Comparison Results for Stochastic Volatility Models via Coupling

David Hobson

October 8, 2007

Abstract

The aim of this paper is to investigate the properties of stochastic volatility models, and to discuss to what extent, and with regard to which models, properties of the classical exponential Brownian motion model carry over to a stochastic volatility setting. The properties of the classical model of interest include the fact that the discounted stock price is positive for all t but converges to zero almost surely, the fact that it is a martingale but not a uniformly integrable martingale, and the fact that European option prices (with convex payoff functions) are convex in the initial stock price and increasing in volatility. We give examples of stochastic volatility models where these properties continue to hold, and other examples where they fail.

The main tool is a construction of a time-homogeneous autonomous volatility model via a time change.

1 Introduction

Notwithstanding the success of the Samuelson-Black-Scholes model, it is a truth, universally acknowledged, that the model fails to capture many observed features of financial data. Evidence of this failure manifests itself in (at least) two ways. Firstly, an analysis of the historical time series shows that volatility is not constant, and secondly, and more importantly from the derivative pricing perspective, the prices of vanilla traded options exhibit smiles and skews, so that the market does not price consistently using the Black-Scholes model.

There have been many responses to these facts in the literature including Garch models and their generalisations, level-dependent volatility models, (Cev models and displaced diffusion models), jump-diffusion models and local-volatility models. However, probably the most popular and widespread extension of the exponential Brownian motion model is to stochastic volatility models, in which the asset price process is augmented by an auxiliary volatility process which is itself random.

The exponential Brownian motion model is characterised by its simple structure. This structure leads to lots of nice but potentially misleading properties — for example the discounted price process is a true martingale which remains strictly positive — and simple comparative statics, including the fact that the price of a call option is increasing in volatility.

The aim of this paper is to study the analogue of these questions in stochastic volatility models. We answer questions of the form *can the price process hit zero?*, *does the discounted price process converge?*, *are discounted prices true-martingales?*, *are option prices convex?* and *are option prices monotonic in the model parameters?*. Since we are interested in stochastic volatility models for the purpose of derivative pricing, we work under a martingale measure. Note, however, that in a stochastic volatility model there can be no unique equivalent martingale measure, and that the selection of a particular pricing measure is a modelling choice. One of the reasons for studying these properties of stochastic volatility models is to understand the impact of this choice.

In a stochastic volatility setting the question about whether S is a true martingale was studied by Sin [33], Lewis [26] and Andersen and Piterbarg [1]. Issues related to the convexity of the option pricing function were studied in Bergman et al [5], see also Janson and Tysk [23] who introduced the notion of convexity preserving models in multi-dimensions. Henderson [13] and Henderson et al [15] proved a comparison theorem - namely that option prices were monotonic in the market price of volatility risk - between option prices under different martingale measures. This paper extends and complements this work.

We will solely consider stochastic volatility models, but it should be noted that related questions arise in other classes of models, and have been studied elsewhere in the literature. For example Henderson and Hobson [14] and Ekström and Tysk [9] investigate the properties of jump diffusion models, and Bergenthum and Rüschendorf [4] have some comparison results for general semi-martingales.

The main results of this paper are a construction of the solution to a stochastic volatility model (Theorem 3.1), the use of this construction to derive results describing when the discounted asset price can hit zero, and when it is a martingale (Theorem 4.2) and a comparison theorem for option prices in different stochastic volatility models (Theorem 6.4). These theoretical results are augmented by discussion of several examples, including both famed volatility models from the literature, and new models which illustrate the various phenomena.

Notation: We will use the following notation consistently throughout the paper: for a diffusion process Z_t, B_t^Z will be the Brownian motion which drives Z in the stochastic differential equation (SDE) representation; $H_z^Z := \inf\{u \geq 0 : Z_u = z\}$ will be the first hitting time of level z by Z; given $\rho \in (-1, 1)$ and B_t^Z, $\rho^\perp$ will denote $\rho^\perp = \sqrt{1-\rho^2}$, and $B_t^{Z,\perp}$ (often abbreviated to $B_t^\perp$) will denote a Brownian motion which is independent of B_t^Z. All Brownian motions are normalised so that $B_0 = 0$.

2 Stochastic Volatility Models

We work on a model $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ supporting two Brownian motions, and satisfying the usual conditions.

