Passport Options with Stochastic Volatility

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Abstract

A passport option is a call option on the profits of a trading account. In this article we investigate the robustness of passport option pricing by incorporating stochastic volatility. The key feature of a passport option is the holders' optimal strategy. It is known that in the case of exponential Brownian motion the strategy is to be long if the trading account is below zero and short if the account is above zero. Here we extend this result to models with stochastic volatility where the volatility is defined via an autonomous SDE. It is shown that for certain models of this type, the form of the optimal strategy remains unchanged. This means that pricing is robust to misspecification of the underlying model.

A second aim of this article is to investigate some of the biases which become apparent in a stochastic volatility regime. Using an analytic approximation we are able to obtain comparisons for passport option prices using the exponential Brownian motion model and some well known stochastic volatility models. This is illustrated by a number of numerical examples. One conclusion is that fair prices are generally *lower* in a model with stochastic volatility than in a model with constant volatility.

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1 Introduction

Passport options are a relatively new type of option which give the holder a zero-strike call option on the value of a trading account. The holder of the option trades dynamically on an underlying. At exercise, if the trades have realised a profit, the holder receives the proceeds of the trading account; if the account is in deficit then the losses are bourne by the option seller. In return for the option the holder makes an upfront payment, the option premium, to the seller. Since the holder undertakes trades over the life of the option, pricing becomes complex and must involve identifying the "best" trading strategy for the holder.

Passport options were introduced in the paper of Hyer *et al* [16] and have since been studied by Andersen *et al* [2], Ahn *et al* [1] and Lipton [18] using pde methods. These methods use a scaling which relies on the fact that the underlying is modelled by an exponential Brownian motion. More recently Henderson and Hobson [10], Henderson [9], Shreve and Večeř [22] and Delbaen and Yor [4] have used probabilistic methods to study the passport problem, giving different insights into the solution of the problem. One of the interesting features of the solution is the choice of trading strategy made by the option holder. In Henderson and Hobson [10], the strategy obtained in [16] for exponential Brownian motion was shown to remain optimal for a wide class of diffusion models. This is an interesting generalisation as it means the passport option set-up is robust to model misspecification.

In this paper, we show that this robustness feature can be taken further to include stochastic volatility models. Not only can we allow for diffusion models with level-dependent volatility, we can also use models where volatility is driven by a second stochastic differential equation.

We answer the question: what is the optimal strategy for general stochastic volatility models? We consider a certain class of models which includes many of the popular models in the literature. We use probabilistic methods to analyse the problem, and in particular the coupling of stochastic processes.

Related work on volatility and passport options in Henderson [8] shows that the price of a passport option is increasing in volatility, given only the same diffusion assumption mentioned earlier. That paper addresses hedging issues and considers what happens when the seller of a passport option follows the "wrong" model.

It is relevant to consider stochastic volatility models as they have evolved in response to empirical evidence that call prices are inconsistent with the constant volatility assumption. See the survey articles Frey [6] and Hobson [12]. These models are incomplete, requiring some choice of the appropriate measure to use for pricing. Interestingly, we show this choice of measure does not affect the optimal strategy for a passport option, although it will affect the price.

It is convenient in the case where there is no correlation between the asset price process and the volatility process to express the price of the passport option as an integral of a lookback option over the variance of the log-price. In this set-up we can obtain prices analytically if we can find a way to describe the distribution of the variance. We price the passport option under the Hull and White model [14] and the Stein and Stein [23] (and Scott [21]) models using power series methods. In this way we show that implementing stochastic volatility models for pricing passport options is quite straightforward. The results are compared to the prices obtained in the "Black-Scholes" type model. We discover that the Black Scholes passport option price does not necessarily underestimate the passport price with uncertain volatility. Instead the relationship between the price in the Black-Scholes model and a stochastic volatility alternative depends on the convexity of the lookback option price and the parameter values considered. In particular the typical method of adjusting the price upwards to account for (volatility) uncertainty is not appropriate for passport options.

The remainder of the paper is structured as follows. The next section discusses stochastic volatility models and defines a class of models which we use later. Those models in the literature fitting into this class are outlined. We introduce the passport option problem in §3 and in §4 prove the form of the optimal strategy for the class of models. The following section discusses pricing the passport option in our stochastic volatility framework. Then §6 prices the passport option under the Hull and White [14], Scott [21] and Stein and Stein [23] models and compares prices to the simple exponential Brownian motion model. Section 7 concludes, and an appendix contains proofs of results delayed from the main text.

2 Stochastic volatility models

We will consider a broad class of models, which covers many of the popular models in the literature and which allows for dependence on the asset price in the coefficients of the asset price process. In §4 we derive the form of the optimal strategy for this class of model. The models used in §6 for computations will fall into this class.

Let P_u and ξ_u be defined via the pair of SDE's

$$\begin{aligned} \frac{dP_u}{P_u} &= a_P(u,\xi_u,P_u)du + \sigma_P(u,\xi_u,P_u)dB_u \\ d\xi_u &= b(u,\xi_u)du + \eta(u,\xi_u)dW_u \end{aligned} \tag{1}$$

where $a_P(u,\xi_u,P_u) = \mu_P(u,\xi_u,P_u)\sigma_P(u,\xi_u,P_u)$ and B_u, W_u are independent $\mathbb{P}$-Brownian motions. Here, P_u is the asset price and ξ is an autonomous stochastic process which governs the volatility of the asset price process. Such stochastic volatility models are known to be incomplete, as the introduction of a second source of uncertainty via W means that options cannot be priced uniquely.

We could simplify the class of models in (1) by removing the P dependence in the equation for the asset price. This gives

$$\begin{aligned} \frac{dP_u}{P_u} &= a(u,\xi_u)du + \sigma_P(u,\xi_u)dB_u \\ d\xi_u &= b(u,\xi_u)du + \eta(u,\xi_u)dW_u \end{aligned} \tag{2}$$

where $a(u,\xi_u) = \mu(u,\xi_u)\sigma_P(u,\xi_u)$, and covers many of the models that are studied in the literature.

We wish to work with the discounted asset price $S_u = e^{-ru}P_u$ and obtain the framework in which S_u is a martingale. Here, and throughout this paper, we assume the rate of interest r is constant for simplicity. Denote the set of equivalent martingale measures which result in S_u being a $\mathbb{Q}$-martingale by $Q = \{\mathbb{Q} \in Q\}$. In the standard way (see for example Frey [6]) we see that S and ξ solve the following stochastic differential equations under $\mathbb{Q}$:

$$\begin{aligned} dS_u &= S_u \sigma(u, \xi_u, S_u) dB_u^{\mathbb{Q}} \\ d\xi_u &= [b(u, \xi_u) + \eta(u, \xi_u)\lambda_u]du + \eta(u, \xi_u) dW_u^{\mathbb{Q}} \end{aligned} \tag{3}$$

where $\sigma(u, \xi, s) = \sigma_P(u, \xi, e^{ru}s)$, and $B_u^{\mathbb{Q}}$ and $W_u^{\mathbb{Q}}$ are independent $\mathbb{Q}$ - Brownian motions. We assume throughout that σ, η, b have sufficient continuity properties to ensure the pair of SDE's in (3) have a weak solution unique in law, with the strong Markov property. (See, for example, Chapter V of Rogers and Williams, [20] and in particular Theorems 11.2 and 24.1. Note that we often apply these theorems to the pair $(\log P, \xi)$.)

