \lambda \leq B_T. \end{cases} \quad (13)$$

Then taking expectations,

$$\mathbb{P}(M_T \geq x) \leq \frac{1}{x - \lambda} \left[ \mathbb{E}(B_{T \wedge H_x}) + \frac{c(\lambda) - |m| - (m + \lambda)}{2} \right]. \quad (14)$$

If  $m \leq 0$  then by Theorem 5(vii) and the minimality of  $T$  we know  $\mathbb{E}(B_{T \wedge H_x}) = 0$  and so

$$\mathbb{P}(M_T \geq x) \leq \frac{1}{2} \frac{c(\lambda) - \lambda}{x - \lambda}.$$

Conversely if  $m > 0$ , by Theorem 5(iii) applied to  $-B$ ,

$$m = \mathbb{E}(B_T) \geq \mathbb{E}(B_{T \wedge H_x}) \quad (15)$$

and so

$$\mathbb{P}(M_T \geq x) \leq \frac{1}{x - \lambda} \left[ m + \frac{c(\lambda) - 2m - \lambda}{2} \right] = \frac{1}{2} \frac{c(\lambda) - \lambda}{x - \lambda}.$$

Since  $\lambda$  was arbitrary in either case, (8) must hold.

It remains to show that  $T_{max}$  attains equality in (8), embeds  $\mu$  and is minimal. We begin by showing that it does attain equality in (8). Since

$$\frac{c(\lambda) - \lambda}{x - \lambda} = 1 + \frac{c(\lambda) - x}{x - \lambda}$$

the infimum in (8) is attained by a value  $\lambda^*$  with the property that a tangent of  $c$  at  $\lambda^*$  intersects the line  $y = x$  at  $(x, x)$ . By the definition of  $b$  we can choose  $\lambda^* = b(x)$ . In particular, since  $\{M_{T_{max}} < x\} \subseteq \{B_{T_{max}} \leq b(x)\}$  and  $\{M_{T_{max}} \geq x\} \subseteq \{B_{T_{max}} \geq b(x)\}$ , the stopping time  $T_{max}$  attains equality almost surely in (12) and (13). Assuming that  $T_{max}$  is minimal, we are then done for  $m \leq 0$ . If  $m > 0$  we do not always have equality in (15). If  $x < m$  then  $\mathbb{E}(B_{T_{max} \wedge H_x}) = x$ , but then  $\lambda^* = -\infty$  and so the term  $\mathbb{E}(B_{T_{max} \wedge H_x}) - m)/(x - \lambda^*)$  in (14) is zero. As a result equality is again attained in (8). Otherwise, if  $x \geq m$  then  $T_{max} \geq H_m$  and the properties of the classical Azema-Yor embedding ensure that  $\mathbb{E}(B_{T_{max} \wedge H_x}) = m$  and there is equality both in (15) and (8).

Fix a value of  $y$  which is less than the supremum of the support of  $\mu$ . Let  $b^{-1}$  be the left-continuous inverse of  $b$ . Then, since we have equality in (8), we deduce:

$$\begin{aligned} \mathbb{P}(B_{T_{max}} \geq y) &= \mathbb{P}(M_{T_{max}} \geq b^{-1}(y)) \\ &= \frac{1}{2} \left[ 1 + \frac{c(b^{-1}(y)) - b^{-1}(y)}{b^{-1}(y) - b(b^{-1}(y))} \right] \\ &= \frac{1}{2}(1 - c'_-(y)) \end{aligned}$$

where  $c'_-$  is the left-derivative of  $c$ . It is easy to see from the definition (7) that this last expression equals  $\mu([y, \infty))$ . Hence  $T_{max}$  embeds  $\mu$ .

Now we consider minimality of  $T_{max}$ . It is well known that in the centred case  $T_{max}$  is uniformly integrable, and hence by Theorem 2,  $T_{max}$  is minimal. Suppose  $m > 0$ . By an analogue of Theorem 5(v), in order to prove minimality it is sufficient to show that  $\lim_{x \uparrow \infty} x \mathbb{P}(T_{max} > H_x) \rightarrow 0$ . But, by the arguments in the previous paragraph

$$2x \mathbb{P}(T_{max} > H_x) = 2x \mathbb{P}(M_{T_{max}} > x) = x [1 - c'_-(b(x))]$$

This last quantity is exactly the height above 0, when it crosses the  $y$ -axis, of the tangent to  $c$  which passes through  $(x, x)$ . As  $x \uparrow \infty$  this height decreases to zero.