Consider the bi-variate model for the price of a traded asset P and a (non-traded) auxiliary process V under the physical measure $\mathbb{P}$:

$$\begin{aligned} P_0 &= p & dP_t &= \eta_t dB_t^{\mathbb{P},P} + \mu_t dt \\ V_0 &= v & dV_t &= a_t dB_t^{\mathbb{P},V} + b_t^{\mathbb{P}} dt \\ & & dB_t^{\mathbb{P},P} dB_t^{\mathbb{P},V} &= \rho_t dt. \end{aligned} \tag{1}$$

We will be concerned with questions related to option pricing in which case it is natural to work under an equivalent martingale measure $\mathbb{Q}$, and further it will be convenient to work with discounted asset prices (we write S_t for P_t discounted by the bond price). Then the model of interest becomes

$$\begin{aligned} S_0 &= s & dS_t &= \eta_t dB_t^S \\ V_0 &= v & dV_t &= a_t dB_t^V + b_t dt \\ & & dB_t^S dB_t^V &= \rho_t dt, \end{aligned}$$

where B^S and B^V are $\mathbb{Q}$-Brownian motions, b is related to $b^{\mathbb{P}}$ through the associated change of measure, and $s = p$. If we assume that the pair (S, V) is a bi-variate diffusion then

$$\begin{aligned} S_0 &= s & dS_t &= \eta(S_t, V_t, t) dB_t^S \\ V_0 &= v & dV_t &= a(S_t, V_t, t) dB_t^V + b(S_t, V_t, t) dt \\ & & dB_t^S dB_t^V &= \rho(S_t, V_t, t) dt. \end{aligned}$$

Under the further assumptions that the model is time-homogeneous, that V is autonomous (in a strong form such that a, b and ρ are functions of V alone) and that η factorises into measurable functions σ, g such that $\eta(S_t, V_t) \equiv \sigma(S_t) g(V_t)$ (and then, by a change of variable if necessary, we may assume that g is the identity function) we are left with our final model (on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{Q})$)

$$\begin{aligned} S_0 &= s & dS_t &= \sigma(S_t) V_t dB_t^S \\ V_0 &= v & dV_t &= \alpha(V_t) dB_t^V + \beta(V_t) dt \\ & & dB_t^S dB_t^V &= \rho(V_t) dt. \end{aligned} \tag{2}$$

Here we have written α and β instead of a or b purely to distinguish this model from previous versions. Many papers model the instantaneous variance V^2 rather than the volatility V, but this is a simple re-parameterisation. From the context it is natural to assume that the pair (S_t, V_t) has state space the first quadrant. For reasons of limited liability we assume that if the local martingale S reaches zero, then it is absorbed there. If V can reach zero, then we need to add appropriate boundary conditions of which the most natural is to assume that zero is a reflecting boundary.

Clearly, at each stage above the rewriting of the problem is only valid under some technical conditions, and the addition of further assumptions is at some considerable loss of generality. For example, we *assume* the existence of an equivalent martingale measure. Moreover, it is unlikely that a time-homogeneous model will be able to fit an initial term-structure of volatility. However, most stochastic volatility models in the literature are special cases of the formulation in (2), written under some pricing measure $\mathbb{Q}$, and for this reason we will use (2) as our starting point. Indeed most models take $\sigma(S) = S$ and take the correlation to be constant.

Hull and White [20] modelled V as an exponential Brownian motion. Wiggins [34] and Scott [32] added a mean reversion co-efficient so that the logarithm of the volatility followed an Ornstein-Uhlenbeck (OU) process. (Scott [32] also considered a model in which volatility itself followed an OU process, but this is slightly outside the scope of our paper since V can go negative, which raises issues about whether V is measurable with respect to the observable filtration generated by S.) Hull and White [21], see also Heston [16], proposed a model in which the volatility is given by a Bessel process with an additional mean reversion co-efficient. Lewis suggested a model with $\alpha(v) = v^2$ which is useful for several counter-examples.

All the models in the previous paragraph assume that σ is a constant multiple of S. There are also a small number of models which incorporate a leverage effect. Johnson and Shanno [24] and Melino and Turnbull [28] assume that $\sigma(S) = S^\alpha$. The SABR model of Hagan et al [11] is also of this form, and in the SABR model $\ln V$ is modelled as a Brownian motion. If we are outside the log-linear case then we need some regularity assumptions on σ:

Assumption 2.1. σ is positive and continuous on the positive reals.

By convention $\sigma(s)$ is identically zero for $s \leq 0$.

3 The main coupling

Our aim is to construct a pair (S_t, V_t) on a suitable probability space such that

$$\begin{aligned} S_0 &= s > 0 & dS_t &= \sigma(S_t)V_t dB_t^S \\ V_0 &= v \geq 0 & dV_t &= \alpha(V_t)dB_t^V + \beta(V_t)dt \\ & dB_t^S dB_t^V = \rho(V_t)dt, \end{aligned} \tag{3}$$

in such a way that we can provide useful couplings, from which it will be possible to derive comparison results.

Theorem 3.1. *Suppose that $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t>0}, \mathbb{Q})$ is a Brownian filtration, satisfying the usual conditions. Suppose that the* SDE

$$\begin{aligned} X_0 &= s & dX_t &= \sigma(X_t)dB_t^X \\ Y_0 &= v & dY_t &= \frac{\alpha(Y_t)}{Y_t}dB_t^Y + \frac{\beta(Y_t)}{Y_t^2}dt \\ & dB_t^X dB_t^Y = \rho(Y_t)dt \end{aligned} \tag{4}$$

has a unique strong solution, up to the first explosion time ε.

Define $\Gamma_t = \int_0^t Y_s^{-2} ds$, and set $A \equiv \Gamma^{-1}$.