The parameter λ_u, often called the market price of risk, is a free parameter. In fact, as stated by Frey [6], market incompleteness is equivalent to non-uniqueness of the market price of risk process.

Our models (1) (and (2)) cover many of the stochastic volatility models studied in the literature. Hull and White's (1987) model [14] assumes that the square of volatility follows exponential Brownian motion and so takes $a(u, \xi_u) = \bar{\mu}, \sigma_P(u, \xi_u) = \sqrt{\xi_u}, \eta(u, \xi_u) = \delta\xi_u$ and $b(u, \xi_u) = \bar{b}\xi_u$ with $\bar{b}, \delta, \bar{\mu}$ constants in (2) to give

$$\begin{aligned} \frac{dP_u}{P_u} &= \bar{\mu} du + \sqrt{\xi_u} dB_u \\ d\xi_u &= \bar{b}\xi_u du + \delta\xi_u dW_u. \end{aligned} \tag{4}$$

Wiggins [24] models the log of volatility as an arithmetic Ornstein Uhlenbeck process and has $a(u, \xi_u) = \bar{\mu}, \sigma_P(u, \xi_u) = e^{\frac{1}{2}\xi_u}, \eta(u, \xi_u) = \delta$ and $b(u, \xi_u) = \bar{v} - \kappa\xi_u$ with $\bar{v}, \bar{\mu}, \kappa, \delta$ constants. Other models of this type are those of Scott [21] and Stein and Stein [23]. They each use same stochastic differential equation for ξ (volatility an arithmetic Ornstein Uhlenbeck process) and use exponential Brownian motion for the asset price:

$$\begin{aligned} \frac{dP_u}{P_u} &= \bar{\mu} du + \xi_u dB_u \\ d\xi_u &= \beta(\alpha - \xi_u) du + \gamma dW_u. \end{aligned} \tag{5}$$

with γ, β, α constants. Here $\sigma(u, \xi_u) = \xi_u$.

Each of the models mentioned so far fits into the class in (1). More general models such as the Markovian model of Hofmann *et al* [13] allow for dependence on the asset price in the coefficients b and η as well as correlated Brownian motions. Examples of such models are analysed in Heston [11], who derives an efficient method for calculating option prices using characteristic functions. Another stochastic volatility model is the Hull and White (1988) [15] model.

It allows for non-zero correlation between the Brownian motions and volatility follows a mean-reverting process. Models with correlation between B and W are beyond the scope of this paper.

3 The Passport Option under Stochastic Volatility

A passport option is a call option on a trading account where the holder (buyer) of the option undertakes a trading strategy which is subject to a constraint. The constraint takes the form of a limit on the number of shares held. At expiry T, the holder receives from the option seller either the positive gains from trading or nothing if a loss was made. A key problem in the pricing of passport options is determining the holders' optimal strategy. Andersen *et al* [2] and Hyer *et al* [16] show that when the price of the underlying follows exponential Brownian motion the holder should invest up to the allowed limit, buying when the value of the trading account is negative and selling otherwise. This is shown to hold for more general diffusion models (with non-decreasing diffusion coefficient) by Henderson and Hobson [10]. We will work in the set-up of Henderson and Hobson [10] and give the essential definitions here.

First, if the buyer holds q_u units of the underlying at time u then their gains from trade ψ_u are defined by

$$d\psi_u(q) = r(\psi_u(q) - q_u P_u)du + q_u dP_u.$$

The second term corresponds to gains from investment in the risky asset and the first term corresponds to the return on uninvested monies which is paid into the trading account. Here we assume that the price dynamics in (1) hold and that ψ_0, the initial wealth, can be non-zero.

Upon discounting by the interest rate r we have the 'discounted' gains $G_u := e^{-ru}\psi_u$ which follow

$$dG_u(q) = q_u \sigma(u, \xi_u, S_u) S_u dB_u^{\mathbb{Q}} = q_u dS_u. \tag{6}$$

Note that $G(q)$ is a local martingale under the measure $\mathbb{Q}$ (in fact a martingale for each q), see Chapter 2, Henderson [9] for details.

Given that the investor follows the admissible strategy $(q_u)_{\{0\le u\le T\}}$ the passport option pricing problem is to find the strategy q_u^* such that the price (at time 0) is maximised:

$$\sup_{|q_u|\le D} \mathbb{E}^{\mathbb{Q}} G_T^+(q). \tag{7}$$

Here $\mathbb{E}^{\mathbb{Q}}$ denotes expectation under a martingale measure $\mathbb{Q}$. Note that to specify the price completely we need to choose a measure $\mathbb{Q}$ by selecting a λ_u, the market price of risk. Further, as part of the specification for a passport option, the strategy q_u is bounded by the constant D. This is to ensure the seller does not risk unlimited losses. The problem is linear in the constraint so henceforth we shall take $D = 1$.

When we extend the model for the underlying price process beyond exponential Brownian motion, the key result in the pricing of passport options is the fact that the price can be identified with the price of a lookback option. In particular, in Henderson and Hobson [10] (see also Delbaen and Yor [4]) it is shown that the price of the passport option reduces to

$$\frac{1}{2} \sup_{|v_u| \leq 1} \mathbb{E}^{\mathbb{Q}}(\overline{M}_T(v) - |G_0|)^+ + G_0^+ \tag{8}$$

where $M_p(v) = \int_0^p v_r dS_r$ and $\overline{M}_z(v) = \sup_{0 \leq r \leq z} M_r(v)$. Further, v in the supremum of (8) is related to the strategy q in (7) by $v_u = -q_u \text{sgn}(G_u)$. For completeness we prove this result as Theorem 8.3 in the Appendix. Our goal therefore is to find the optimising value of v. This is the subject of the next section.

4 The Optimal Strategy for Stochastic Volatility models

We first state the results.

Theorem 4.1 *Let $M_p(v) = \int_0^p v_r dS_r$ and let $\overline{M}_z(v) = \sup_{0 \leq r \leq z} M_r(v)$. For the class of stochastic volatility models in (1), or equivalently (3), if $s\sigma(u, \xi, s)$ is non-decreasing in s (for all u, ξ) then, for any k, $\sup_{|v_u| \leq 1} \mathbb{E}^{\mathbb{Q}}(\overline{M}_T(v) - k)^+$ is attained by $v_u \equiv 1$.*

Corollary 4.2 *For the given class of stochastic volatility models in (3), provided that $s\sigma(u, \xi, s)$ is non-decreasing in s, the optimal strategy in the passport option pricing problem is $q_u = -sgn(G_u)$.*

Remark 4.3 Note that in both the above results the condition that $s\sigma(u, \xi, s)$ is non-decreasing in s is trivially satisfied if the model is of the simpler form (2) with no dependence of $\sigma(.)$ on S.

We shall proceed to prove the form of the optimal strategy by extending the techniques in Henderson and Hobson [10].

