Optimal stopping of the maximum process: a converse to the results of Peskir

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Abstract

Peskir, (and also Meilijson and Obłój) considered the following optimal stopping problem: find, for an increasing function F and a positive function λ,

$$\sup_{\tau} \mathbb{E}\left[F(S_\tau) - \int_0^\tau \lambda(B_u)du\right], \tag{1}$$

where S is the maximum process of Brownian motion. In this article we are interested in the converse: find, for an increasing function F and a suitable function λ,

$$\sup_{\tau} \mathbb{E}\left[\int_0^\tau \lambda(B_u)du - F(S_\tau)\right].$$

In the non-degenerate cases the optimal stopping rule is of the form stop the first time that S_t reaches γ or B_t falls below $g(S_t)$ where γ, a positive constant, and g, a negative function, are both to be chosen. The optimal function g is characterised as the solution to non-linear differential equation, which is very similar to that used by Peskir to characterise the solution to (1), however we derive this differential equation in a completely different way.

1 Introduction

Given a one-dimensional time-homogeneous diffusion X, Peskir [9] (see also Meilijson [5] and Obłój [6]) studies the problem of finding

$$\sup_{\tau} \mathbb{E}\left[F(S^X_\tau) - \int_0^\tau \lambda(X_s)ds\right] \tag{2}$$

where $S^X_u = \sup_{s\le u}\{X_s\}$, F is an increasing reward function, λ is a positive cost function, and the supremum is taken over a suitable class of stopping times for which the expected value is well defined. By a change of scale and a time-change, the problem can be reduced to the Brownian case $X \equiv B$, whence we write S as shorthand for S^B. Peskir finds that the optimal stopping rule

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is the first exit time of the bivariate process (B_t, S_t) from a region, and that this region can be characterised by its boundary function which is the solution to a first-order ordinary differential equation. The optimal stopping rule is a member of the Azéma-Yor class of stopping times (see Azéma-Yor [1]). This is not unexpected since amongst the stopping rules for which the law of B_τ is fixed, the Azéma-Yor stopping rule maximises (the law of) the maximum.

In this article we consider a converse to (2): we wish to find, for F an increasing continuous function,

$$\sup_\tau \mathbb{E}\left[\int_0^\tau \lambda(B_s)ds - F(S_\tau)\right]. \tag{3}$$

(We show in Section 4.2 how the solution to this problem can be extended to the diffusion case.) If λ is non-positive then clearly $\tau = 0$ is optimal. Conversely, if λ is non-negative then it would never be optimal to stop when $B_t < S_t$, and the optimal stopping rule, if any, would be to set τ equal to the first hitting time of some judiciously chosen positive level. Instead, to get an interesting class of non-degenerate problems we assume that λ is positive on the positive half-line, and negative of the negative half-line. Clearly one case of interest is when λ is anti-symmetric. At first sight, given that there is an additional cost associated with the maximum process, it looks as if the anti-symmetry must imply that the optimal stopping rule must be to stop instantly — however this need not always be the case, and the option to stop can lead to positive value.

In solving (2) for $X = B$ and with $F(s) = s$ and $\lambda > 0$, Peskir finds that the optimising stopping rule is of the form $\tau = \inf_u\{B_u \leq g(S_u)\}$ for some increasing function g which solves the ordinary differential equation

$$\frac{d}{ds}g(s) = \frac{1}{2\lambda(g(s))(s - g(s))}. \tag{4}$$

It remains to determine the initial point $g(0)$. Peskir determines this point by the *maximality principle*, namely that the optimal stopping rule is characterised by the largest $g(0)$ such that $g(s) < s$ for all s. As Peskir observes, the maximality principle is a convenient reformulation of the notion that the value function is associated with the smallest superharmonic majorant of the payoff.

In contrast, our solution of (3) takes the form $\tau = \inf_u\{B_u \leq g(S_u) \text{ or } S_u \geq \gamma\}$ for some negative function g and non-negative γ. Here g solves the same differential equation (4), except that now g is decreasing, since λ is negative on the negative real line. Again, it is necessary to make an appropriate choice from the family of solutions to (4). On this occasion each solution is associated with a level γ at which stopping occurs at the maximum, and the problem is to choose the optimal γ. Whereas the stopping rules for the problem (2) can be identified with the Azéma-Yor solution to the Skorokhod embedding problem, our solution to (3) can be related to the Perkins [8] solution of the Skorokhod problem, in the sense that as for the Perkins embedding the optimal stopping rule involves stopping only when the process reaches a new maximum, or a new minimum. However the parallel is imprecise, and not all Perkins-style embeddings can arise from solutions to (3).

Our method of solution and general approach is different to that in Peskir [9]. Peskir writes down the Hamilton-Jacobi-Bellman equation for the value function and invokes the principle of smooth fit to derive the equation (4) which characterises the optimal stopping rule. The proof is completed via a verification argument. Instead, we use excursion theoretic arguments to write down the value function for a class of stopping rules, we then find the maximum value via calculus of variations. Again this gives us a candidate optimal stopping strategy. Our arguments could also be applied to the original problem (2) and vice-versa. A third approach would be to use duality and Azéma martingales.

The first solution of a problem of the type (2) was given by Dubins and Schwarz [3] in the case $X \equiv B$, $F(s) = s$ and $\lambda(x) = \lambda > 0$. This was extended to Bessel processes in Dubins et al [4], and to more general diffusions by Peskir [9]. Peskir restricts attention to the case $F(s) = s$, and continuous non-negative functions λ, but his results extend to more general increasing continuous functions F by a suitable transformation. Meilijson [5] treats the case of drifting Brownian motion and fixed positive costs $\lambda > 0$, but allows for discontinuous functions F. Obłój [6] combines and extends these two frameworks to give a solution for a general increasing reward function F and fairly general cost functions λ.

The remainder of the paper is organised as follows. In the next section we discuss the intuition behind the choice of the problem, and the form of the optimal solution. We show that it is possible to write down the value function for a certain class of strategies, and that it is possible to find the form of the optimal solution by considering a perturbation about this optimum. In Section 3 we verify that the candidate solution discovered in this way is indeed optimal, and in the final Section we give some examples and extensions.

2 Calculus of variations and the optimal stopping rule.

In this section we describe a class of stopping rules, and give the intuition as to why the optimal rule should belong to this class. For each element of this class we can write down the associated value function, and then using the calculus of variations we can find the optimal element of the class.

Our problem is to find

$$\sup_{\tau} \mathbb{E}\left[\int_0^{\tau} \lambda(B_s)ds - F(S_\tau)\right] \tag{5}$$

for suitable functions λ and F. Alternatively we can consider the objective function in (3) but with the supremum replaced by an infimum.

Assumption 2.1 *λ is a bounded measurable function such that $x\lambda(x) \geq 0$, such that $\lambda(x) < 0$ for $x < 0$ and such that $\int_x^0 y\lambda(y)dy$ increases to infinity as x tends to minus infinity. F is a continuously differentiable, increasing function $F(s) = \int_0^s f(y)dy$, where f is bounded.*

Write $r(x) = \lambda^+(x)$ and $c(x) = \lambda^-(x)$ and define $R_t = \int_0^t \lambda(B_s)I_{\{B_s>0\}}ds = \int_0^t r(B_s)ds$ and $C_t = -\int_0^t \lambda(B_s)I_{\{B_s<0\}}ds = \int_0^t c(B_s)ds$. Then R_t and C_t are increasing (reward and cost) functions and the problem becomes to find

$$\sup_{\tau:\mathbb{E}[C_\tau+F(S_\tau)]<\infty} \mathbb{E}\left[R_\tau - C_\tau - F(S_\tau)\right], \tag{6}$$

where we restrict attention to stopping times τ such that $\mathbb{E}[C_\tau + F(S_\tau)] < \infty$ to ensure that the expression is well defined.