Then $S_t \equiv X_{A_t}$ and $V_t \equiv Y_{A_t}$ solve (3).

More precisely, let $\zeta = \lim_{t \uparrow \varepsilon} \Gamma_t \leq \infty$, so that $A_\zeta = \varepsilon$, and for $t \leq \zeta$ set $\mathcal{F}_t = \mathcal{G}_{A_t}$ and define

$$B_t^S = \int_0^{A_t} \frac{dB_u^X}{Y_u} \qquad B_t^V = \int_0^{A_t} \frac{dB_u^Y}{Y_u}.$$

Then, for $t \leq \zeta$, B_t^S and B_t^V are $\mathcal{F}_t$-Brownian motions and $(S_t \equiv X_{A_t}, V_t \equiv Y_{A_t})$ is a weak solution to (3).

Proof. Note that Γ is strictly increasing and continuous (at least until Y hits zero or infinity). Hence A is well defined and $A_{\Gamma_t} = t = \Gamma_{A_t}$.

Set

$$M_t = \int_0^t \frac{dB_u^Y}{Y_u}$$

and let

$$B_t^V = M_{A_t} = \int_0^{A_t} \frac{dB_u^Y}{Y_u} = \int_0^t \frac{dB_{A_w}^Y}{Y_{A_w}}.$$

Then, $\langle M \rangle_t = \Gamma_t$, and by the Dambis-Dubins-Schwarz theorem (Karatzas and Shreve [25][Theorem 3.4.6, p174], see also Revuz and Yor [29][V.1.6, p181]), B_t^V is a $\mathcal{F}_t$-Brownian motion. Furthermore, $V_t \equiv Y_{A_t}$ solves

$$\begin{aligned} dV_t &= \alpha(Y_{A_t})\frac{dB^Y_{A_t}}{Y_{A_t}} + \beta(Y_{A_t})\frac{dA_t}{Y^2_{A_t}} \\ &= \alpha(V_t)dB_t^V + \beta(V_t)dt. \end{aligned}$$

By an identical argument we can conclude that B_t^S is a $\mathcal{F}_t$-Brownian motion and $dS_t = dX_{A_t} = \sigma(X_{A_t})dB^X_{A_t} = \sigma(S_t)V_t dB_t^S$.

Finally

$$\langle B^S, B^Y \rangle_t = \int_0^{A_t} \frac{d\langle B^X, B^Y\rangle_u}{Y_u^2} = \int_0^{A_t} \rho(Y_u)d\Gamma_u = \int_0^t \rho(V_s)ds.$$

□

At a first reading of Theorem 3.1, the explosion time ε should be taken to be the first time that either X or Y hits 0 or ∞. However, with $\mathbb{Q}$-probability one X does not explode to infinity, and if X hits zero, then thereafter we can define X and S to be identically zero. Further the assumption that σ is positive and continuous on $\mathbb{R}_+$ ensures that X_t cannot converge to a positive value as $t \uparrow \infty$. (Note that S_t may still converge if A_t converges.)

The only remaining cases are when Y hits zero or infinity. If the first explosion time occurs when Y hits zero, and if at that point Γ is finite, then it may be possible to extend the construction beyond this moment. If V_t is assumed to be instantaneously reflecting at zero, then it is natural to choose the solution to (4) for which Y_t is also instantaneously reflecting, provided such a solution exists. The important consideration is whether Γ is well defined beyond the first hit of Y_t on zero, and this can be checked via a scale and speed analysis.

Finally, it is possible that Y explodes to infinity, in which case it is not possible to determine the behaviour of Y from the SDE alone, and it is necessary to specify additional boundary conditions. There is an analytic condition involving the co-efficients of the SDE for Y which determines exactly when it is possible for Y to explode, see Rogers and Williams [30][Theorem 52.1].

To avoid issues of this type, except where otherwise stated in the examples we make the following assumption throughout the rest of the paper;

Assumption 3.2. The process Y given in (4) does not explode; moreover, if H_0^Y is finite then either $\Gamma_{H_0^Y}$ is infinite or $\Gamma_{H_0^Y+}$ is finite. In particular we do not have $\Gamma_{H_0^Y} < \Gamma_{H_0^Y+} = \infty$.

Under Assumption 3.2, and given our comments about zero being absorbing for S, we may take the first explosion time ε in Theorem 3.1 to be the first explosion time of Γ:

$$\varepsilon = \sup\{u \geq 0 : \Gamma_u < \infty\}.$$

By Assumption 3.2, if $\varepsilon < \infty$ then $\zeta = \Gamma_\varepsilon = \infty$. Hence, the construction in Theorem 3.1 is valid for all time.

Note that V will explode if Y explodes, or if Y diverges to infinity and the time-change A explodes. By assumption we have excluded the former possibility, but not the latter. However, if A explodes then S converges to zero, at least under our assumption that σ is continuous and positive on $\mathbb{R}_+$.

Issues of this type become important in some of the examples discussed below, beginning with the Bessel process model.