Now suppose  $m < 0$ . By Theorem 5(v), in order to prove minimality it is sufficient to show that as  $x \downarrow -\infty$ ,  $|x| \mathbb{P}(T_{max} > H_x) \rightarrow 0$ . We have

$$|x| \mathbb{P}(T_{max} > H_x) = |x| \mathbb{P}(H_x < H_{b^{-1}(x)}) = \frac{|x| b^{-1}(x)}{|x| + b^{-1}(x)} < b^{-1}(x).$$

It is easy to see from the representation of  $b$  that  $b^{-1}(x)$  tends to zero as  $x \rightarrow -\infty$ .  $\square$

## 4 An embedding to maximise the modulus

Jacka (1988) shows how to embed a centred probability distribution in a Brownian motion so as to maximise  $\mathbb{P}(\sup_{t \leq T} |B_t| \geq y)$ . Our goal in this section is to extend this result to allow for non-centered target distributions with mean  $m \neq 0$ . In fact we solve a slightly more general problem. Let  $h$  be a measurable function; we will construct a stopping time  $T_{mod}$  which will maximise  $\mathbb{P}(\sup_{t \leq T} |h(B_t)| \geq y)$  simultaneously for all  $y$  where the maximum is taken over the class of all minimal stopping times which embed  $\mu$ . The reason for our generalisation will become apparent in the application in the next section.

Without loss of generality we may assume that  $h$  is a non-negative function with  $h(0) = 0$  and such that for  $x > 0$  both  $h(x)$  and  $h(-x)$  are increasing. To see this, observe that for arbitrary  $h$  we can define the function

$$\tilde{h}(x) = \begin{cases} \max_{0 \leq y \leq x} |h(y)| - |h(0)| & x \geq 0; \\ \max_{x \leq y \leq 0} |h(y)| - |h(0)| & x < 0. \end{cases}$$

Then  $\tilde{h}$  has the desired properties and since

$$\sup_{s \leq T} |h(B_s)| = \sup_{s \leq T} \tilde{h}(B_s) + |h(0)|$$

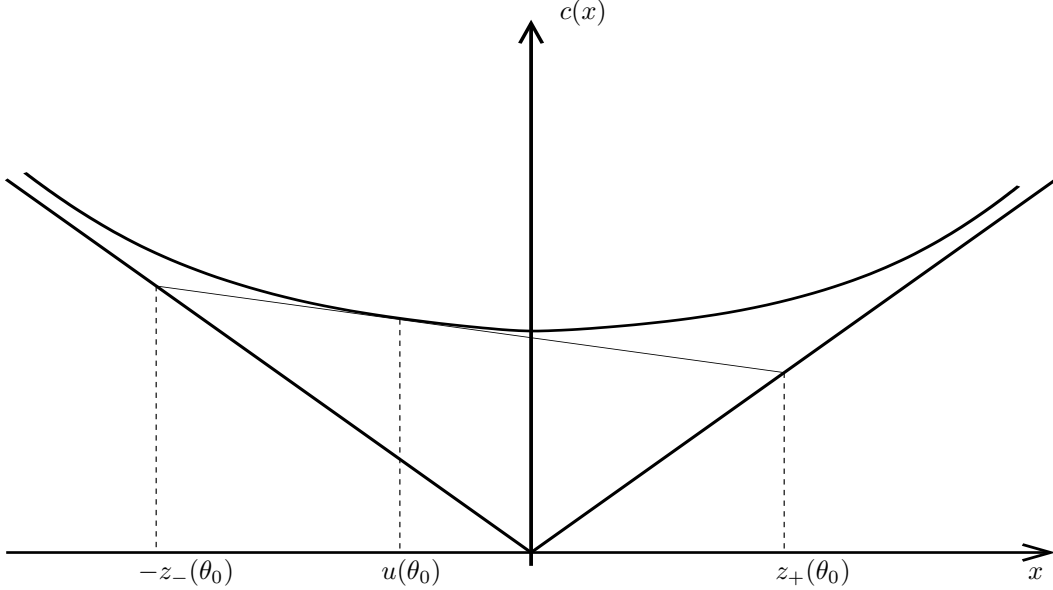


Figure 2:  $c(x)$  for a distribution  $\mu$  showing the construction of  $z_+(\theta_0)$  and  $z_-(\theta_0)$ . The slope of the tangent is  $\theta_0$  where  $\theta_0$  has been chosen such that (assuming  $h$  is continuous)  $h(z_+(\theta_0)) = h(-z_-(\theta_0))$ .

the optimal embedding for  $\tilde{h}$  will be an optimal embedding for  $h$ .