If $0 < B_t < S_t$ then $R_u - C_u - F(S_u)$ is non-decreasing, at least until B first hits zero or reaches its current maximum. For this reason it is never optimal to stop if B is positive but strictly below its maximum (unless r is identically zero on the interval $(0, S_t)$). Now suppose $B_t = S_t$. If we

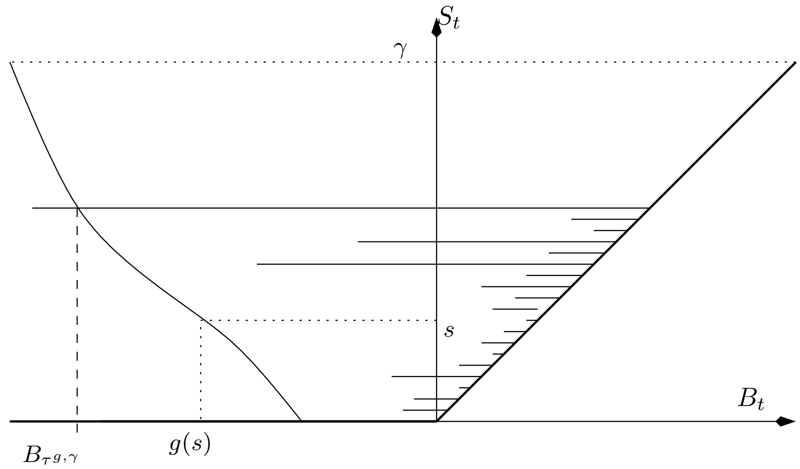

Figure 1: Representation of the general stopping rule $\tau^{g,\gamma}$.

continue then the risk that the maximum will increase is counterbalanced by the fact that there is a non-negative reward from continuing. Hence, depending on the current value of $B \equiv S$, and also on λ it may or may not be optimal to stop. By the strong Markov property, at such a moment it is optimal to stop with probability zero or one, and the optimal stopping rule never depends on the path-history to date except through the current maximum. Thus, for $B > 0$ it is optimal to stop the first time B hits some level γ, where γ, possibly infinite, is to be determined.

Now suppose $B < 0$. Then $\lambda < 0$, so in the short term $R_u - C_u - F(S_u)$ is decreasing. This suggests that stopping immediately may be advantageous. The reason why this may not be optimal is that in the future B will return to the positive half-space where the rewards are positive and the functional is increasing, at least until B reaches a new maximum. When $B < 0$, the expected cost until B hits 0 is decreasing in the current value, and hence it is clear that for fixed S the optimal strategy must be of the form: stop if $B_t < g$ for some threshold $g \equiv g(S_t)$. Finally, if we compare the location of this threshold for different values of the current maximum, we see that when S is large, the costs associated with expected increases in S will be small, and so there is less incentive to stop. In particular g should be a decreasing function of S.

The above discussion is motivation to consider stopping rules of the form

$$\tau(g, \gamma) \equiv \tau^{g,\gamma} = \inf\{u : B_u \leq g(S_u)\} \wedge H_\gamma. \tag{7}$$

where $\gamma > 0$, g is a decreasing negative function and H denotes the first hitting time: $H_s = \inf\{u : B_u = s\}$. Let $W^{g,\gamma} = \mathbb{E}[\int_0^{\tau(g,\gamma)} \lambda(B_s)ds - F(S_{\tau(g,\gamma)})]$, and define

$$V^{g,\gamma}(x, s) = \mathbb{E}^{x,s}\left[\int_0^{\tau(g,\gamma)} \lambda(B_s)ds - F(S_{\tau(g,\gamma)})\right], \tag{8}$$

where the superscript (x, s) is the initial value of the two-dimensional Markov process (B_u, S_u).

Let $\mathcal{T}$ be the set of all stopping times for which $\mathbb{E}[C_\tau + F(S_\tau)] < \infty$. For $s > 0$ let $\mathcal{T}_s$ be the subset of $\mathcal{T}$ for which $\tau \leq H_s$. Finally, let $\mathcal{S}_s$ be the subset of $\mathcal{T}_s$ consisting of those stopping times for which $(\tau < H_s) \equiv (B_\tau < 0)$. Thus for $\tau \in \mathcal{S}_s$ stopping is only allowed when the Brownian motion takes a negative value, or when the Brownian motion first hits s.

For fixed positive γ we begin by finding the optimal rule in (5) when the infimum is calculated over stopping rules which are elements of $\mathcal{S}_\gamma$. We then show how to choose the best $\gamma = \gamma^*$, and extend the result to the set of admissible stopping rules $\mathcal{T}_{\gamma^*}$. Finally, we show that this stopping rules leads to a solution which is globally optimal in $\mathcal{T}$.

2.1 The value function for a stopping rule of the given form.

Suppose g and γ are fixed, and define $\Gamma^g(s) = \int_0^s dz/(z-g(z))$. In this section, since g is fixed, we abbreviate Γ^g to Γ.

Lemma 2.2 *(i) For $0 < x < \gamma$, $\mathbb{P}(\tau(g,\gamma) > H_x) = e^{-\Gamma(x)}$.*
(ii)

$$W^{g,\gamma} = \int_0^\gamma e^{-\Gamma(y)} \frac{dy}{y-g(y)} \left\{ 2\int_{g(y)}^y \lambda(z)(z-g(y))dz - F(y) \right\} - F(\gamma)e^{-\Gamma(\gamma)}. \tag{9}$$

(iii) For $s \le \gamma$ and $x \le g(s)$, we have $V^{g,\gamma}(x,s) = -F(s)$. For $s \le \gamma$ and $x > g(s)$,

$$\begin{aligned} V^{g,\gamma}(x,s) &= \frac{(x-g(s))}{s-g(s)} e^{\Gamma(s)} \left[\int_s^\gamma e^{-\Gamma(y)} \frac{dy}{y-g(y)} \left\{ 2\int_{g(y)}^y \lambda(z)(z-g(y))dz - F(y) \right\} - F(\gamma)e^{-\Gamma(\gamma)} \right] \\ &\quad - \frac{F(s)(s-x)}{s-g(s)} + 2\int_{g(s)}^s \lambda(y) \frac{(x\wedge y - g(s))(s - x\vee y)}{s-g(s)} dy. \end{aligned}$$

Proof: (i) By Lévy's identification of the pair $(L_t, |B_t|)$ with $(S_t, S_t - B_t)$ and the fact that the excursions of Brownian motion form a Poisson process, we deduce that the excursions of Brownian motion down below the maximum also form a Poisson process; the proof is completed using the fact that the local-time rate of excursions which hit $\pm a$ is a^{-1}. See Rogers [12] or Rogers and Williams [13, VI.42-59] for further details.
(ii) From the above representation we have

$$\begin{aligned} \mathbb{E}[F(S_{\tau(g,\gamma)})] &= \int_0^\gamma \mathbb{P}(\tau(g,\gamma) > H_s) F(s) \mathbb{P}(S_{\tau(g,\gamma)} \in ds | \tau(g,\gamma) > H_s) + F(\gamma)\mathbb{P}(\tau(g,\gamma) \ge H_\gamma) \\ &= \int_0^\gamma e^{-\Gamma(s)} F(s) \frac{ds}{s-g(s)} + F(\gamma)e^{-\Gamma(\gamma)} \end{aligned}$$

Also, if $L(y,t)$ is the local time at y at time t of Brownian motion, then for $y, w \in (x,z)$, and Brownian motion started at w

$$\mathbb{E}^w[L(y, H_x \wedge H_z)] = 2\frac{(z - w\vee y)(w \wedge y - x)}{z-x}.$$