Remark 3.3. It is possible to give Lipschitz conditions on the stochastic differential equations for Y which guarantee existence of a strong solution, but these conditions typically rule out several examples of interest in finance. Indeed, since Y is a one-dimensional diffusion, there are weaker conditions for the existence of a strong solution, see for example Revuz and Yor [29][Theorem IX.3.5].

Remark 3.4. The construction in Theorem 3.1 generates a weak solution for the pair (S, V). However, this is the appropriate form of solution in finance, in that the statistical properties of the price process are specified, but never the driving Brownian motion.

4 Transience Convergence and Martingale properties for log-linear models

Our goal in this section is to use the construction of the previous section to discuss some of the issues raised in the introduction about stochastic volatility models, namely can S hit 0 in finite time; does S_t converge to a positive limit; is S_t a true martingale; is S_t a uniformly integrable martingale? In this section we concentrate on the log-linear case in which $\sigma(s) = \sigma s$. Then by absorbing the constant σ into the volatility process we may assume without loss of generality that $\sigma(s) = s$.

To date the literature has mainly concentrated on the third of these questions concerning whether the price process is a true martingale or merely a local martingale. This problem has been considered by Sin [33], Lewis [26] and Andersen and Piterbarg [1]. As Lewis [26] has shown (see also Heston et al [17] and Cox and Hobson [6]), if S is a strict local martingale then put-call parity fails and care is needed over other 'obvious' properties of option prices.

The clever idea in Sin [33] can be summarised as follows. Suppose $\sigma(S) = S$ and ρ is constant, and write $B_t^V = \rho B_t^S + \rho^\perp B_t^\perp$ where $B^\perp$ is independent of B^S. Suppose that S given by $dS = S_t V_t dB_t^S$ is a true martingale under $\mathbb{Q}$ on $[0, T]$. Then $\tilde{\mathbb{Q}}$ defined by $d\tilde{\mathbb{Q}}/d\mathbb{Q} = S_T$ on $\mathcal{F}_T$ is a probability measure which is absolutely continuous with respect to $\mathbb{Q}$, and under $\tilde{\mathbb{Q}}$, $dV_t = a(V_t)dB_t^V + (\beta(V_t) + \rho\alpha(V_t)V_t)dt$. However, this introduces a potential contradiction if the behaviour of the autonomous diffusion V is different under $\mathbb{Q}$ and $\tilde{\mathbb{Q}}$, for example, if V explodes with positive probability under $\tilde{\mathbb{Q}}$ but not under $\mathbb{Q}$. For certain examples, Sin is able to complete the analysis to derive an if and only if condition.

Theorem 4.1 (Sin [33])**.** *Consider the stochastic volatility model in (2). Suppose $\sigma(s) = s$, $\alpha(v) = \alpha v$, $\beta(v) = \beta_0 - \beta_1 v$, and $\rho(v) = \rho$ with α and β_1 positive. Then S_t is a true martingale if and only if $\rho \leq 0$.*

The methods of Sin give a general approach for considering the true martingale question for log-linear models. Andersen and Piterbarg [1] exploited these ideas to prove the martingale property in other models including the Bessel process model below. Conversely, Lewis [26] gives examples where the martingale property could be shown to fail.

We can use the construction of Theorem 3.1 to make statements about the martingale property, and about the transience and convergence properties of S_t. Recall that $\varepsilon = \sup\{u : \Gamma_u < \infty\}$, and $\zeta = \lim_{t\uparrow\varepsilon} \Gamma_t$. By Assumption 3.2, if ε is finite then ζ is infinite. Write Ω as the disjoint union $\Omega = \Omega_\zeta \cup \Omega_\varepsilon \cup \Omega_\infty$ where

$$
\begin{aligned}
\Omega_\zeta &= \{\omega : \varepsilon = \infty, \zeta < \infty\} \\
\Omega_\varepsilon &= \{\omega : \varepsilon = H_0^Y < \infty, \zeta = \infty\} \\
\Omega_\infty &= \{\omega : \varepsilon = \infty, \zeta = \infty\}
\end{aligned}
$$

Note that the event $\{\omega : \varepsilon = H_0^Y < \infty, \zeta < \infty\}$ is ruled out either by Assumption 3.2, or because ε is not the first explosion time of Γ.

Theorem 4.2. *Suppose $\sigma(s) = s$. Then, modulo null sets,*
on Ω_ζ we have that S hits zero in finite time,
on Ω_ε we have that S_t converges and the limit S_∞ is strictly positive, and
on Ω_∞ we have that S_t is positive for all t, but tends to zero.

The discounted price process $(S_t)_{t\leq T}$ is a true martingale if and only if $\lim_{\gamma\uparrow\infty} e^\gamma \mathbb{Q}(\sup_{t\leq A_T} B_t^X - t/2 > \gamma) \to 0$. Furthermore, S_t is a uniformly integrable martingale if and only if $\lim_{\gamma\uparrow\infty} e^\gamma \mathbb{Q}(\sup_{t\leq\varepsilon} B_t^X - t/2 > \gamma) \to 0$. Sufficient conditions for the martingale and uniformly integrability properties are given by $\mathbb{E}[e^{A_T/2}] < \infty$ and $\mathbb{E}[e^{\varepsilon/2}] < \infty$ respectively.