So suppose that  $h$  has the properties listed above. We want to find an embedding of  $\mu$  in  $B$  which is minimal and which maximises the law of  $\sup_{t \leq T} h(B_t)$ . (Since  $h$  is non-negative we can drop the modulus signs.) Suppose also for definiteness that  $\mu$  has a finite, positive mean  $m = \int_{\mathbb{R}} x\mu(dx) > 0$ . In fact our construction will also be optimal when  $m = 0$  (the case covered by Jacka (1988)), but in order to avoid having to give special proofs for this case we will omit it.

Recall the definitions of  $c$  and  $u$  from the previous section. Define

$$z_+(\theta) := \frac{c(u(\theta)) - \theta u(\theta)}{1 - \theta}, \quad (16)$$

$$z_-(\theta) := \frac{c(u(\theta)) - \theta u(\theta)}{1 + \theta}, \quad (17)$$

so that  $-z_-(\theta)$  is the  $x$ -coordinate of the point where the tangent to  $c$  with slope  $\theta$  intersects the line  $y = -x$ , and set

$$\theta_0 := \inf\{\theta \in [-1, 1] : h(z_+(\theta)) \geq h(-z_-(\theta))\},$$

as pictured in Figure 2. Our optimal stopping time will take the following form. Run the process until it hits either  $z_+(\theta_0)$  or  $-z_-(\theta_0)$ , and then embed the restriction of  $\mu$  to  $[u(\theta_0), \infty)$  or  $(-\infty, u(\theta_0)]$  respectively (defining the target measures more carefully when there is an atom at  $u(\theta_0)$ ). For the embeddings in the second part, we will use the constructions described in Section 3.

To be more precise about the measures we embed in the second step, define

$$p := \mathbb{P}(H_{z_+(\theta_0)} < H_{-z_-(\theta_0)}) = \frac{z_-(\theta_0)}{z_+(\theta_0) + z_-(\theta_0)},$$

and note

$$\theta_0 = \frac{z_+(\theta_0) - z_-(\theta_0)}{z_+(\theta_0) + z_-(\theta_0)} = 1 - 2p.$$

Let  $\mu_+$  (respectively  $\mu_-$ ) be the measure obtained by conditioning a random variable with law  $\mu$  to lie in the upper  $p^{\text{th}}$  (respectively lower  $(1-p)^{\text{th}}$ ) quantile of its distribution.

Recall that we have taken  $m > 0$  and that

$$c(y) = \int |w - y| \mu(dw) + |m| = 2 \int_{\{w > y\}} (w - y) \mu(dw) + y \quad (18)$$

$$= 2 \int_{\{w < y\}} (y - w) \mu(dw) + 2m - y \quad (19)$$

Then using (18) in the definition (16) we have that

$$\begin{aligned} z_+(\theta_0) &= \frac{1}{2p} \left( 2 \int_{\{w > u(\theta_0)\}} (w - u(\theta_0)) \mu(dw) + 2pu(\theta_0) \right) \\ &= \frac{1}{p} \int_{\{w > u(\theta_0)\}} w \mu(dw) + u(\theta_0) \left( 1 - \frac{1}{p} \mu((u(\theta_0), \infty)) \right) \end{aligned}$$

In particular  $z_+(\theta_0)$  is the mean of  $\mu_+$ , since  $\mu_+(\{u(\theta_0)\}) = 1 - \frac{1}{p} \mu((u(\theta_0), \infty))$ . When we repeat the calculation for  $z_-(\theta_0)$  using (19) we find that

$$-z_-(\theta_0) = \frac{1}{1-p} \int_{\{w < u(\theta_0)\}} w \mu(dw) + u(\theta_0) \left( 1 - \frac{1}{1-p} \mu((-\infty, u(\theta_0))) \right) - \frac{m}{(1-p)}.$$

Note that  $-z_-(\theta_0)$  is strictly smaller than the mean of  $\mu_-$ .