Then, for $y \le z$

$$\mathbb{E}^z[L(y, H_x \wedge H_{z+\Delta}) - L(y, H_x \wedge H_z)] = 2\Delta \frac{y-x}{z-x} + O(\Delta^2).$$

Taking $x = g(z)$ and integrating against z we have

$$\begin{aligned} \mathbb{E}[L(y, \tau(g,\gamma))] &= \int_0^\gamma \mathbb{P}(\tau(g,\gamma) > H_s) \mathbb{E}[L(y, H_{g(s)} \wedge H_{s+ds}) - L(y, H_{g(s)} \wedge H_s)] \\ &= \int_0^\gamma e^{-\Gamma(s)} 2 \frac{y-g(s)}{s-g(s)} I_{\{g(s)<y<s\}} ds \end{aligned}$$

Finally, by the representation of local time as an occupation measure, we have that (Revuz and Yor [11, VI.1.6]), $\mathbb{E}[\int_0^{\tau(g,\gamma)} \lambda(B_s)ds] = \int_{\mathbb{R}} \lambda(y)\mathbb{E}[L(y,\tau(g,\gamma))]$.
(iii) A straightforward extension of part (ii) gives that for $s \le \gamma$,

$$\begin{aligned}\mathbb{E}^{s,s}[F(S_{\tau(g,\gamma)})] &= \int_s^\gamma \mathbb{P}(\tau^{g,\gamma} > H_w | \tau^{g,\gamma} > H_s) F(w) \mathbb{P}(S_{\tau(g,\gamma)} \in dw | \tau^{g,\gamma} > H_w) \\ &\quad + F(\gamma)\mathbb{P}(\tau^{g,\gamma} \ge H_\gamma | \tau^{g,\gamma} > H_s) \\ &= \int_s^\gamma e^{-(\Gamma(w)-\Gamma(s))} F(w) \frac{dw}{w-g(w)} + F(\gamma)e^{-(\Gamma(\gamma)-\Gamma(s))}.\end{aligned}$$

Similarly

$$\mathbb{E}^{s,s}[L(y,\tau(g,\gamma))] = e^{\Gamma(s)} \int_s^\gamma e^{-\Gamma(w)} 2\frac{y-g(w)}{w-g(w)} I_{\{g(w)<y<w\}} dw$$

and we have

$$V^{g,\gamma}(s,s) = e^{\Gamma(s)}\left[\int_s^\gamma e^{-\Gamma(w)} 2dw \int_{g(w)}^w \lambda(y)\frac{y-g(w)}{w-g(w)}dy - \int_s^\gamma e^{-\Gamma(w)}F(w)\frac{dw}{w-g(w)} - F(\gamma)e^{-\Gamma(\gamma)}\right].$$

The expression in the statement of the Lemma follows from the identity

$$V^{g,\gamma}(x,s) = \frac{(x-g(s))}{s-g(s)}V^{g,\gamma}(s,s) + \frac{(s-x)}{s-g(s)}V^{g,\gamma}(g(s),s) - \mathbb{E}^{x,s}\left[\int_0^{H_s \wedge H_{g(s)}} \lambda(B_u)du\right].$$

□

We collect some important properties of V which will be used in the sequel. These properties are easily verified by calculation.

Lemma 2.3 *For $g(s) < x < s$ and $s < \gamma$, $V^{g,\gamma}$ is such that: (i) $V^{g,\gamma}(\gamma,\gamma) = -F(\gamma)$, (ii) $V^{g,\gamma}(g(s),s) = -F(s)$, (iii) $(\partial/\partial s)V^{g,\gamma}(s,s) = 0$, (iv) $(\partial^2/\partial x^2)V^{g,\gamma}(x,s) = -2\lambda(x)$.*

2.2 Optimisation via calculus of variations.

For fixed γ, and for stopping times in $\mathcal{S}_\gamma$, we want to deduce the optimal form of the function g via calculus of variations. This optimum will depend on γ, so that we should (and later shall) write g^*_γ for the optimal function. However, since γ is fixed for the present we suppress the subscript.

Suppose that the optimal g^* exists, and for a general g write $g(s) = g^*(s) + \epsilon\eta(s)$. If $g^* \equiv g^*_\gamma$ is optimal , then $W^{g^*+\epsilon\eta,\gamma} - W^{g^*,\gamma} \le 0$ and by considering the first order term in an expansion with respect to ϵ we have

$$\begin{aligned} 0 &= \int_0^\gamma \Lambda(s)\left[-\int_0^s \frac{\eta(u)du}{(u-g^*(u))^2} + \frac{\eta(s)}{(s-g^*(s))} - \frac{2\eta(s)\int_{g^*(s)}^s \lambda(y)dy}{2\int_{g^*(s)}^s \lambda(y)(y-g^*(s))dy - F(s)}\right]ds \\ &\quad + F(\gamma)\int_0^\gamma \frac{\eta(u)du}{(u-g^*(u))^2}e^{-\Gamma^*(\gamma)} \end{aligned} \tag{10}$$

where Γ^* is shorthand for Γ^{g^*} and

$$\Lambda(s) = e^{-\Gamma^*(s)}\frac{1}{s-g^*(s)}\left\{2\int_{g^*(s)}^s \lambda(y)(y-g^*(s))dy - F(s)\right\}.$$

Define $\Xi(s) = \int_0^s (\eta(u)/(u - g^*(u))^2)du$, so that $\eta(s) = \Xi'(s)(s - g^*(s))^2$. Then (10) can be rewritten as

$$0 = \int_0^\gamma \Lambda(s)\Xi(s)ds - F(\gamma)e^{-\Gamma^*(\gamma)}\Xi(\gamma) + \int_0^\gamma \Xi'(s)e^{-\Gamma^*(s)} \left(F(s) + 2\int_{g^*(s)}^s \lambda(y)(s-y)dy \right) ds.$$

Integrating this last term by parts

$$\begin{aligned} 0 &= \left[\Xi(s)e^{-\Gamma^*(s)} \left(F(s) + 2\int_{g^*(s)}^s \lambda(y)(s-y)dy \right) \right]_0^\gamma - F(\gamma)\Xi(\gamma)e^{-\Gamma^*(\gamma)} \\ &\quad + \int_0^\gamma \Xi(s)ds \left[\Lambda(s) - \frac{d}{ds}\left(e^{-\Gamma^*(s)} \left(F(s) + 2\int_{g^*(s)}^s \lambda(y)(s-y)dy \right) \right) \right]. \end{aligned}$$

Since the above equality must hold for all η we have that $\Xi(s)$ and $\Xi(\gamma)$ are arbitrary, and it follows that we must have both

$$\int_{g^*(\gamma)}^\gamma \lambda(y)(\gamma - y)dy = 0 \tag{11}$$

and

$$\Lambda(s) - \frac{d}{ds}\left(e^{-\Gamma^*(s)} \left(F(s) + 2\int_{g^*(s)}^s \lambda(y)(s-y)dy \right) \right) = 0,$$

this second condition being equivalent to

$$\frac{d}{ds}g^*(s) = \frac{f(s)}{2\lambda(g^*(s))(s - g^*(s))} = \frac{-f(s)}{2c(g^*(s))(s - g^*(s))}. \tag{12}$$

Thus, for each fixed γ we have a conjectured optimal stopping time amongst the class of rules of the form (7), namely the associated function g^* must satisfy the ordinary differential equation (12) subject to the terminal value condition (11).