Proof. We have that $S_t = se^{B^X_{A_t} - A_t/2}$, and clearly, $B^X_t - t/2 \to -\infty$. On Ω_ζ we have that $A_t \uparrow \infty$ as $t \uparrow \zeta$ and hence $S_\zeta = 0$.

On Ω_ε we have $\varepsilon < \infty$ and $S_\infty = se^{B^X_\varepsilon - \varepsilon/2} > 0$.

Otherwise, on Ω_∞, A_t is finite for each t and $S_t = se^{B^X_{A_t} - A_t/2}$ is positive for each t but tends to zero as t and A_t increases to infinity.

The statements about martingales are direct applications of Théorème 1a in Azéma et al [3] and the Novikov condition. □

Consider the price of a put option with strike K. For $T \le T'$ we have

$$\mathbb{E}[(K - S_{T'})^+ | \mathcal{F}_T] \ge (K - \mathbb{E}[S_{T'} | \mathcal{F}_T])^+ \ge (K - S_T)^+$$

where the two inequalities follow by Jensen's inequality and the supermartingale property of the local martingale S_t. It follows that put prices are increasing in maturity and

$$\lim_{T \uparrow \infty} \mathbb{E}[(K - S_T)^+] = \mathbb{E}[(K - S_\infty)^+]$$

A similar result holds for any bounded decreasing convex payoff function.

Now consider a call option. Then, although we have $\mathbb{E}[(S_{T'} - K)^+ | \mathcal{F}_T] \ge (\mathbb{E}[S_{T'} | \mathcal{F}_T] - K)^+$ it is not necessarily the case that $\mathbb{E}[S_{T'} | \mathcal{F}_T] \ge S_T$ and monotonicity in maturity of option prices does not follow, unless S is a true martingale. Furthermore, even in the martingale case we only have that $\lim_{T \uparrow \infty} \mathbb{E}[(S_T - K)^+] = \mathbb{E}[(S_\infty - K)^+]$ if $(S_t)_{t \ge 0}$ is uniformly integrable.

4.1 Examples

4.1.1 The lognormal volatility model.

Consider the following model introduced by Hull and White [20]: the version of (3) with

$$\sigma(s) = s, \qquad \alpha(v) = av, \qquad \beta(v) = bv, \qquad \rho(v) = \rho,$$

with $a > 0$. Then $dX = XdB^X$ and $dY = adB^Y + (b/Y)dt$. If we set $Z = Y/a$ then

$$dZ_t = dB^Y_t + \frac{b}{a^2 Z_t} dt$$

so that Z is a Bessel process of dimension $\phi = 1 + 2b/a^2$. Then Y can hit zero if and only if $\phi < 2$, or equivalently $b < a^2/2$. It follows from Proposition A.1 that if $b < a^2/2$ then $\Gamma_t = a^2 \int_0^t Z_s^{-2} ds$ explodes the first time that Y hits zero, and $\mathbb{Q}(\Omega_\varepsilon) = 1$; moreover Assumption 3.2 is automatically satisfied. Then S_t converges to a positive limit and $S_\infty = \exp(B^X_{H^Y_0} - H^Y_0/2)$.

Conversely, if $b \geq a^2/2$, then (with probability one) Y does not hit zero, and Γ does not explode. On the other hand, again by Proposition A.1 nor does Γ converge: even when $b > a^2/2$ and Y tends to infinity almost surely, it only grows at rate $\sqrt{t}$. It follows that $S_t \to 0$ almost surely.

Now suppose that we modify the drift condition to become $\beta(v) \leq 0$, so that

$$\sigma(s) = s, \quad \alpha(v) = a, \quad \beta(v) \leq 0 \quad \rho(v) = \rho.$$

Then $Y_t \leq v + aB_t^Y$, $\mathbb{Q}(H_0^Y < \infty) = 1$ and

$$\int_0^{H_0^Y} \frac{ds}{Y_s^2} \geq \int_0^{H^{B^Y}_{-v/a}} \frac{ds}{(v + aB_s^Y)^2} = \infty$$

so that $\mathbb{Q}(\Omega_\varepsilon) = 1$. Write $B_t^X = \rho B_t^Y + \rho^\perp B_t^\perp$, where $B^\perp$ is independent of B_t^Y. Then, by Theorem 4.2, S_t converges and $S_\infty = s \exp(\rho B^Y_{H_0^Y} + \rho^\perp B^\perp_{H_0^Y} - H_0^Y/2) > 0$.