We now describe the candidate stopping time  $T_{mod} \equiv T_{mod}^h$ . Note that this stopping time will depend implicitly on the function  $h$  via  $z_{\pm}(\theta_0)$ . Let

$$T_0 := \inf\{t > 0 : B_t \notin (-z_-(\theta_0), z_+(\theta_0))\},$$

and define

$$T_{mod} := \begin{cases} T_{max}^{\mu_+} \circ \theta_{T_0} & B_{T_0} = z_+(\theta_0) \\ T_{min}^{\mu_-} \circ \theta_{T_0} & B_{T_0} = -z_-(\theta_0). \end{cases}$$

Here we use  $\theta_{T_0}$  to denote the shift operator, and  $T_{max}^{\mu_+}$  is the stopping time constructed in Section 3 for a zero-mean target distribution, so that  $T_{max}^{\mu_+}$  is a standard Azema-Yor embedding of the centred target law  $\mu_+$ . (Recall that  $z_+(\theta_0)$  is the mean of the corresponding part of the target distribution.) Similarly  $T_{min}^{\mu_-}$  is the stopping time applied to  $-B$  started at  $-z_-(\theta_0)$  which maximises the law of the maximum of  $-B$ . In this case the mean of the target law  $\mu_-$  is larger than  $-z_-(\theta_0)$  so that in order to define  $T_{min}^{\mu_-}$  we need to use the full content of Section 3 for embeddings of non-centred distributions.

The following theorem asserts that this embedding is indeed an embedding of  $\mu$ , that it is minimal, and that it has the claimed optimality property.

**Theorem 14.** *Let  $\mu$  be a target distribution such that  $m > 0$ . Then within the class of minimal embeddings of  $\mu$  in  $B$ , the embedding  $T_{mod}$  as defined above has the property that it maximises*

$$\mathbb{P}\left(\sup_{t \leq T} h(B_t) \geq x\right)$$

*simultaneously for all  $x$ .*

*Proof.* By construction  $T_{mod}$  embeds  $\mu$ . We need only show that it is optimal and minimal.

For  $x \leq h(-z_-(\theta_0)) \wedge h(z_+(\theta_0))$  we know the probability that  $\{\sup_{t \leq T_{mod}} h(B_t) \geq x\}$  is one and so, for such  $x$ ,  $T_{mod}$  is clearly optimal. Indeed if  $h$  is discontinuous at  $-z_-(\theta_0)$  or  $z_+(\theta_0)$  slightly more can be said. Note first that if  $z_+(\theta_0)$  coincides with the supremum of the support of  $\mu$ , then by Theorem 5(iii) and the minimality of  $T_{mod}$  (see below), the stopped Brownian motion can never go above  $z_+(\theta_0)$ . Hence we may assume that  $h$  is constant on the interval to the right of the supremum of its support.

Define

$$L = \left(\lim_{y \uparrow -z_-(\theta_0)} h(y)\right) \wedge \left(\lim_{y \downarrow z_+(\theta_0)} h(y)\right)$$

and take  $x \leq L$ . Then either  $B_{T_0} = z_+(\theta_0)$  or  $B_{T_0} = -z_-(\theta_0)$ . If  $B_{T_0} = z_+(\theta_0)$  then either  $M_{T_{mod}} > z_+(\theta_0)$  almost surely and

$$\max_{0 \leq t \leq T_{mod}} h(B_t) \geq \lim_{y \downarrow z_+(\theta_0)} h(y) \geq L$$

or  $z_+(\theta_0)$  is the supremum of the support of  $\mu$  and

$$\max_{0 \leq t \leq T_{mod}} h(B_t) = h(B_{T_0}) \geq L.$$

Similar considerations apply for  $B_{T_0} = -z_-(\theta_0)$  except that then  $-J_{T_{mod}} > z_-(\theta_0)$  in all cases. We deduce that for  $x \leq L$

$$\mathbb{P}\left(\sup_{t \leq T_{mod}} h(B_t) \geq x\right) = 1$$

and hence  $T_{mod}$  is optimal for such  $x$ .

So suppose that  $x > L$ . For any stopping time  $T$  embedding  $\mu$ , the following holds:

$$\mathbb{P}\left(\sup_{s \leq T} h(B_s) \geq x\right) \leq \mathbb{P}(h(M_T) \geq x) + \mathbb{P}(h(J_T) \geq x). \quad (20)$$

We will show that the embedding  $T_{mod}$  attains the maximal values of both terms on the right hand side, and further that for  $T_{mod}$  the two events on the right hand side are disjoint. Hence  $T_{mod}$  is optimal.