Assumption 2.4 *Assume that on each subinterval I of $(0,\infty)$, c is bounded below by a positive constant c_I. Assume further that the positive functions f and c are sufficiently regular that for each $K > 0$, and for every starting point $(s_0, g(s_0)) \in [0, K] \times \mathbb{R}^-$ (where by convention $0 \in \mathbb{R}^-$) the ordinary differential equation (12) has a unique non-explosive solution defined in the domain $0 \le s \le K$ and $g^* \le 0$, and that such solutions are continuous in the starting point.*

Let h be the inverse function to g^*. Then h solves

$$\frac{d}{dx}h(x) = \frac{-2c(x)(h(x) - x)}{f(h(x))}. \tag{13}$$

If $F(s) = s$, so that $f \equiv 1$ then we can write down solutions to (13). We have

$$\begin{aligned} h(x) &= h(\beta)e^{-2\int_\beta^x c(z)dz} + \int_\beta^x 2yc(y)e^{-2\int_y^x c(z)dz}dy, \\ &= x + (h(\beta) - \beta)e^{-2\int_\beta^x c(z)dz} - \int_\beta^x e^{-2\int_y^x c(z)dz}dy. \end{aligned} \tag{14}$$

It is clear that h exists, and is a decreasing function (provided we only consider the domain $h > 0 > x$) which increases to infinity as x tends to minus infinity under our assumption that $\int_x^0 |y|c(y)dy \uparrow \infty$. Hence g^* is also well defined for all positive s, as long as $g^* < 0$.

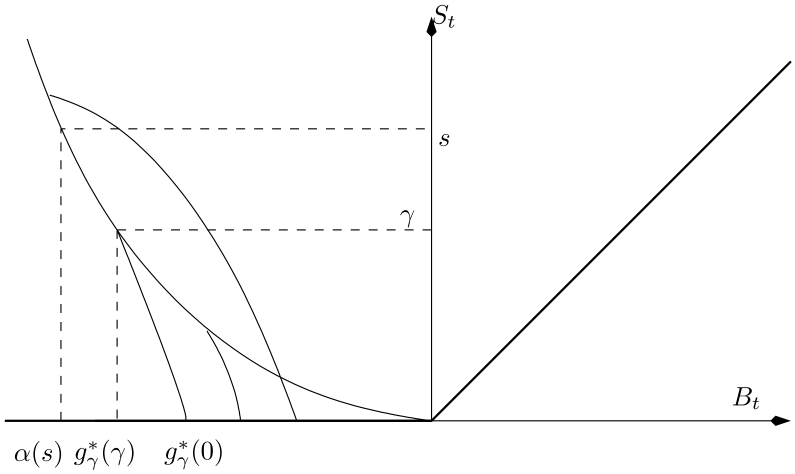

Figure 2: Representation of the function α and some of the family of solutions g^*_γ.

3 Constructing the optimal solutions.

In this section we use the intuition and the candidate stopping rule constructed in the previous section to find the optimal stopping rule. The proof proceeds by considering the problem (6) over increasing families of stopping rules.

3.1 Optimality for $\tau \in \mathcal{S}_\gamma$.

For $s \leq \gamma$ define $\alpha(s)$ to be the solution in $\mathbb{R}^-$ to

$$\int_{\alpha(s)}^{s} \lambda(y)(s-y)dy = 0. \tag{15}$$

The rationale behind the definition of α is that $\mathbb{E}^{\alpha(s),s} \int_0^{H_{\alpha(s)-\Delta} \wedge H_s} \lambda(B_u)du = O(\Delta^2)$, so that to first order the expected reward minus expected cost at (α, s) from continuing until the Brownian motion reaches $\alpha - \Delta$ or s is zero.

It follows from the Assumption 2.1 that α exists and is unique. Further, provided $\int_0^s r(y)dy > 0$,

$$s\int_0^s r(y)dy > \int_0^s (s-y)r(y)dy = \int_{\alpha(s)}^0 c(y)(s-y)dy > s\int_{\alpha(s)}^0 c(y)dy$$

and hence it follows from differentiating (15) that α is decreasing.

Lemma 3.1 *Let g^* solve (12) on $s \leq \gamma$ subject to the terminal value condition $g^*(\gamma) = \alpha(\gamma)$. Hence (11) is satisfied. Suppose that this solution has $g^*(0) < 0$.*

Then, for all $s \leq \gamma$, $(\partial/\partial x)V^{g^*,\gamma}(g^*(s), s) = 0$, and for $s \leq \gamma$ and $x \leq 0$, $V^{g^*,\gamma}(x,s) \geq -F(s)$. Further, $W^{g^*,\gamma} = -2\int_{g^*(0)}^0 yc(y)dy > 0$ and $V^{g^*,\gamma}(s,s) = 2\int_{g^*(s)}^{\alpha(s)}(s-y)c(y)dy - F(s)$.

Proof:

For the duration of the proof we write V for $V^{g^*,\gamma}$. We have that

$$\frac{\partial V}{\partial x} = \frac{F(s)}{s-g^*(s)} - \frac{2}{s-g^*(s)}\int_{g^*(s)}^{x} \lambda(y)(y-g^*(s))dy + \frac{2}{s-g^*(s)}\int_x^s \lambda(y)(s-y)dy + \frac{V(s,s)}{s-g^*(s)}.$$

In particular, using the fact that $V(\gamma,\gamma) = -F(\gamma)$ and $g^*(\gamma) = \alpha(\gamma)$,

$$\left.\frac{\partial V}{\partial x}\right|_{\alpha(\gamma),\gamma} = \frac{2}{\gamma - \alpha(\gamma)} \int_{g^*(\gamma)}^{\gamma} \lambda(y)(s-y)dy = 0.$$

Now define

$$U(s) = \left.\frac{\partial V}{\partial x}\right|_{g^*(s),s} = \frac{F(s)}{s-g^*(s)} + \frac{2}{s-g^*(s)} \int_{g^*(s)}^{s} \lambda(y)(s-y)dy + \frac{V(s,s)}{s-g^*(s)}. \qquad (16)$$

Then $U(\gamma) = 0$ and

$$\frac{dU}{ds} = \frac{f(s)}{s-g^*(s)} + \frac{dg^*}{ds}\left(2c(g^*(s)) + \frac{U(s)}{s-g^*(s)}\right) = \frac{dg^*}{ds}\frac{U(s)}{s-g^*(s)}.$$

It follows that $U(s) \equiv 0$, for $s \geq 0$.

From Lemma 2.3 we have that V is convex in x on $\mathbb{R}^-$. Given the derivative condition at $g^*(s)$ it follows that V is increasing in x on $(g^*(s),0)$ and hence $V(x,s) \geq -F(s)$ for $x<0$. Finally, from (16)

$$0 = (s-g^*(s))\left.\frac{\partial V}{\partial x}\right|_{g^*(s),s} = F(s) + V(s,s) + 2\int_{g^*(s)}^{s} \lambda(y)(s-y)dy$$

so that, using (15) and the fact that $\lambda(x) = -c(x)$ for $x<0$, $V(s,s) = 2\int_{g^*(s)}^{\alpha(s)} c(y)(s-y)dy - F(s)$. The expression for $W^{g^*,\gamma}$ follows on setting $s=0$. □

Remark 3.2 The condition $(\partial/\partial x)V^{g^*,\gamma}(g^*(s),s) = 0$ is the smooth fit condition, and arises from the optimality property of the boundary g.

Proposition 3.3 *Suppose that g^* defined via the differential equation (12), and subject to $g^*(\gamma) = \alpha(\gamma)$, is such that $g^*(0) < 0$. Define $M_t = V^{g^*,\gamma}(B_{t\wedge H_\gamma}, S_{t\wedge H_\gamma}) + \int_0^{t\wedge H_\gamma}\lambda(B_s)ds$. Then M_t is a supermartingale, and a martingale for $t \leq \tau^{g^*,\gamma}$.*

Further, for all $\tau \in \mathcal{S}_\gamma$

$$W^{g^*,\gamma} \geq \mathbb{E}\left[\int_0^\tau \lambda(B_s)ds - F(S_\tau)\right],$$

with equality for $\tau = \tau(g^*,\gamma)$.