Now suppose also that $\rho < 0$. Then, for $t \leq H_0^Y$, B_t^Y is bounded below by $-v/a$ and $\exp(\rho B^Y_{H_0^Y})$ is bounded. Furthermore, $A_T \leq \varepsilon \leq H_0^Y \leq H^{B^Y}_{-v/a}$ and for sufficiently large γ,

$$\begin{aligned}\mathbb{Q}\left(\sup_{t<\varepsilon}(\rho B_t^Y + \rho^\perp B_t^\perp - t/2) > \gamma\right) \quad &\leq \quad \mathbb{Q}\left(\sup_{t<\infty}(\rho^\perp B_t^\perp - t/2) > \gamma + \rho v/a\right) \\ &= \quad e^{-(\gamma+\rho v/a)/(1-\rho^2)}\end{aligned}$$

and then $e^\gamma \mathbb{Q}(\sup_{t<\varepsilon}(\rho B_t^Y + \rho^\perp B_t^\perp - t/2) > \gamma) \to 0$. Hence, by Theorem 4.2, if $\beta(v) \leq 0$ and $\rho < 0$ then $(S_t)_{t \leq \infty}$ is a uniformly integrable martingale. It follows that if we take the lognormal volatility model with $\beta(v) = bv$ for $b < 0$, and if $\rho < 0$, then the model has features which distinguish it from the exponential Brownian motion model. For example, if we consider put options with maturity T and payoff $(K - S_T)^+$ then unlike in the exponential Brownian case, the prices of such options do not tend to K with T.

4.1.2 The Bessel process model

For this model, variants of which were introduced by Hull and White [21] and Heston [16], we have

$$\sigma(s) = s, \quad \alpha(v) = a, \quad \beta(v) = a\left(\frac{\gamma}{v} - \delta v\right), \quad \rho(v) = \rho$$

where a is a positive parameters. (Hull and White [21] assume that $\delta = 0$, whereas Heston [16] takes $\gamma = 0$.) Then

$$dY_t = \frac{a}{Y_t} dB_t^Y + a\left(\frac{\gamma}{Y_t^3} - \frac{\delta}{Y_t}\right) dt$$

and if we set $Z = Y^2/2a$ and $\phi = \gamma/a \ + 3/2$ then

$$dZ_t = dB_t^Y + \left(\frac{\phi - 1}{2Z_t} - \delta \right) dt,$$

so that Z is the radial part of a ϕ-dimensional Ornstein-Uhlenbeck process (suitably interpreted when ϕ is not an integer). If we define $\tilde{\mathbb{Q}}$ via $d\tilde{\mathbb{Q}}/d\mathbb{Q}|_{\mathcal{F}_t} = \exp(\delta B_t^Y - \delta^2 t/2)$ then $\tilde{B}_t^Y \equiv B_t^Y - \delta t$ is a $\tilde{\mathbb{Q}}$-Brownian motion, and Z_t is a $\text{Bes}(\phi)$ process under $\tilde{\mathbb{Q}}$.

If $\phi < 2$ then Z (and hence Y) can and will hit zero in finite time, but in contrast to the lognormal model, for the Bessel process model we have $\Gamma_{H_0^Y} = 2a \int_0^{H_0^Y} Z_s^{-1} ds < \infty$, see Proposition A.1. We now get two cases depending on whether $\phi > 1$, or $\phi \leq 1$. If $\phi > 1$ then by taking Y instantaneously reflecting at zero we can continue the process beyond the first hit of Y on zero in such a way that Γ increases to infinity almost surely, but does not explode, see Corollary A.2. Using this extension we have $S_t > 0$ for all t, but $S_\infty = 0$ almost surely. However, if $\phi \leq 1$ then even though $\Gamma_{H_0^Y} < \infty$ the first explosion time ε for Γ is H_0^Y. In this case Assumption 3.2 fails, and it is not possible to extend the construction in Theorem 3.1 beyond H_0^Y.

Otherwise, if $\phi \geq 2$ then $Y^2 \equiv 2aZ$ does not hit zero, and is a positive recurrent diffusion on state space $(0, \infty)$. Hence Γ_t increases to infinity almost surely, but does not explode. In this case $\mathbb{Q}(\Omega_\infty) = 1$ and again $S_t > 0$ for all t, but $S_\infty = 0$ almost surely.

4.1.3 Lewis's 3/2 model

Lewis [26] proposes a model for the squared volatility $U \equiv V^2$ of the form $dU = 2\alpha U^{3/2} dW_t + (2\beta + \alpha^2) U^2 dt$ which reduces to

$$\sigma(s) = s, \qquad \alpha(v) = av^2, \qquad \beta(v) = bv^3, \qquad \rho(v) = \rho$$

This translates to $dY_t = aY_t dW_t + bY_t dt$ so that Y_t is exponential Brownian motion, and is positive and finite for all t. If $b \leq a^2/2$, then $\Gamma_\infty = \infty$ and S is positive for all time, but tends to zero as $t \uparrow \infty$. The more interesting case is when $b > a^2/2$. In this case $Y_t^{-2} \downarrow 0$ and moreover $\Gamma_\infty < \infty$; then A hits infinity in finite time, and S hits zero in finite time almost surely. (It is also true that V explodes in finite time.)