Proof of Theorem 4.1
Let $N_t = N_0 + \int_0^t v_u dS_u$, let $\overline{N}_0$ be a given constant, and let $\overline{N}_t = \max\{\overline{N}_0, \sup_{0 \leq p \leq t} N_p\}$. Then with $\overline{S}_r = \sup_{0 \leq u \leq r} S_u$ we let

$$\begin{aligned} Z_t &= \mathbb{E}_t^{\mathbb{Q}}(\max[\overline{N}_t, N_t + \sup_{t < p \leq T}(S_p - S_t)]) \\ &= \mathbb{E}_t^{\mathbb{Q}}(\max[\overline{N}_t - N_t + S_t, \sup_{t < p \leq T} S_p]) + (N_t - S_t) \\ &= f(\overline{N}_t - N_t + S_t, S_t, \xi_t, T - t) + (N_t - S_t) \end{aligned}$$

where f is the expected value of $\overline{S}_T$ given the information available at time t:

$$f(x, y, z, T - t) = \mathbb{E}_t^{\mathbb{Q}}(\overline{S}_T \mid \overline{S}_t = x, S_t = y, \xi_t = z). \tag{9}$$

Denote $\tilde{b}(u,\xi_u) = b(u,\xi_u) + \eta(u,\xi_u)\lambda_u$. Since $f(\overline{S}_t, S_t, \xi_t, T-t)$ is a martingale, using Itô's lemma we know that

$$f_1 d\overline{S} = 0 = (f_3\tilde{b}(t,\xi_t)dt - f_4 dt + \tfrac{1}{2}f_{22}(dS)^2 + \tfrac{1}{2}f_{33}(d\xi)^2). \tag{10}$$

We show that Z_t is a supermartingale, and a martingale under the optimal control $v = 1$. Using Itô's lemma

$$\begin{aligned} dZ &= (dN - dS) + (f_1(1-v) + f_2)dS + f_3\eta(t,\xi_t)dW^{\mathbb{Q}} + [f_3\tilde{b}(t,\xi_t)dt - f_4 dt \\ &\quad + \tfrac{1}{2}f_{22}(dS)^2 + \tfrac{1}{2}f_{33}(d\xi)^2] + \tfrac{1}{2}f_{11}(1-v)^2(dS)^2 + f_{12}(1-v)(dS)^2. \end{aligned}$$

After accounting for martingale terms and terms which sum to zero, using (10), we need to show $(\frac{1}{2}(1-v)^2 f_{11} + (1-v)f_{12}) \leq 0$ for all $v \in [-1,1]$. (It is clear that if $v = 1$ then Z is a martingale). It is convenient to use the representation

$$f(x,y,z,T-t) = x + \mathbb{E}_t^{\mathbb{Q}}\left(\sup_{t\leq r\leq T} S_r - x\right)^+$$

where $S_t = y$. The arguments we use will be the same for all t so it is sufficient to consider the case $t = 0$. We want to show

$$\begin{aligned} f_{12} &= -\frac{\partial}{\partial y}\mathbb{Q}\left(\sup_{0\leq r\leq T} S_r^y \geq x\right) \leq 0 \text{ and} \\ f_{11} + f_{12} &= -\frac{\partial}{\partial y}\mathbb{Q}\left(\sup_{0\leq r\leq T} S_r^y - y \geq x\right) \leq 0. \end{aligned}$$

where the notation S^y specifies the initial value $S_0^y = y$. These two inequalities are proved in the appendix. Theorem 4.1 follows.

5 Pricing the Passport Option

We now return to pricing the option in (8). Fix a pricing measure $\mathbb{Q} \in Q$, see the remarks in the next section about suitable choice of $\mathbb{Q}$. By Theorem 4.1, we have $v^* = 1$ and the price becomes

$$\tfrac{1}{2}\mathbb{E}^{\mathbb{Q}}(\overline{S}_T - S_0 - |G_0|)^+ + G_0^+. \tag{11}$$

Importantly, these results hold for any $\mathbb{Q} \in Q$. This is interesting as it means that the strategy in Henderson and Hobson [10] which held for a certain class of models, remains optimal for quite a range of stochastic volatility models also. The optimal strategy is therefore fairly robust to model mis-specification.

As it is stated above, calculation of (11) is a numerical problem. We are pricing an option (a fixed-strike lookback call) under stochastic volatility. Henceforth we will restrict ourselves to models of the type (2). Recall that we are assuming no correlation between the asset price and volatility.

If the asset price P and volatility ξ are uncorrelated, and the drift and diffusion coefficient of ξ do not depend on P then ξ and B are independent.

This is the case for models of type (2). Of course, for models with P dependence in the diffusion coefficient, we have proved the optimal strategy but techniques to calculate prices will be more complicated. This is left as further work and we concentrate on the case in (2). We have shown a large number of models in the literature fit into this class.

Conditional on the path $(\xi_s)_{\{0\leq s\leq u\}}$, $\int_0^u \sigma(s,\xi_s)dB_s$ is Gaussian with zero mean and variance

$$V_u = \int_0^u \sigma^2(s,\xi_s)ds. \tag{12}$$

Then $S_u = S_0 e^{\tilde{Y}_u}$ with

$$\tilde{Y}_u = \int_0^u \sigma(s,\xi_s)dB_s^{\mathbb{Q}} - \int_0^u \tfrac{1}{2}\sigma^2(s,\xi_s)ds.$$

Under $\mathbb{Q}$, and conditional on $(\xi_s)_{\{0\leq s\leq u\}}$, $\tilde{Y}_u$ is Gaussian with mean $-\frac{1}{2}V_u$ and variance V_u. Then with $K = S_0 + |G_0|$ and total variance V_T,

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}(\overline{S}_T - K)^+ &= \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left((\overline{S}_T - K)^+ \mid \int_0^T \sigma^2(s,\xi_s)ds\right)\right] \\ &= \mathbb{E}^{\mathbb{Q}}[LB(S_0;V_T;K)] = \int_{V_T=v} LB(S_0;v;K)f^{\mathbb{Q}}(dv)\end{aligned} \tag{13}$$

where $f^{\mathbb{Q}}$ is the density function for the distribution of the total variance and $LB(S_0;v;K)$ is the time 0 price of a fixed strike lookback call option with strike K, expiry T and volatility $\sqrt{v/T}$ under exponential Brownian motion.

The lookback price is

$$LB(S_0;v;K) = \mathbb{E}^{\mathbb{Q}}(S_0 e^{\sup_{0<u\leq T}(\sqrt{\frac{v}{T}}B_u^{\mathbb{Q}} - \frac{1}{2}(\frac{v}{T})u)} - K)^+$$

and standard arguments, see Goldman *et al* [7], yield

$$LB(S_0;v;K) = S_0[N(d) + \sqrt{v}(N'(d) + dN(d))] - KN(d-\sqrt{v})$$

where

$$d = \frac{-\ln(K/S_0) + \frac{1}{2}v}{\sqrt{v}}.$$

Putting $K = S_0 + |G_0|$ gives

$$\begin{aligned}LB(S_0;v;S_0+|G_0|) &= S_0[N(d) - N(d-\sqrt{v}) \\ &\quad + \sqrt{v}(N'(d) + dN(d))] - |G_0|N(d-\sqrt{v}).\end{aligned} \tag{14}$$

Therefore, it is sufficient to characterise the law of V_T to derive the law of the option price, as discussed by Hobson [12], Frey [6] and Ball and Roma [3]. Several techniques have been proposed in the literature, involving computing the moment generating function of V_T. Both Hull and White [14] and Stein and Stein [23] present such methods.