Proof:
Let

$$N_t = V^{g^*,\gamma}(B_{t\wedge H_\gamma}, S_{t\wedge H_\gamma}) + \int_0^{t\wedge H_\gamma} \lambda(B_u) I_{\{g^*(S_u)\leq B_u \leq S_u\}} du.$$

Then $M_t = N_t - \int_0^{t\wedge H_\gamma} c(B_u) I_{\{B_u < g^*(S_u)\}du}$ and the supermartingale property for M will follow if N_t is a martingale. But it is easy to see from Itô's Lemma and the properties of $V^{g^*,\gamma}$ derived in Lemmas 2.3 and 3.1 that N_t is a local martingale. The true martingale property follows from the fact that $V^{g^*,\gamma}$ is bounded and, for example,

$$R_{t\wedge H_\gamma} = \int_0^{t\wedge H_\gamma} r(B_s)ds \leq \int_0^\gamma r(y)L(y,H_\gamma)dy.$$

This last quantity has finite expectation since r is bounded.

Using the supermartingale property we have that for any $\tau \leq H_\gamma$,

$$W^{g^*,\gamma} = V^{g^*,\gamma}(0,0) = M_0 \geq \mathbb{E}[M_\tau] = \mathbb{E}\left[V(B_\tau, S_\tau) + \int_0^\tau \lambda(B_u)du\right] \geq \mathbb{E}\left[R_\tau - C_\tau - F(S_\tau)\right].$$

Here we use the fact that $V^{g^*,\gamma}(x,s) \geq -F(s)$ for $x \leq 0$ and $s \leq \gamma$. In particular, the validity of the proof relies on the fact that we are searching over $\tau \in \mathcal{S}_\gamma$. $\square$

In summary, provided we restrict attention to τ such that $\tau \leq H_\gamma$ and such that on $\tau < H_\gamma$ we must have $B_\tau < 0$ then

$$\sup_{\tau \in \mathcal{S}_\gamma} \mathbb{E}\left[R_\tau - C_\tau - F(S_\tau)\right] = W^{g^*,\gamma}.$$

However, in general $\tau(g^*_\gamma, \gamma)$ is not optimal for $\tau \in \mathcal{T}_\gamma$ because it may be optimal to choose to stop at some earlier moment when $B_t = S_t$. Our task is to choose the optimal γ and then to show that for this γ the restriction to stopping times with $B_\tau < 0$ on $\tau < H_\gamma$ is redundant.

3.2 Optimality for $\tau \in \mathcal{T}_{\gamma^*}$.

Fix K with $0 < K < \infty$. (Later we will let $K \uparrow \infty$, but the case $K < \infty$ is of independent interest in the study of transient diffusions, see Section 4.2.) The immediate goal is to consider the maximisation problem (5) over stopping times $\tau \in \mathcal{T}_K$, firstly by choosing a key level γ^* and proving optimality over $\mathcal{T}_{\gamma^*}$, and then by extending the result to $\mathcal{T}_K$.

Let $\mathcal{G}_K = \{\gamma \leq K : g^*_\gamma(0) \leq 0\}$. Note that $0 \in \mathcal{G}_K$ so that $\mathcal{G}_K$ is non-empty. Let $\gamma^*_K = \arg\min\{g^*_\gamma(0); \gamma \leq K\}$. More precisely, to cover the case where the arg min is not uniquely defined we set

$$\gamma^* \equiv \gamma^*_K = \sup\{\gamma \leq K : g^*_\gamma(0) = \min_{\xi \leq K} g^*_\xi(0)\}. \tag{17}$$

If $\gamma^* = 0$ then $\tau = 0$ is the only element of $\mathcal{T}_0$ and therefore optimal. The choice of γ^* and subsequently $g^*_{\gamma^*}$ plays the same role as the choice of a particular function g^* via the maximality principle in Peskir [9].

We write g^*_* as shorthand for $g^*_{\gamma^*_K}$. Until further notice, since K is fixed we omit it from the notation. Note that to date $g^*_\gamma(s)$ has been defined on $(s \leq \gamma)$, but we can extend the definition to $(\gamma < s \leq K)$, by assuming that g^*_γ continues to satisfy (12) on (γ, K).

Lemma 3.4 *For $0 \leq s \leq K$, $g^*_*(s) \leq \alpha(s)$.*

Proof:
It follows from Assumption 2.4 that if both $g^*_{\gamma_1}(s)$ and $g^*_{\gamma_2}(s)$ solve (12) and $g^*_{\gamma_1}(s) < g^*_{\gamma_2}(s)$ for some s, then $g^*_{\gamma_1}(s) < g^*_{\gamma_1}(s)$ for all s.

Now suppose $g^*_*(s) > \alpha(s)$ for some s. Then $g^*_s(s) = \alpha(s) < g^*_*(s)$ and thus $g^*_s(0) < g^*_*(0)$, contradicting the choice of γ^*. $\square$

Lemma 3.5 *For $s \leq \gamma^*$ we have $V^{g^*_*,\gamma^*}(x,s) \geq -F(s)$.*

Proof:
By the concavity in x of $V^{g,\gamma}(x,s)$ for $0 < x < s$ (Lemma 2.3) and the fact that $V^{g^*,\gamma}(0,s) \geq -F(s)$ (Lemma 3.1) it is sufficient to show that $V^{g^*_*,\gamma^*}(s,s) \geq -F(s)$.

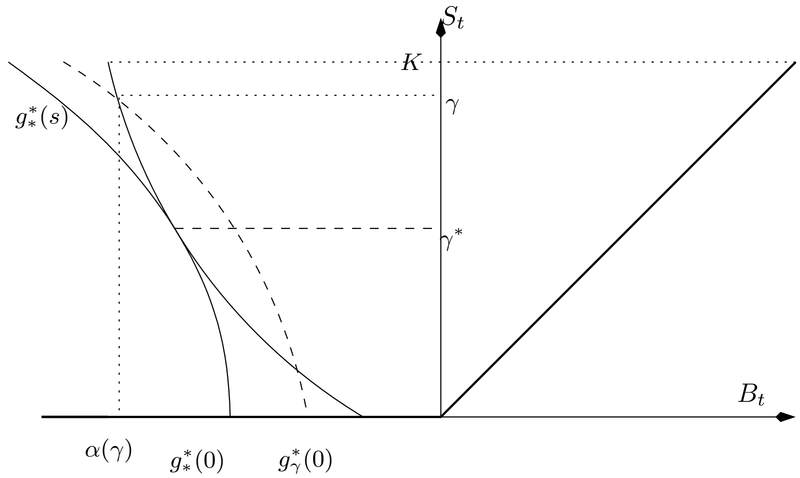

Figure 3: The optimal choice of γ^* minimises $g^*_\gamma(0)$.

But, again by Lemma 3.1,

$$V^{g^*_*,\gamma^*}(s,s) = 2\int_{g^*_*(s)}^{\alpha(s)} c(y)(s-y)dy - F(s)$$

and the integral in this expression is positive by Lemma 3.4. $\square$

Corollary 3.6 $\tau^{g^*_{\gamma^*},\gamma^*}$ *is optimal in* $\mathcal{T}_{\gamma^*}$ *and*

$$\sup_{\tau\in\mathcal{T}_{\gamma^*}} \mathbb{E}\left[\int_0^\tau \lambda(B_s)ds - F(S_\tau)\right] = W^{g^*_*,\gamma^*}.$$

Proof:
The proof proceeds exactly as in the proof of Proposition 3.3, except that now $V^{g^*_*,\gamma^*}(x,s) \geq -F(s)$ for all $x \leq s$, so that $W^{g^*_*,\gamma^*} \geq \mathbb{E}[R_\tau - C_\tau - F(S_\tau)]$ for all $\tau \leq H_{\gamma^*}$.