Alternatively, if we assume that the volatility itself satisfies a stochastic-differential equation of the same form as that for U above, or in other words if our standard notation we take

$$\sigma(s) = s, \qquad \alpha(v) = av^{3/2}, \qquad \beta(v) = bv^2, \qquad \rho(v) = \rho$$

then $dY_t = a\sqrt{Y} dB_t^Y + bdt$. Then $R_t = (2/a)\sqrt{Y_t}$ is a Bessel process of dimension $\phi = 4b/a^2$, and $\Gamma_t = 16a^{-4} \int_0^t R_s^{-4} ds$.

If $b < a^2/2$ then $\phi < 2$, Y hits zero in finite time (almost surely) and $\Gamma_{H_0^Y} = \infty$. It follows that $S_\infty > 0$. Conversely, if $\phi > 2$ then $\Gamma_\infty < \infty$, A explodes, and S hits zero in finite time almost surely. Finally, if $b = a^2/2$, then Γ does not explode, but does diverge, and S_t is positive but tends to zero almost surely.

4.1.4 A further tractable model

Consider the model for which

$$\sigma(s) = s, \quad \alpha(v) = av, \quad \beta(v) = a\delta v^2, \quad \rho(v) = \rho$$

where a is positive.

Then

$$dY_t = a(dB_t^Y + \delta dt)$$

is linear Brownian motion. As in Example 4.1.1, Γ explodes the first time, if ever, that Y hits zero. Hence, S_t does not hit zero in finite time, and it converges to a limit which is positive if H_0^Y is finite. (In particular, if $\delta > 0$ then $0 < \mathbb{Q}(\Omega_\varepsilon) < 1$, and the $\mathbb{Q}$-probability that S has a positive limit S_∞ lies strictly between 0 and 1.)

Note that V can hit zero (and we take zero to be a reflecting boundary), and, in the case $\delta > 0$, V will explode to infinity in finite time. Conversely, although Y will hit zero with positive probability it does not explode to infinity. In the case $\delta > 0$ then Y_t increases to infinity almost surely, and then Γ_∞ is finite, so that the time-change A_t explodes.

For this example we are interested in whether S_t is a uniformly integrable martingale. We have

$$S_t = s \exp(B_{A_t}^X - A_t/2)$$

where B_t^X can be decomposed into two independent Brownian motions: $B_t^X = \rho B_t^Y + \rho^\perp B_t^\perp$. In particular

$$S_\infty = s \exp(\rho B_{H_0^Y}^Y - \rho^2 H_0^Y/2) \exp(\rho^\perp B_{H_0^Y}^\perp - (1-\rho^2) H_0^Y/2)$$

It follows from Lemma 4.3 below that S is a uniformly integrable martingale if and only if $\delta + \rho \le 0$. To see this set $B^Y \equiv -W$, $\delta = \mu$ and $\theta = \rho$ so that $H_0^Y = \inf\{u \ge 0 : W_u = v/a + \mu t\}$, where $v = Y_0$.

Lemma 4.3. *Let W and $W^\perp$ be independent $\mathbb{P}$-Brownian motions. For positive z and general μ define*

$$\tau \equiv H_0^{w,\mu} = \inf\{u : W_u = z + \mu u\}$$

For $(\theta, \theta^\perp)$ a constant non-zero vector set

$$M_t = M_t^{(\theta,\phi)} = e^{\theta W_t + \phi W_t^\perp - (\theta^2 + \phi^2)t/2}.$$

Then $M_{t\wedge\tau}$ is uniformly integrable and $\mathbb{E}[M_\tau] = 1$ if and only if $\mu \le \theta$.

Proof. Suppose first that $\phi = 0$, but that θ is non-zero, and write N_t for $M_t^{(\theta,0)}$.

Let $\hat{N}_t$ denote the stopped martingale $N_{t\wedge\tau}$, let $\eta_n = \inf\{u \geq 0 : \hat{N}_u \geq e^n\}$, and define $\hat{\mathbb{P}}^n$ via

$$\left.\frac{d\hat{\mathbb{P}}^n}{d\mathbb{P}}\right|_{\mathcal{F}_{t\wedge\tau}} = \hat{N}_{t\wedge\eta_n}$$

It is easy to see that $\mathbb{E}[N_\tau] \equiv \mathbb{E}[\hat{N}_\infty] = 1$ if and only if $\hat{N}_t$ is uniformly integrable. Then, by Lemma B.1 in the Appendix either of these conditions is equivalent to the condition $\hat{\mathbb{P}}^n(\eta_n < \infty) \to 0$.