By the definition of  $\theta_0$ ,  $x > (h(z_+(\theta_0))) \vee (h(-z_-(\theta_0)))$ . It follows that

$$\mathbb{P}(h(M_{T_{mod}}) \geq x) = p\mathbb{P}(h(M_{T_{mod}}) \geq x | B_{T_0} = z_+(\theta_0))$$

and by the definition of  $T_{mod}$  and the properties of  $T_{max}$ , we deduce

$$\begin{aligned} \mathbb{P}(h(M_{T_{mod}}) \geq x) &= p\mathbb{P}(h(M_{T_{max}^\mu}) \geq x | M_{T_{max}^\mu} \geq z_+(\theta_0)) \\ &= \mathbb{P}(h(M_{T_{max}^\mu}) \geq x) \end{aligned}$$



where here  $T_{max}^\mu$  is the embedding of Section 3 applied to  $\mu$ . A similar calculation can be done for the minimum. In particular  $T_{mod}$  inherits its optimality property from the optimality of its constituent parts  $T_{max}^{\mu+}$  and  $T_{min}^{\mu-}$ .

Finally we note that  $T_{mod}$  is indeed minimal. Consider the family of stopping times  $S \leq T_{mod}$ . On the set where  $B_{T_0} = -z_-(\theta_0)$  we have that  $B_S^+$  is bounded, whereas on the set  $B_{T_0} = -z_-(\theta_0)$  the minimality of  $T_{min}^{\mu-}$  ensures that  $B_S^+$  is uniformly integrable. Hence, combining these two cases,  $B_S^+$  is a uniformly integrable family and the minimality of  $T_{mod}$  follows from 5(vi) applied to a target distribution with positive mean.  $\square$

**Remark 15.** If the restrictions of  $h$  to  $\mathbb{R}_+$  and  $\mathbb{R}_-$  are strictly increasing then  $T_{mod}$  will be essentially the unique embedding which attains optimality in Theorem 14. If however  $h$  has intervals of constancy then other embeddings may also maximise the law of  $\sup_{t \leq T} |h(B_t)|$ .

## 5 Embeddings in diffusions

Our original motivation in considering the embeddings of the previous sections was their use in the investigation of the following question:

Given a regular (time-homogeneous) diffusion  $(X_t)_{t \geq 0}$  and a target distribution  $\mu_X$ , find (if possible) a minimal stopping time which embeds  $\mu_X$  and which maximises the law of  $\sup_{t \leq T} X_t$  (alternatively  $\sup_{t \leq T} |X_t|$ ) among all such stopping times.

Note that in the martingale (or Brownian) case it is natural to consider centred target laws, at least in the first instance. However in the non-martingale case this restriction is no longer natural, and as we shall see below is completely unrelated to whether it is possible to embed the target law in the diffusion  $X$ . It was this observation which led us to consider the problem of embedding non-centred distributions in  $B$ .

The key idea (see Azéma and Yor (1979b)) is that we can relate the problem of embedding in a diffusion to the case where we are dealing with Brownian motion via the scale function. There exists a continuous, increasing function  $s$  such that  $s(X_t)$  is a local martingale, and hence a time-change of Brownian motion. Then the requirement  $X_{T^x} \sim \mu_X$  translates to finding an embedding of a related law in a Brownian motion  $B$ , and the criterion of maximising  $\sup_{t \leq T} X_t$  also has an equivalent statement in terms of  $B$ .

We first recall the properties of the scale function (see e.g. Rogers and Williams (2000, V.45)). If  $(X_t)_{t \geq 0}$  is a regular (time-homogeneous) diffusion on an interval  $I \subseteq \mathbb{R}$  with absorbing or inaccessible endpoints and vanishing at zero, then there exists a continuous, strictly increasing scale function  $s : I \rightarrow \mathbb{R}$  such that  $Y_t = s(X_t)$  is a diffusion in natural scale on  $s(I)$ . We may also choose  $s$  such that  $s(0) = 0$ . In particular  $Y_t$  is (up to exit from the interior of  $s(I)$ ) a time change of a Brownian motion with strictly positive speed measure. For definiteness we write  $Y_t = B_{\tau_t}$ .