$\square$

3.3 Optimality for $\tau \in \mathcal{T}_K$.

The extension to $\mathcal{T}_K$ relies on the following existence lemma concerning the functions α and g, the proof of which is delayed until the appendix.

Lemma 3.7 *Suppose* $\gamma^* < K$*. Then there exists a positive function* $\tilde{r}$ *defined on* $[0,K]$ *such that* $\tilde{r} \geq r$ *and* $\tilde{r}(x) = r(x)$ *for* $x \leq \gamma^*$ *and such that* $\tilde{\alpha}$ *defined via*

$$\int_0^s \tilde{r}(y)(s-y)dy = \int_{\tilde{\alpha}(s)}^0 c(y)(s-y)dy$$

is such that $g^*_*(s) \leq \tilde{\alpha}(s) \leq \alpha(s)$ *and* $g^*_*(K) = \tilde{\alpha}(K)$.

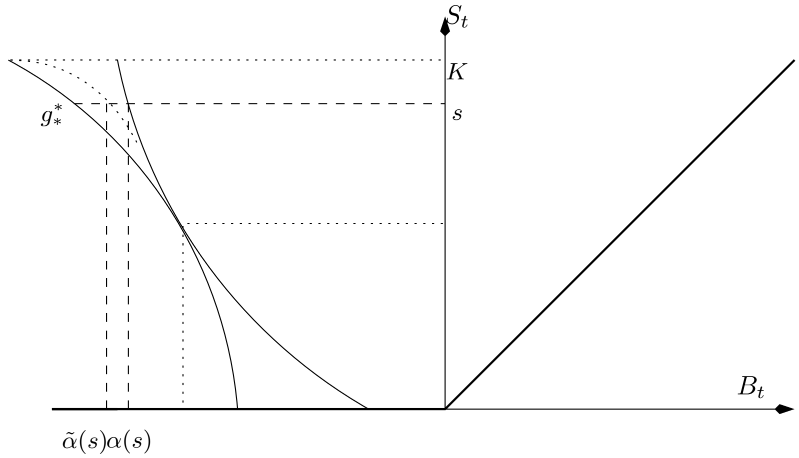

Figure 4: The function $\tilde{\alpha}$ agrees with α except on a neighbourhood of K.

Theorem 3.8 *$\tau(g^*_*, \gamma^*)$ is optimal in $\mathcal{T}_K$ and*

$$\sup_{\tau \in \mathcal{T}_K} \mathbb{E}\left[\int_0^\tau \lambda(B_s)ds - F(S_\tau)\right] = W^{g^*_*, \gamma^*}.$$

Proof:
Let $\tilde{r}$ be as given in Lemma 3.7 and define $\tilde{R}_t = \int_0^t \tilde{r}(B_u)du$ and

$$\tilde{W} = \sup_{\tau \in \mathcal{T}_K} \mathbb{E}[\tilde{R}_\tau - C_\tau - F(S_\tau)].$$

Observe that using $\tilde{r}$ in place of r changes the definition of α, and hence changes the boundary conditions on the functions g^*_γ, but leaves the differential equation (12) unchanged.

Then, with $\tilde{\gamma}^*$ denoting the optimal choice of γ for the modified problem and with $\tilde{g}^*_*$ the associated stopping boundary, $\tilde{\gamma}^* = K$ and $\tilde{g}^*_*(0) = \tilde{g}^*_K(0) = \tilde{g}^*_{\gamma^*}(0) = g^*_{\gamma^*}(0) = g^*_*(0)$. In particular, by Corollary 3.6

$$\tilde{W} = -2\int_{\tilde{g}^*_*(0)}^0 yc(y)dy = -2\int_{g^*_*(0)}^0 yc(y)dy = W^{g^*_*, \gamma}$$

and

$$W^{g^*_*, \gamma} = \tilde{W} = \sup_{\tau \in \mathcal{T}_K} \mathbb{E}[\tilde{R}_\tau - C_\tau - S_\tau] \geq \sup_{\tau \in \mathcal{T}_K} \mathbb{E}[R_\tau - C_\tau - S_\tau] \geq \sup_{\tau \in \mathcal{T}^*_\gamma} \mathbb{E}[R_\tau - C_\tau - S_\tau] = W^{g^*_*, \gamma}$$

and the theorem is proved. $\square$

3.4 Optimality for $\tau \in \mathcal{T}$.

Theorem 3.9 *Let $\gamma^*_\infty = \lim_{K \uparrow \infty} \gamma^*_K$, and let $g^*_{\gamma^*_\infty}(0) = \lim_{K \uparrow \infty} g^*_{\gamma^*_K}(0)$. Then*

$$\sup_{\tau \in \mathcal{T}} \mathbb{E}[R_\tau - C_\tau - F(S_\tau)] = -2\int_{g^*_{\gamma^*_\infty}(0)}^0 yc(y)dy.$$

Furthermore, if $g^*_{\gamma^*_\infty}(0) > -\infty$ then $\tau^{g^*_{\gamma^*_\infty}, \gamma^*_\infty}$ is the optimal stopping rule.

Observe that, even if $\gamma^*_\infty = \infty$, it may be the case that $g^*_{\gamma^*_\infty}(0) > -\infty$, whence $g^*_{\gamma^*_\infty}$ can be defined as the initial value problem solution to (12).

Proof:

For $\tau \in \mathcal{T}$, let $\tau_K = \tau \wedge H_K$. Then, by monotone convergence, and the definition of $\mathcal{T}$

$$\begin{aligned} \mathbb{E}[R_\tau - C_\tau - F(S_\tau)] &= \lim_{K\uparrow\infty} \mathbb{E}[R_{\tau_K} - C_{\tau_K} - F(S_{\tau_K})] \\ &\leq \lim_{K\uparrow\infty} \sup_{\tau\in\mathcal{T}_K} \mathbb{E}[R_\tau - C_\tau - F(S_\tau)] \\ &= -\lim_{K\uparrow\infty} 2\int_{g^*_{\gamma^*_K}(0)}^0 yc(y)dy \\ &= -2\int_{g^*_{\gamma^*_\infty}(0)}^0 yc(y)dy. \end{aligned}$$

If $\gamma^*_\infty < \infty$, then $W^{g^*_{\gamma^*_\infty},\gamma^*_\infty} = 2\int_{g^*_{\gamma^*_\infty}(0)}^0 yc(y)dy$ so that $\tau(g^*_{\gamma^*_\infty}, \gamma^*_\infty)$ is optimal.

Otherwise it is possible to find a sequence γ^*_K increasing to infinity such that $g^*_{\gamma^*_K}(0) \downarrow g^*_{\gamma^*_\infty}(0)$ and then, with $\tau^*_K = \tau(g^*_{\gamma^*_K}, \gamma^*_K)$,

$$\mathbb{E}[R_{\tau^*_K} - C_{\tau^*_K} - F(S_{\tau^*_K})] \to 2\int_{g^*_{\gamma^*_\infty}(0)}^0 yc(y)dy.$$

$\square$

4 Examples and Extensions

4.1 Examples

We give two examples. In the first example the optimal stopping rule is to stop immediately if τ is constrained to satisfy $\tau \leq H_K$ for sufficiently small K. If there is no constraint then even though the reward/cost function is antisymmetric, stopping rules can be designed for which the value function is arbitrarily large.