Hull and White [14] use power series expansion methods and require the first few central moments of the distribution of the variance. An alternative used by Stein and Stein [23] is to use Fourier inversion methods. Ball and Roma [3] note that these two techniques can be extended to any price which is conditionally lognormal and the moment generating function of the variance possesses a known analytic form. They also remark that the power series expansion provides an alternative and insightful way to compute option prices within the Stein and Stein [23] model. We adopt this approach which also has the advantage of easy comparison between the two models.

The techniques of Hull and White [14] are generalised in the later paper Hull and White [15] to cope with non-zero correlation. Alternatively, Heston [11] looks at the Fourier transforms of conditional probabilities that the option expires in the money, also an analytic approach. Other approaches taken to solve for prices include Monte Carlo simulation (used by Scott [21]), and numerical methods in Wiggins [24].

In the next section we calculate the price using a power series approach. We then price the passport option using the Hull and White [14] and Stein and Stein [23] models for a number of parameter sets.

6 Calculating Passport Option Prices Analytically using Power Series Expansion methods

We will adapt the method used by Hull and White [14] to calculate the price given in (13) and obtain the passport option price to be

$$\tfrac{1}{2}\int_v LB(S_0; v; S_0 + |G_0|) f^{\mathbb{Q}}(dv) + G_0^+. \tag{15}$$

This will be done for the price dynamics in (4), the Hull and White [14] model and the Stein and Stein [23] model in (5). We need to choose a λ_u under which the distribution of the total variance is calculated. There are many different ways to do this in the literature, either on economic arguments, or for tractability. We follow a popular method here and take the "minimal martingale measure" of Follmer and Schweizer [5], corresponding to $\lambda_u = 0$. Henceforth this particular pricing measure will be denoted $\mathbb{Q}_0$ with expectation operator $\mathbb{E}^0$.

We note that the option price is a nonlinear function of the total variance $V \equiv V_T$ and so we expand $LB(S_0; v; K)$ in a Taylor series about the expected value $\mathbb{E}^0 V$ of V:

$$\begin{aligned}\int_v LB(S_0; v; K) f^{\mathbb{Q}_0}(v) dv &= LB(S_0; \mathbb{E}^0 V; K) + \tfrac{1}{2} \frac{\partial^2 LB(.)}{\partial v^2}\bigg|_{\mathbb{E}^0(V)} \mathrm{Var}(V) \\ &+ \frac{1}{6}\frac{\partial^3 LB(.)}{\partial v^3}\bigg|_{\mathbb{E}^0(V)} \mathrm{Skew}(V) + \textit{higher order terms}\ldots \end{aligned} \tag{16}$$

It is possible to compute explicit expressions for the derivatives of the lookback price and the moments for both models under consideration.

Firstly we compare the Hull and White [14] model to exponential Brownian motion, then in §6.2 we examine the Stein and Stein [23] model. The exponential Brownian motion prices are calculated using (11).

6.1 The Hull White model

Changing measure for the Hull and White model in (4), we have

$$\begin{aligned} \frac{dS_u}{S_u} &= \sqrt{\xi_u} dB_u^{\mathbb{Q}_0} \\ d\xi_u &= \bar{b}\xi_u du + \delta\xi_u dW_u^{\mathbb{Q}_0} \end{aligned}$$

where $dW^{\mathbb{Q}_0}, dB^{\mathbb{Q}_0}$ are independent $\mathbb{Q}_0$-Brownian motions.

The moments of $V = \int_0^T \xi_u du$ for the Hull and White model with $\bar{b} = 0$ are given below. These are as stated in Hull and White [14].

$$\begin{aligned} \mathbb{E}^0 V &= \xi_0 T \\ \mathbb{E}^0 (V^2) &= \frac{2(e^{\delta^2 T} - \delta^2 T - 1)}{\delta^4} \xi_0^2 \\ \mathbb{E}^0 (V^3) &= \frac{e^{3\delta^2 T} - 9e^{\delta^2 T} + 6\delta^2 T + 8}{3\delta^6} \xi_0^3. \end{aligned}$$

We first calculate the price of a passport option for an example with $S_0 = 100, G_0 = 10, T = 1, \delta = 0.2$. The results are given in Table 6.1 and were calculated using Maple. We use the first three terms in (16) although omitting the third term makes a negligible difference.

ξ_0	EBM	$HW model$
0.0	10.0	10.0
0.01	10.97628	10.97602
0.04	14.53221	14.5233
0.09	18.88084	18.8643
0.25	28.78122	28.75127
0.36	34.23635	34.20029
0.64	46.12905	46.08185

Table 6.1: Passport option price for Hull and White model and Black Scholes (EBM) using $S_0 = 100, G_0 = 10, T = 1, \delta = 0.2$, and with volatility ranging from 0 to 0.8. Note that the volatility parameter used in the calculation of the exponential Brownian motion price is $\sqrt{\xi_0}$ so that the expected total variance is the same in each model.

We can display the percentage difference in the prices on a graph. The difference refers to the Hull and White price of the passport option minus the price for the simple exponential Brownian motion model expressed as a percentage of the Black Scholes price. This convention will be used in all further graphs.

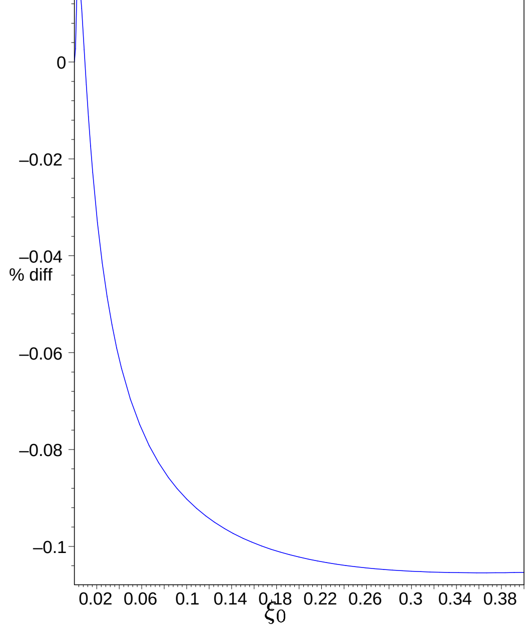

Figure 1: *Percentage difference in the passport option prices using the stochastic volatility model (Hull and White) and exponential Brownian motion with* $S_0 = 100$, $T = 1$, $G_0 = 10$, $\delta = 0.2$ *and* $0 \leq \xi_0 \leq 0.4$.

From Figure 1 we see that the stochastic volatility price is lower than the exponential Brownian motion (Black Scholes) price apart from for very small ξ_0 values. This is unusual in that often the introducion of stochastic volatility increases the price of an option. This means that the standard practice of adjusting prices upwards to allow for the uncertainty of volatility is inappropriate for the passport option. This can be explained by looking at the convexity of the lookback pricing function in (14).

Recall the lookback option price under stochastic volatility in (13) is

$$\mathbb{E}^0 \left[LB(S_0, V, K) \right]$$

and we multiply by a half and add G_0^+ to get the passport option price. If the function $LB(.)$ is locally concave in v then $\mathbb{E}^0 LB(S_0, V, K) < LB(S_0, \mathbb{E}^0 V, K)$ and so the stochastic volatility price is lower than the Black Scholes price. Conversely, if $LB(.)$ is convex in v then the inequality is reversed. Numerically evaluating the second derivative of $LB(.)$ with respect to v showed that for our parameters, the sign changed from positive to negative when $v = 0.01$. Using $\mathbb{E}^0 V = \xi_0$ we can relate this to a sign change at $\xi_0 = 0.01$.