We suppose also that our target distribution  $\mu_X$  is concentrated on the interior of  $I$ . Then we may define a measure  $\mu = \mu_Y$  on  $s(I)^\circ$  by:

$$\mu(A) = \mu_Y(A) = \mu_X(s^{-1}(A)), \quad A \in s(I)^\circ, \text{ Borel}$$

The original problem of embedding  $\mu_X$  in  $X$  is equivalent to the problem of embedding  $\mu_Y$  in  $Y$  before  $Y$  exits the interval  $s(I)^\circ$ . Since  $Y$  is a time change of a Brownian motion, we need only consider the problem of embedding  $\mu_Y$  in a Brownian motion before exit from  $s(I)^\circ$ . If  $T$  is an embedding of  $\mu$  in  $B$  then  $T^X \equiv \tau^{-1}(T)$  is simultaneously an embedding of  $\mu_Y$  in  $Y$  and  $\mu_X$  in  $X$ .

The first question is when does any embedding exist? If we define  $m = \int_I s(x)\mu_X(dx)$ , then the following lemma (see Pedersen and Peskir (2001) and Cox and Hobson (2004)) gives us necessary and sufficient conditions for an embedding to exist.

**Lemma 16.** *There are three different cases:*

- (i)  $s(I)^\circ = \mathbb{R}$ , in which case  $X$  is recurrent and we can embed any distribution  $\mu_X$  on  $I^\circ$  in  $X$ .
- (ii)  $s(I)^\circ = (-\infty, \alpha)$  (respectively  $(-\alpha, \infty)$ ) for some  $\alpha > 0$ . Then we may embed  $\mu_X$  in  $X$  if and only if  $m$  exists and  $m \geq 0$  (resp.  $m \leq 0$ ).
- (iii)  $s(I)^\circ = (\alpha, \beta)$ ,  $\alpha < 0 < \beta$ . Then we may embed  $\mu_X$  in  $X$  if and only if  $m = 0$ .

In each case it is clear that:

$$T^X \text{ is minimal for } X \iff T^X \text{ is minimal for } Y \iff T \text{ is minimal for } B,$$

where  $T = \tau(T^X)$ . Further, since  $\mu = \mu_Y$  is concentrated on  $s(I)^\circ$ , if the stopping time  $T$  is minimal then  $T$  will occur before the Brownian motion leaves  $s(I)^\circ$  (this is a consequence of Theorem 5 in case (ii) and Theorem 2 in case (iii)), and then  $T^X$  will be less than the first explosion time of  $X$ .

It is now possible to apply the results of previous sections to deduce a series of corollaries about embeddings of  $\mu_X$  in  $X$ . Suppose that  $\mu_X$  can be embedded in  $X$  or equivalently that  $\mu_Y$  can be embedded in  $B$ , before the Brownian motion leaves  $s(I)^\circ$ . Suppose further that in the recurrent case where  $s(I)^\circ = \mathbb{R}$  the law  $\mu = \mu_Y$  is integrable. Let  $T_{max}$  and  $T_{mod}^h$  be the optimal embeddings of  $\mu$  in  $B$  as defined in Sections 3 and 4. (Observe that from now on we make the dependence of  $T_{mod}^h$  on  $h$  explicit in the notation.) Then we can define  $T_{max}^X$  and  $T_{mod}^{X,h}$  by

$$T_{max}^X = \tau^{-1} \circ T_{max} \quad T_{mod}^{X,h} = \tau^{-1} \circ T_{mod}^h.$$

**Corollary 17.**  $T_{max}^X$  is optimal in the class of minimal embeddings of  $\mu_X$  in  $X$  in the sense that it maximises

$$\mathbb{P} \left( \max_{t \leq T} X_t \geq y \right)$$

uniformly in  $y \geq 0$ .

**Corollary 18.**  $T_{mod}^{X,h}$  is optimal in the class of minimal embeddings of  $\mu_X$  in  $X$  in the sense that it maximises

$$\mathbb{P}\left(\max_{t \leq T}(h \circ s)(X_t) \geq y\right)$$

uniformly in  $y \geq 0$ .

**Corollary 19.**  $T_{mod}^{X,|s^{-1}|}$  is optimal in the class of minimal embeddings of  $\mu_X$  in  $X$  in the sense that it maximises

$$\mathbb{P}\left(\max_{t \leq T}|X_t| \geq y\right)$$

uniformly in  $y \geq 0$ .

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