For the second example, the optimal stopping problem (6) has a finite well-defined solution, even when the stopping times are unconstrained.

Example 4.1 *Suppose $F(s) = s$ and $\lambda(x) = \xi sgn(x)$ with $\xi > 0$.*

It is easy to see that $\alpha(s) = -(\sqrt{2}-1)s$. Further, using (13) the family of functions g^*_γ are given the inverses of the functions $h(x) = h(0)e^{-2\xi x} + \{e^{-2\xi x} - (1-2\xi x)\}/2\xi$ restricted to the appropriate domain $h \geq 0, x \leq 0$.

Suppose we consider stopping rules $\tau \leq H_K$ for $K < \infty$. Let $\hat{z}$ be the positive solution of $e^{2(\sqrt{2}-1)z} - 2\sqrt{2}z - 1 = 0$. (Then $y = \hat{z}/\xi$ is the y-coordinate of the intersection in the top-left quadrant of the plane between the functions $h_0(x) = \{e^{-2\xi x} - (1-2\xi x)\}/2\xi$ and $\beta(x) = -(\sqrt{2}+1)x$. Here β is the inverse to α. See Figure 5.) If $K \geq \hat{z}/\xi$, then $\tau = \tau^{g^*_K, K}$ is optimal. Conversely, if $K \leq \hat{z}/\xi$, we have that $\mathcal{G}_K = \{0\}$ and $\tau = 0$ is optimal. The optimality of $\tau \equiv 0$ over $\mathcal{T}_K$ is guaranteed by Theorem 3.8.

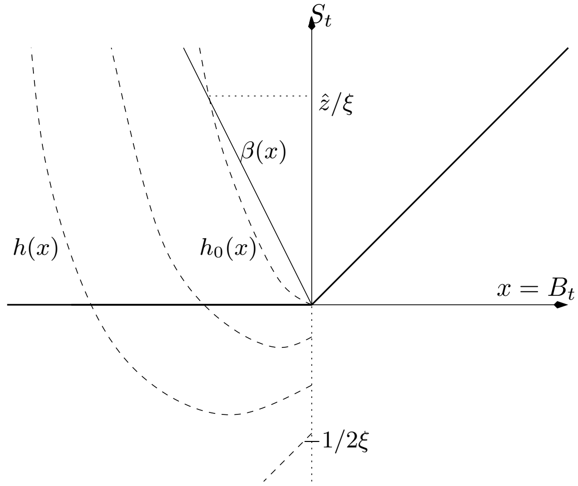

Figure 5: Schematic representation of the functions $\beta(x)$ and the family $h(x)$ for Example 4.1. Also shown is the point of intersection between $\beta(x)$ and $h_0(x)$.

Example 4.2 *Suppose $F(s) = s$ and $c(x) = 1/(1 + |x|)$. Let $b(y) = (|y|^3 + |y|)/3$ for $y < 0$. Then $b(y) > 0$. Let $a : \mathbb{R}^+ \to \mathbb{R}^-$ be the inverse to b. Finally, set*

$$r(x) = \frac{3}{(1 + 3a(x)^2)(1 - a(x))} + \frac{3}{(1 + 3a(x)^2)} \frac{d}{da} \frac{a(a^2 + 4)}{(1 - a)(1 + 3a^2)}\bigg|_{a=a(x)}$$

The choice of c has been made so that the family of solutions h to (13) has a simple form. At the same time we choose a simple form for β, the inverse to α. In this case β is chosen to equal b. Finally, we use c and β to identify the reward function r using the identity

$$\int_0^{\beta(x)} r(y)(\beta(x) - y)dy = \int_x^0 c(y)(\beta(x) - y)dy.$$

In this case we have $h(x) = x^2 + (1 - x)^2 h(0)$, and $\alpha(s) = a(s)$. We can now solve for g_*^* by finding the initial value $h_0 < 0$ for which the curve $h_0(x) = x^2 + (1 - x)^2 h_0$ just touches $b(x) = -(x + x^3)/3$. See Figure 6.

4.2 Extension to Diffusion Processes

Suppose X_t is a regular time-homogeneous diffusion, and consider the problem of finding

$$\sup_\tau \mathbb{E}\left[\int_0^\tau \lambda^X(X_u)du - F^X(S_\tau^X)\right]. \tag{18}$$

Then, as we shall show, it is possible to reduce the problem to one concerning Brownian motion. The idea has been used many times before, including in the Skorokhod embedding context by Azéma and Yor [1], Pedersen and Peskir [7] and Cox and Hobson [2].

Let Ψ be the scale function of X (chosen to be increasing and to satisfy $\Psi(X_0) = 0$) so that $Y_t = \Psi(X_t)$ is a local martingale, and indeed a time-change of Brownian motion, perhaps on a

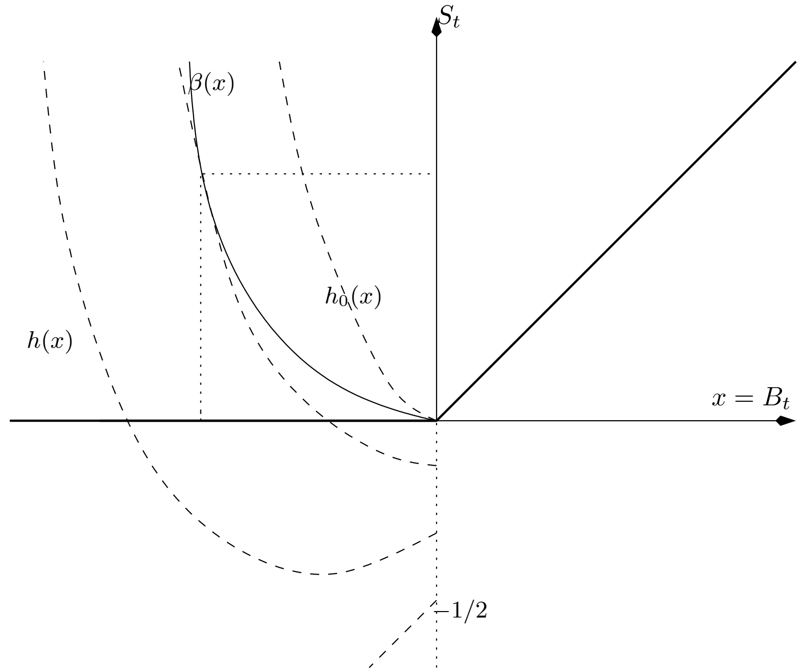

Figure 6: Representation of the functions $\beta(x)$ and the family $h(x)$ for Example 4.2.

suitably enriched probability space. Then $dY_t = \sigma(Y_t)dW_t$ and $Y_t = B_{A_t}$ for an increasing process $A_t = \int_0^t \sigma(Y_u)^2 du$.

Let $S_t^Y = \Psi(S_t^X)$ be the maximum process of Y_t, and let S_t be the maximum process for B. Define $\lambda^Y(y) = \lambda^X(\Psi^{-1}(y))$, $\lambda(y) = \lambda^Y(y)/\sigma(y)^2$ and $F(s) = F^X(\Psi^{-1}(s))$ Then $F^X(S_t^X) = F(S_t^Y) = F(S_{A_t})$ and

$$\int_0^\tau \lambda^X(X_u)du = \int_0^\tau \lambda^Y(Y_u)du = \int_0^{A_\tau} \lambda(B_v)dv.$$

Thus (18) becomes

$$\sup_{A_\tau \leq H_{\Psi(\infty)}} \mathbb{E}\left[\int_0^{A_\tau} \lambda(B_u)du - F(S_{A_\tau})\right] \equiv \sup_{\tau \leq H_{\Psi(\infty)}} \mathbb{E}\left[\int_0^{\tau} \lambda(B_u)du - F(S_\tau)\right]$$

and we have reduced the problem (18) to one of the form (5) for Brownian motion. Note that if $\Psi(\infty) < \infty$ it is appropriate to consider Brownian stopping times satisfying $\tau \leq H_K$ for $K = \Psi(\infty)$.