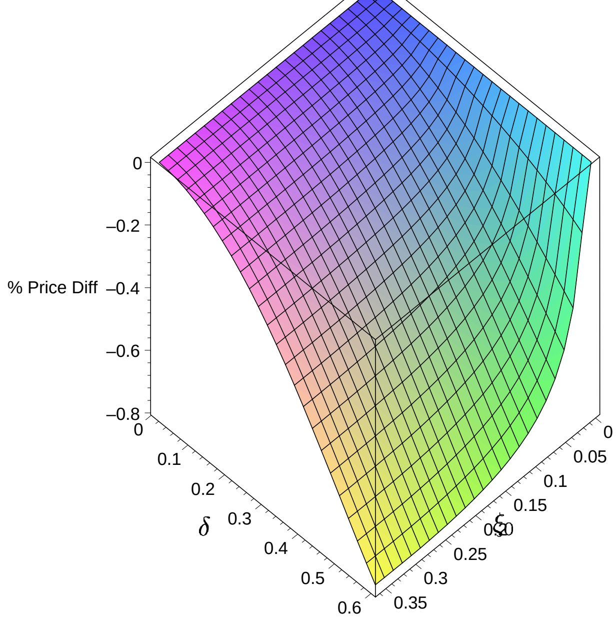

Figure 2: *Percentage difference in the passport option prices using the Hull and White model and exponential Brownian motion with* $S_0 = 100$, $T = 1$, $G_0 = 10$, $0 \leq \xi_0 \leq 0.36$ *and* $0 \leq \delta \leq 0.6$.

Figure 2 shows the relationship between the stochastic volatility option price and the price under an exponential Brownian motion model for a range of values of the parameter δ. Observe first that the positive differences of Figure 1 are no longer visible due to the coarser scale of the graph. Also, as $\delta \to 0$ the Hull White price approaches the exponential Brownian motion price, as is to be expected. Further, even for values of δ about 0.6, the percentage difference is still only -0.8% for ξ_0 around 0.36.

Now we consider the effect of changing the initial value of the trading strategy G_0. If we replace $G_0 = 10$ with $G_0 = -10$ then the only change in the passport option price (15) is the change in value G_0^+. Thus Table 6.1 can be modified to allow for $G_0 = -10$ by simply subtracting 10 from each term. In particular, the results for negative initial values can be deduced directly from those with positive initial values. However, because passport option values are much smaller in the negative case, it is inappropriate to display results in percentage terms.

Now consider the effect of varying the absolute value of G_0. The results for the Hull White model with $G_0 = 50$ are presented in Figure 3. The region over which the constant volatility model underestimates the passport price in a stochastic volatility regime is much wider. This is because when $G_0 = 50$ the range of values for volatility over which the lookback pricing function is convex is also much wider than when $G_0 = 10$ (see Figure 7). In particular, for low values of ξ_0, $LB(S_0, v, S_0 + |G_0|)$ is convex at $v = \mathbb{E}^0 V = \xi_0 T$, so the effect of stochastic volatility is to increase the option price. It remains true in

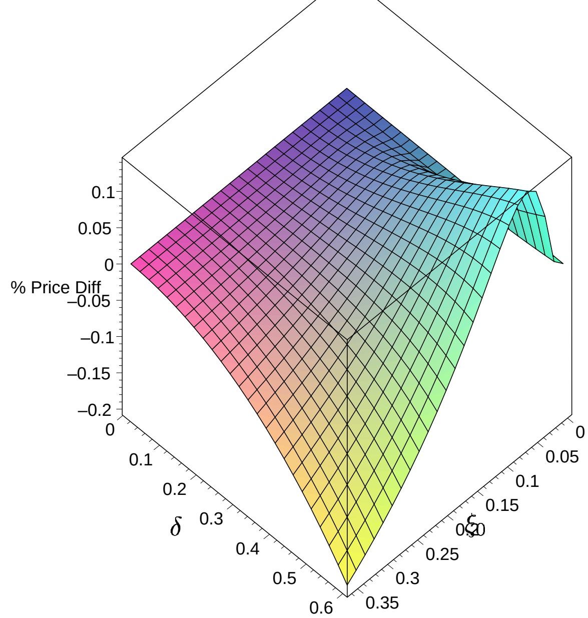

Figure 3: *Percentage difference in the passport option prices using the Hull and White model and exponential Brownian motion with* $S_0 = 100$, $T = 1$, $G_0 = 50$, $0 \leq \xi_0 \leq 0.36$ *and* $0 \leq \delta \leq 0.6$.

this example that for large values of ξ_0, $\mathbb{E}^0 V$ is large and stochastic volatility decreases the price. Indeed, this statement will always be true: for any value of G_0, a sufficiently large value of the initial volatility ξ_0 will lead to the passport option price being overestimated by the Black Scholes model. Finally, the magnitude of the effects becomes larger as δ increases. This is because as δ increases, the variance of volatility increases and this has a direct impact on price differentials as given in the second term on the right-hand-side of (16).

6.2 The Stein and Stein model

We consider the Stein and Stein model [23] in (5) under the pricing measure:

$$\begin{aligned} \frac{dS_t}{S_t} &= \theta_t dB_t^{\mathbb{Q}_0} \\ d\theta_t &= -\beta(\theta_t - \alpha)dt + \gamma dW_t^{\mathbb{Q}_0} \end{aligned} \tag{17}$$

where $W^{\mathbb{Q}_0}$, $B^{\mathbb{Q}_0}$ are independent Brownian motions, and θ is playing the role of ξ from earlier sections.

For this model we have $V \equiv V_T = \int_0^T \theta_u^2 du$ and the driving Markov process θ represents volatility rather than variance. The first moment for the Stein and Stein model is given below, for brevity the others are omitted.

$$\mathbb{E}^0 V = \left(\frac{\gamma^2}{2\beta} + \alpha^2\right) T + \frac{2\alpha}{\beta}(\theta_0 - \alpha)(1 - e^{-\beta T}) + (1 - e^{-2\beta T}) \left(\frac{\theta_0^2 + \alpha^2 - 2\alpha\theta_0}{2\beta} - \frac{\gamma^2}{4\beta^2}\right)$$

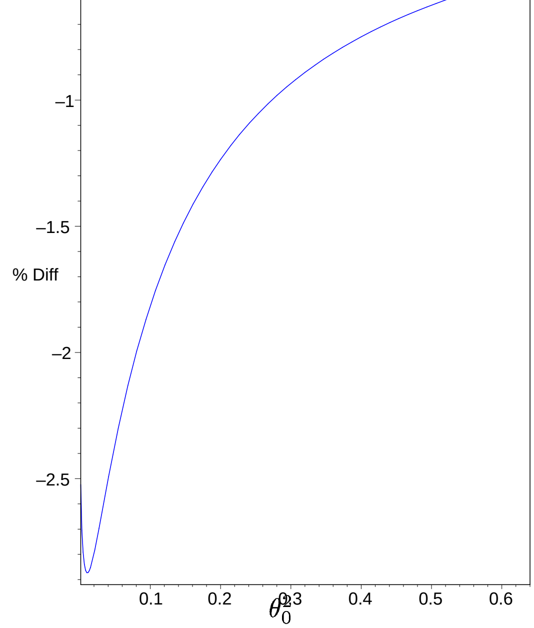

Figure 4: *Percentage difference in the passport option prices using the Stein and Stein model and exponential Brownian motion with* $S_0 = 100$, $T = 1$, $G_0 = 10$, $\beta = 1$, $\alpha = 0.2$ $\gamma = 0.2$ *and* $0 \leq \theta_0^2 \leq 0.64$.