Example 4.3 *Suppose* $X_0 = 0$, $dX = dW + dt/2$, $F^X(s) = 1 - e^{-s}$, $\lambda^X(x) = \xi e^{-2x} sgn(x)$.

In this case $\Psi(x) = 1 - e^{-x}$, and the problem (18) reduces to the problem in Example 4.1 with $K = 1$. Then if $\xi \leq \hat{z}$, the problem is degenerate and $\tau \equiv 0$ is optimal. Alternatively, if $\xi > \hat{z}$ (18) has a non-trivial solution and positive value function.

4.3 Reward functions based on the terminal value

Consider the problem of finding

$$\sup_{\tau} \mathbb{E}\left[D(B_\tau) - F(S_\tau)\right] \tag{19}$$

for a suitable function D. Suppose D is such that $\lambda = D''/2$. Then, by Itô's formula $D(B_\tau) = D(B_0) + \int_0^\tau D'(B_s)dB_s + \int_0^\tau \lambda(B_s)ds$ so that, as Peskir [9] has observed, provided the local martingale term is a true martingale, (19) is equivalent to

$$\sup_{\tau} \mathbb{E}\left[\int_0^\tau \lambda(B_s)ds - F(S_\tau)\right].$$

The functions D for which we get a non-degenerate solution to (19) are concave on $\mathbb{R}^-$ and convex on $\mathbb{R}^+$.

4.4 Conclusions and Extensions

In this paper we have considered a converse problem to that studied in Peskir [9]. Using different methods to Peskir we have derived the optimal stopping rule, and the value function for the problem (3). Our method is based on explicit calculation of the value function for any suitably nice boundary, and then an optimisation (using calculus of variations) over possible boundaries.

In his papers Peskir [9, 10] also studies several extensions and generalisations of (2). Firstly, in [9], and by considering the case $X \equiv |B|$ and λ constant, he derives (and rederives) many interesting inequalities relating the maximum of the modulus of Brownian motion evaluated at any stopping time to the first moment of that stopping time. Further, in the sequel, (Peskir [10]) he introduces the optimal Skorokhod embedding problem, which, given Brownian motion and a centred target law μ, is to find $\lambda(x) > 0$ such that the solution to (2) has the property that B_τ has law μ. Peskir restricts attention to the case $X \equiv B$ and $F(s) = s$, but Obłój [6] shows how to extend this result to more general functions F and shows how this additional flexibility can be used to embed atomic measures which were excluded from the analysis in Peskir [10].

It is an open question as to whether the solution to the converse problem (3) leads to similar inequalities and to further solutions of the optimal Skorokhod embedding problem. At first sight the answer seems negative in each case: for the inequalities Peskir is able to extend his work to $|B|$, which is not covered in the setting of this paper; for the optimal constructions in the solution of (3) the stopped process only places mass at a single location in $\mathbb{R}^+$ so we cannot hope to embed all target distributions in this way. In order to embed general distributions on $\mathbb{R}$ it would seem to be necessary to modify the problem (3), perhaps also to incorporate a cost associated with the minimum of Brownian motion, and to exploit more fully links with the Perkins solution to the Skorokhod embedding problem. However, if a term involving the minimum is included then it does not seem to be as simple to write down expressions for the value function for arbitrary stopping rules, as in Lemma 2.2. For this reason the optimal Skorokhod embedding problem seems very challenging in this case.

A Appendix

Proof of Lemma 3.7.

Fix $\epsilon > 0$ and let $\Theta = \int_{g_*^*(K)}^{\alpha(K)} c(y)(K-y)dy$, $\Phi = \int_{g_*^*(K)}^{0} c(y)dy$, and

$$Q = \frac{\sup_{x\in I}\{c(x)(K-x)\}}{\inf_{x\in I}\{c(x)(\hat{K}-x)\}}$$

where $\hat{K} = \max\{K-\epsilon, \gamma^*\}$ and $I = [g_*^*(K), \alpha(\hat{K})]$. By Assumptions 2.1 and 2.4, $1 < Q < \infty$.

Choose δ such that $\delta < \min\{\epsilon, K-\gamma^*, \Theta/(Q||f||_\infty + \Phi)\}$ where $||f||_\infty = \sup\{|f(s)| : \hat{K} < s \leq K\}$. Now define $\tilde{r}$ by $\tilde{r} - r \equiv 0$ on $[0, K-\delta]$ and $\tilde{r} - r \equiv \Delta \equiv \Theta/\delta^2$ on $[K-\delta, K]$. It follows that $\tilde{\alpha}(K) = g_*^*(K)$ as required.

Since $g_*^*(s) \leq \alpha(s) = \tilde{\alpha}(s)$ for $s \leq K-\delta$, it remains to show that for $K-\delta < s \leq K$, $g_*^*(s) \leq \tilde{\alpha}(s)$. For $s > K-\delta/2$,

$$\int_0^s \tilde{r}(y)dy - \int_{\tilde{\alpha}(s)}^0 c(y)dy \geq \int_{K-\delta}^{K-\delta/2} \Delta dy - \Phi \geq \delta\Delta/2 - \Phi = \Theta/\delta - \Phi > Q||f||_\infty.$$

If $\tilde{\alpha}(s) = g_*^*(s)$ for some $s > K-\delta/2$, then

$$\left|\frac{d\tilde{\alpha}(s)}{ds}\right| - \left|\frac{dg_*^*(s)}{ds}\right| > \frac{2Q||f||_\infty - f(s)}{2c(g_*^*(s))(s-g_*^*(s))} > 0$$

so that $\tilde{\alpha}(s) \geq g_*^*(s)$ for $s > K-\delta/2$. Further, since $g_*^*(s) \leq \alpha(\gamma^*)$ for $s > \gamma^*$, we have

$$\begin{aligned}
\inf_{s\in[K-\delta/2,K]}\left|\frac{d\tilde{\alpha}(s)}{ds}\right| &= \inf_{s\in[K-\delta/2,K]} \frac{\int_0^s \tilde{r}(y)dy - \int_{\tilde{\alpha}(s)}^0 c(y)dy}{c(\tilde{\alpha}(s))(s-\tilde{\alpha}(s))} \\
&\geq \sup_{s\in[K-\delta,K]} \frac{||f||_\infty}{c(g_*^*(s))(s-g_*^*(s))} \\
&\geq 2\sup_{s\in[K-\delta,K]}\left|\frac{dg_*^*(s)}{ds}\right|.
\end{aligned}$$

Hence,

$$-\tilde{\alpha}(K) + \tilde{\alpha}(K-\delta/2) \geq \frac{\delta}{2}\inf_{s\in[K-\delta/2,K]}\left|\frac{d\tilde{\alpha}(s)}{ds}\right| \geq \delta \sup_{s\in[K-\delta,K]}\left|\frac{dg_*^*(s)}{ds}\right| \geq -g_*^*(K) + g_*^*(K-\delta),$$

so that $g_*^*(K-\delta) \leq \tilde{\alpha}(K-\delta/2)$. Finally, for $s \in (K-\delta, K-\delta/2)$ we have

$$g_*^*(s) \leq g_*^*(K-\delta) \leq \tilde{\alpha}(K-\delta/2) \leq \tilde{\alpha}(s).$$

□

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