This agrees with the expression given in Ball and Roma [3]. In calculating our approximation to the price of the passport option under stochastic volatility we use the first two terms in the expansion as it is expected (see Ball and Roma [3], Stein and Stein [23]) that the third term will not have a significant effect.

For this model we have broadly similar behaviour to the Hull White model in that again, the convexity of the lookback function determines the deviation from the exponential Brownian motion model. Note again that the exponential Brownian motion price is calculated using a calibrated variance $v = \mathbb{E}^0 V$ to ensure an accurate comparison with the Stein and Stein price. Consider first the situation when $G_0 = 10$, shown in Figures 4 and 5.

First notice that if $\gamma = 0$ the price difference is zero as $\mathbb{E}^0 V = V$ and volatility is deterministic. Secondly, in the Stein and Stein model, the volatility of volatility is independent of θ so even when $\theta_0 = 0$ we still have a difference between the two prices. This is in contrast to the Hull White model where the diffusion coefficient of the process ξ is linear in ξ. In that case, when $\xi_0 = 0$, ξ is deterministic, and indeed identically zero.

We see in Figure 5 that the Stein Stein price is less than the Black Scholes price for all values of θ_0. This can be explained because we have G_0 sufficently small so that $LB(S_0, v, S_0 + |G_0|)$ is concave over all values of v considered. This is because, (as noted earlier) the second derivative of $LB(S_0, v, S_0 + |G_0|)$ with respect to v is negative for values of v above about 0.01, and, whatever the value of θ_0, $\mathbb{E}^0 V$ is bounded below. For our parameter values this lower bound is at 0.033.

Notice that the price differences are larger than those for the Hull White

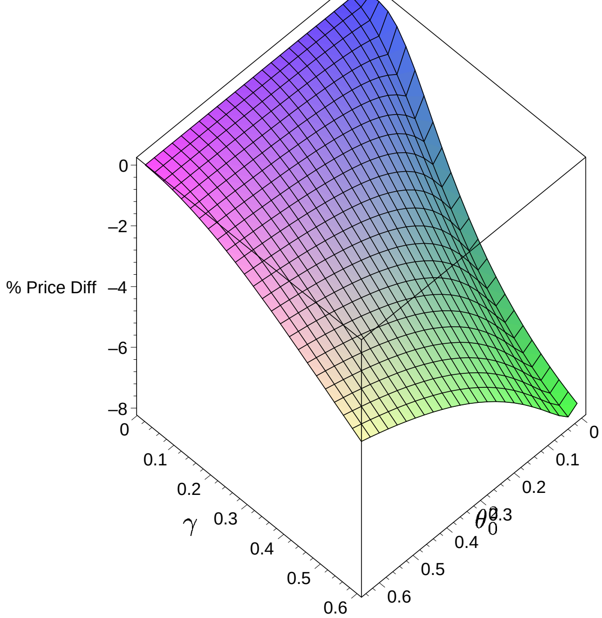

Figure 5: *Percentage difference in the passport option prices using the Stein and Stein model and exponential Brownian motion with* $S_0 = 100$*,* $T = 1$*,* $G_0 = 10$*,* $\beta = 1$*,* $\alpha = 0.2$*,* $0 \leq \theta_0^2 \leq 0.64$ *and* $0 \leq \gamma \leq 0.6$*.*

model, mainly due to parameter choice resulting in a greater $\mathrm{Var}(V)$. Observe in Figure 5 that as θ_0^2 becomes large, the percentage differences decrease. This is because, as v gets large, the concavity of the lookback pricing function decreases. To illustrate the effect on prices of higher values of G_0, we calculate a second example using $G_0 = 50$, given in Figures 6 and 8.

This time we have positive differences for small values of the initial squared volatility θ_0^2. This is because $LB(S_0, v, S_0 + |G_0|)$ is now convex over a wider region. However, θ_0^2 large puts $\mathbb{E}^0 V$ into the concave region. See Figure 7 for the second derivative of $LB(S_0, v, S_0 + |G_0|)$ with respect to v, showing that for low v the function is convex, and higher values of v lead to concavity.

In Figure 8, we see for small θ_0^2 and small γ, increasing γ leads to a larger percentage difference. This is due to increasing $\mathrm{Var}(V)$. However, as γ increases further, the price differences start to get smaller, before ultimately becoming negative. This is because $\mathbb{E}^0 V$ is also increasing and $LB''(S_0, \mathbb{E}^0 V, S_0 + |G_0|)$ switches sign to negative.

In summary, we observe substantially different behaviour for the new model. Whilst generally the effect of stochastic volatility is to depress passport option values the precise nature of this relationship is model dependent.

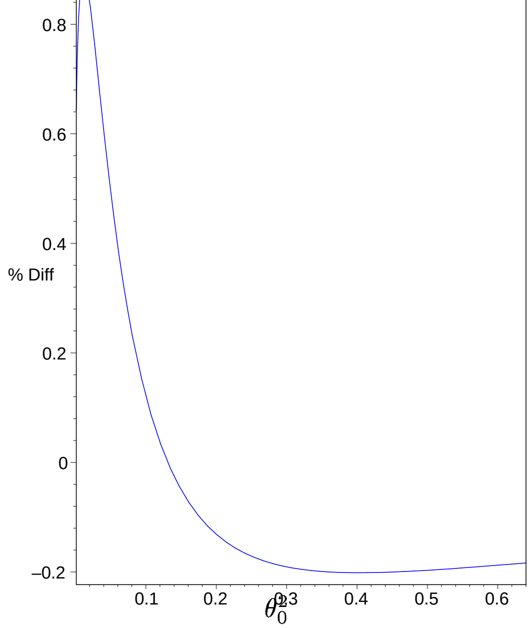

Figure 6: *Percentage difference in the passport option prices using the Stein and Stein model and exponential Brownian motion with* $S_0 = 100$, $T = 1$, $G_0 = 50$, $\beta = 1$, $\alpha = 0.2$ $0 \leq \theta_0^2 \leq 0.64$ *and* $\gamma = 0.2$

7 Conclusion

In this paper we have shown that the optimal strategy for a passport option as found by Hyer *et al* [16] and Andersen *et al* [2] is fairly robust to changes in the model. In particular we have shown that for a wide class of stochastic volatiltiy models, including many popular models in the literature, the optimal position for the option holder is to take a short position in the underlying when the gains process is positive and to take a long position when the gains are negative.

The price we calculate for the option is based on the assumption that the purchaser behaves optimally and follows this strategy. If the option holder acts in a sub-optimal fashion then the option writer can adapt his hedge to cover his obligations and to make riskless profits. See Henderson [9] and Shreve and Večeř [22] for a discussion of the form of the hedge.

The second objective of this article was to investigate the impact of stochastic volatility on passport option prices, and to compare the results with a constant volatility, Black-Scholes world. The key determinant of the difference in prices is the convexity of the price of a related lookback option.

By definition the passport option is a call option on a trading account with strike zero. If the initial value of the trading account is zero, so that in the notation of previous sections $G_0 = 0$, then the price of the lookback option is concave in the total variance parameter v. As a consequence, if the pair consisting of underlying and volatility follow a model as at (1), then the stochastic volatility model will underprice a passport option relative to the constant volatility model. This is the the opposite relationship to the effect of stochastic volatility

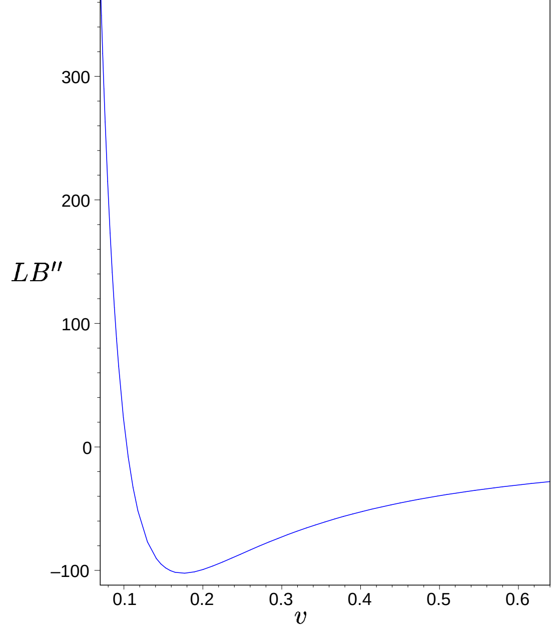

Figure 7: $LB''(100, v, 150)$ *as a function of variance* $0 \le v \le 0.6$ *with* $T = 1$, $G_0 = 50$

on call options.

During the lifetime of the option, and in particular when the trading account contains a non-zero amount the picture is more complicated. In these circumstances the impact of stochastic volatility depends on the model, the parameter values and the ratio of the trading account to the price of the underlying. Although the general pattern remains that stochastic volatility decreases the value of the passport option there are circumstances in which the reverse is true.

In this paper we have concentrated on models in which the underlying asset and volatility process are instantaneously uncorrelated. In this case we have been able to identify the optimal strategy and hence deduce the fair price for the option. When the driving Brownian motions B and W in (1) are correlated the picture is more complicated. From the representation in (18) it remains optimal to attempt to maximise the expected local time of G at zero. Since the local time at 0 depends on the volatility and the price level there is no simple rationale for determining the optimal strategy, and consequently it is impossible to price the passport option exactly.

Instead, in this paper we have concentrated on stochastic volatility models satisfying (1). The Hull-White and Stein-Stein models are of this form. Figures 1 - 8 show the impact of stochastic volatility within these models. We expect that this behaviour will be typical of the general behaviour and can provide a realistic guide to the problem of pricing passport options with stochastic volatility.

Acknowledgement

Thanks are due to Walter Schachermeyer who raised the question of passport

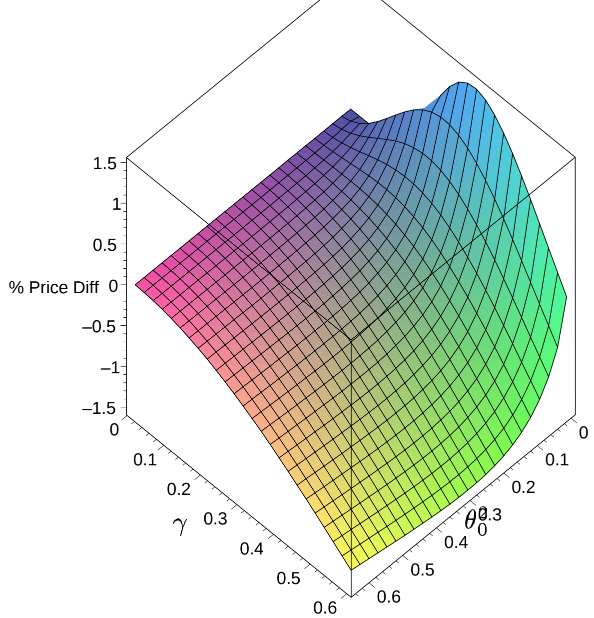

Figure 8: *Percentage difference in the passport option prices using the Stein and Stein model and exponential Brownian motion with* $S_0 = 100$*,* $T = 1$*,* $G_0 = 50$*,* $\beta = 1$*,* $\alpha = 0.2$*,* $0 \leq \theta_0^2 \leq 0.64$ *and* $0 \leq \gamma \leq 0.6$*.*

options with stochastic volatility during a visit by the first author to the University of Technology, Vienna, in May 1999, and people at seminars at Erasmus University, Rotterdam, and an ETH Risk-Day.

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8 Appendix

In the appendix we provide proofs of results which were postponed from the main text.

Lemma 8.1 *Under the model (3),* $\mathbb{Q}(\sup_{0\le r\le T} S_r^y \ge x)$ *is non-decreasing in* y.

Proof: Although S is not Markov, the pair (S,ξ) will be Markov and we exploit the independence of S and $W^{\mathbb{Q}}$ to p rove the result. Take two realisations of the pair $(S^{y_1},\xi),(S^{y_2},\xi)$, with the same realisation of the volatility process ξ and with S starting at y_1 and y_2 respectively where $y_2 > y_1$.

Define the first crossing time of S^{y_1} and S^{y_2} to be

$$\tau = \inf_u \{S_u^{y_2} \le S_u^{y_1}\}$$

and let

$$\tilde{S}_u^{y_1} = \begin{cases} S_u^{y_1} & 0 \le u \le (\tau \wedge T) \\ S_u^{y_2} & (\tau \wedge T) \le u \le T \end{cases}$$

Then by construction $\tilde{S}_u^{y_1} \le S_u^{y_2}$ for all u and all ω. The strong Markov property of the pair gives $(\tilde{S}_u^{y_1},\xi_u) \overset{law}{=} (S_u^{y_1},\xi_u)$. So

$$\mathbb{Q}\left(\sup_{0\le r\le T} S_r^{y_2} \ge x\right) \ge \mathbb{Q}\left(\sup_{0\le r\le T} \tilde{S}_r^{y_1} \ge x\right) = \mathbb{Q}\left(\sup_{0\le r\le T} S_r^{y_1} \ge x\right)$$

as required. □

Lemma 8.2 *Under the model (3), with* $s\sigma(u,z,s)$ *non-decreasing in* s *for each* u *and* z*, we have* $\mathbb{Q}(\sup_{0\le r\le T} S_r^y - y \ge z)$ *is non-decreasing in* y.

Proof: Take $y_2 > y_1$. We want to show

$$\mathbb{Q}\left(\sup_{0\le r\le T} S_r^{y_2} - y_2 \ge z\right) \ge \mathbb{Q}\left(\sup_{0\le r\le T} S_r^{y_1} - y_1 \ge z\right).$$

In particular, for each ω let $\xi(\omega)$ be given as the solution to its autonomous SDE. Define $(\Gamma^i)_{i=1,2}$ to be the solution (up to the first explosion time, if any) of the ordinary differential equation

$$\frac{d\Gamma_s^i}{ds} = \frac{1}{\sigma^2(\Gamma_s^i, \xi_{\Gamma_s^i}, W_s + y_i)(W_s + y_i)^2}.$$

Denote the inverse to Γ^i by A^i and define

$$S_t^{y_i} = y_i + W_{A_t^i}.$$

Then we have that

$$\begin{aligned} \frac{dA_u^i}{du} &= \sigma^2(u, \xi_u, W_{A_u^i} + y_i)(W_{A_u^i} + y_i)^2 \\ &= \sigma^2(u, \xi_u, S_u^{y_i})(S_u^{y_i})^2 \end{aligned}$$

and hence $dS_t^{y_i} = \sigma(t, \xi_t, S_t^{y_i}) S_t^{y_i} dB_t^i$ for some Brownian motion B^i, see Karatzas and Shreve [17, Theorem 3.4.6]. In order to prove that $\mathbb{Q}(\sup_{0 \le s \le A_T^i} W_s(\omega) \ge z)$ is non-decreasing in y_i it will be sufficient to show that $\Gamma_T^1(\omega) \ge \Gamma_T^2(\omega)$, $\forall \omega$. Since $y_2 > y_1$ and $S_u \sigma(u, \xi_u, S_u)$ is non-decreasing in S,

$$\left.\frac{d\Gamma_s^1}{ds}\right|_{s=0} = \frac{1}{\sigma^2(0, \xi_0, y_1)(y_1)^2} \ge \frac{1}{\sigma^2(0, \xi_0, y_2)(y_2)^2} = \left.\frac{d\Gamma_s^2}{ds}\right|_{s=0}$$

Defining $\tau = \inf_u \{\Gamma_u^2(\omega) < \Gamma_u^1(\omega)\}$ we have $\Gamma_\tau^2 = \Gamma_\tau^1 = \gamma$ (say) and

$$\left.\frac{d\Gamma_s^1}{ds}\right|_{s=\tau} = \frac{1}{\sigma^2(\gamma, \xi_\gamma, W_\tau + y_1)(W_\tau + y_1)^2} \ge \frac{1}{\sigma^2(\gamma, \xi_\gamma, W_\tau + y_2)(W_\tau + y_2)^2} = \left.\frac{d\Gamma_s^2}{ds}\right|_{s=\tau}$$

Thus $\Gamma_u^1(\omega) \ge \Gamma_u^2(\omega)$ for all u, uniformly in ω, as required. □

Now we consider the model in (3) with $M_p(v) = \int_0^p v_r dS_r$ and $\overline{M}_z(v) = \sup_{0 \le r \le z} M_r(v)$ and prove the main result relating the price of a passport option to the price of a lookback option. This result extends and corrects the proof of Proposition 3.3b in ([10]).

Theorem 8.3 *If $s\sigma(u, \xi, s)$ is non-decreasing in s, then*

$$\begin{aligned} \sup_{|q| \le 1} \mathbb{E}^{\mathbb{Q}} \left(k + \int_0^T q_r dS_r \right)^+ &= k^+ + \tfrac{1}{2} \sup_{|v| \le 1} \mathbb{E}^{\mathbb{Q}} (\overline{M}_T(v) - |k|)^+ \\ &= k^+ + \tfrac{1}{2} \mathbb{E}^{\mathbb{Q}} (\overline{S}_T - (S_0 + |k|))^+. \end{aligned}$$

Proof: Fix q and set $G_t(q) = k + \int_0^t q_u dS_u$. Using Tanaka's formula (see Revuz and Yor [19, VI Theorem 1.2]) on $|G_t|$ we have

$$(18) \qquad |G_T(q)| = |k| + \int_0^T \operatorname{sgn}(G_u(q)) dG_u(q) + L_T^{G(q)}(0)$$

where $L_T^{G(q)}(0)$ is the local time of process $G(q)$ at level zero until time T.

The Skorokhod lemma (see Revuz and Yor [19, VI Lemma 2.1]) gives

$$\begin{aligned} L_T^{G(q)}(0) &= \left(\sup_{0 \le u \le T} \left(\int_0^u -\mathrm{sgn}(G_s(q)) q_s dS_s \right) - |k| \right)^+ \\ &= \left(\sup_{0 \le u \le T} \left(\int_0^u v_s dS_s \right) - |k| \right)^+ \end{aligned}$$

where

(19) $$v_s = -\mathrm{sgn}(G_s(q)) q_s.$$

So taking expectations in (18) and using the characterisation for the local time above and the definition of $\overline{M}_u(v)$

$$\mathbb{E}^{\mathbb{Q}}|G_T| = |k| + \mathbb{E}^{\mathbb{Q}} \left(\sup_{0 \le u \le T} \left(\int_0^u v_s dS_s \right) - |k| \right)^+ = |k| + \mathbb{E}^{\mathbb{Q}}(\overline{M}_T(v) - |k|)^+.$$

Further, using the martingale property of G,

(20) $$\mathbb{E}^{\mathbb{Q}} G_T^+ = k^+ + \tfrac{1}{2}\mathbb{E}^{\mathbb{Q}}(\overline{M}_T(v) - |k|)^+.$$

Since for any q with $|q| \le 1$ we can define v as at (19) with $|v| \le 1$, it follows that

(21) $$\sup_{|q| \le 1} \mathbb{E}^{\mathbb{Q}} G_T^+(q) \le k^+ + \sup_{|v| \le 1} \tfrac{1}{2}\mathbb{E}^{\mathbb{Q}}(\overline{M}_T(v) - |k|)^+.$$

Given (20) we would like to conclude that there is equality in (21). In fact this is true quite generally, see Delbaen and Yor [4]. The issue is that given $|v| \le 1$ it is not clear that q can be defined via (19) since q appears twice both explicitly and implicitly via G. However from the results in §4 we know that the maximising v in

$$\sup_{|v| \le 1} \tfrac{1}{2}\mathbb{E}^{\mathbb{Q}}(\overline{M}_T(v) - |k|)^+$$

is $v = 1$. Thus it is sufficent to find $\hat{q}$ with

(22) $$\mathbb{E}^{\mathbb{Q}} G_T^+(\hat{q}) = k^+ + \tfrac{1}{2}\mathbb{E}^{\mathbb{Q}}(\overline{M}_T(1) - |k|)^+.$$

Define $\hat{G}$ via $d\hat{G} = -\mathrm{sgn}(\hat{G})dS$. This equation has a (weak) solution under the hypotheses on σ in §2, (see Henderson [9]) so that with (q, G) defined by $\hat{q} = -\mathrm{sgn}(\hat{G}(\hat{q}))$ we have, from (20)

$$k^+ + \tfrac{1}{2}\mathbb{E}^{\mathbb{Q}}(\overline{M}_T(1) - |k|)^+ = \mathbb{E}^{\mathbb{Q}}(\hat{G}_T(\hat{q}))^+ \le \sup_{|q| \le 1} \mathbb{E}^{\mathbb{Q}} G_T^+(q).$$

Thus there is equality everywhere in (21) and hence the theorem is proved.

□