We have that under $\hat{\mathbb{P}}^n$, and for $t \leq \tau \wedge \eta_n$, $\hat{W}_t = W_t - \theta t$ is a Brownian motion. There is a natural consistency condition between the measures $\hat{\mathbb{P}}^n$ which means that we do not need to define a sequence of Brownian motions $\hat{W}^n$, but rather a single process $\hat{W}$ will suffice. Further, if we define $\hat{\mathbb{P}}$ via $d\hat{\mathbb{P}}/d\mathbb{P} = N_t$ on $\mathcal{F}_t$, then the restriction of $\hat{\mathbb{P}}$ to $\mathcal{F}_{t\wedge\tau\wedge\eta_n}$ agrees with $\mathbb{P}^n$, and we can extend $\hat{W}$ to be a $\hat{\mathbb{P}}$-Brownian motion defined for all time.

Then,

$$\begin{aligned}\hat{\mathbb{P}}^n(\eta_n < \infty) &= \hat{\mathbb{P}}^n(\sup_{t\leq\tau} \theta W_t - \theta^2 t/2 \geq n) \\ &= \hat{\mathbb{P}}(\sup_{t\leq\tau} \theta \hat{W}_t + \theta^2 t/2 \geq n).\end{aligned}$$

Note that, under $\hat{\mathbb{P}}$,

$$\tau = \inf\{u : \hat{W}_u = z + (\mu - \theta)u\}$$

so that if we define for all non-zero x and for all ψ

$$\hat{H}_0^{x,\phi} = \inf\{u : \hat{W}_u = x + \psi u\}$$

then

$$\hat{\mathbb{P}}(\sup_{t\leq\tau} \theta\hat{W}_t + \theta^2 t/2 \geq n) = \hat{\mathbb{P}}(\hat{H}_0^{n/\theta,-\theta/2} \leq \hat{H}_0^{z,\mu-\theta})$$

(recall we assuming that θ is non-zero).

The sequence $\hat{H}_0^{n/\theta,-\theta/2}$ is a sequence of finite stopping times which increase to infinity almost surely as n increases to infinity. It follows that $\hat{\mathbb{P}}(\hat{H}_0^{n/\theta,-\theta/2} \leq \hat{H}_0^{z,\mu-\theta}) \to 0$ if and only if $\hat{H}_0^{z,\mu-\theta}$ is finite almost surely, or equivalently if $\mu - \theta \leq 0$.

Now suppose ϕ is non-zero. If we set $N_t = M_t^{(\theta,\phi)}$ then the argument is essentially unchanged and the condition that $M_{t\wedge\tau}$ is uniformly integrable reduces to

$$\hat{\mathbb{P}}(\sup_{t\le\tau} \theta\hat{W}_t + \phi\hat{W}_t^{\perp} + \theta^2 t/2 \ge n) \to 0$$

where $\hat{W}_t^{\perp} := W_t^{\perp} + \phi dt$ is a $\hat{\mathbb{P}}$-Brownian motion. Provided $\mu \le \theta$, τ is finite $\hat{\mathbb{P}}$ almost surely and then $\hat{\mathbb{P}}(\sup_{t\le\tau} \theta\hat{W}_t + \phi\hat{W}_t^{\perp} + \theta^2 t/2 \ge n) \to 0$ as $n \uparrow \infty$. Otherwise, if $\mu > \theta$ then τ is infinite with positive probability and on this set $\sup_{t\le\tau}\{\theta\hat{W}_t + \phi\hat{W}_t^{\perp} + \theta^2 t/2\}$ equals infinity.

Note that if $\theta = \phi = 0$ then $M^{(\theta,\phi)}$ is trivially a uniformly integrable martingale. On this set $\theta\hat{W}_t + \phi\hat{W}_t^{\perp} + \theta^2 t/2$ is identically zero so that $\hat{\mathbb{P}}(\eta_n < \infty) = 0$. □

5 Models with leverage effects

A stylised fact from the finance literature is that as the stock price falls so volatility tends to increase. One way to capture this phenomenon is to use a stochastic volatility model and to insist that the correlation between the Brownian motions driving stock price and volatility is negative. However, another way to capture this phenomenon is to introduce a leverage effect, or in other words to make $\sigma(s)$ a non-linear function of s.

The constant elasticity of variance (CEV) model of Cox and Ross [7] and the displaced diffusion models of Rubinstein [31] both fall into this class. In the CEV model S solves $dS = \sigma S^\theta dB^S$ for some θ with $0 < \theta < 1$; more generally we have diffusion models of the form $dS = \sigma(S)dB^S$. The option pricing properties of these models were studied extensively in Bergman et al [5], see also [10] and [18] and the discussion in the next section.

In this section we are interested in models with both a leverage effect and volatility of the form in (3). Since by Assumption 2.1 we have that σ^{-2} is locally integrable, a necessary and sufficient condition for X to hit zero in finite time is that $\int_{0+} x\sigma(x)^{-2}dx$ is finite. Since X is a time-homogeneous diffusion, in this case H_0^X is finite almost surely. If moreover Γ does not explode, then A increases to infinity almost surely and S hits zero in finite time almost surely. However, if Γ explodes then S may not hit zero.

The Stochastic-Alpha-Beta-Rho (SABR) model introduced by Hagan et al [11] is a combination of a CEV model for the discounted stock price with an exponential Brownian motion for volatility and is of the form

$$\sigma(s) = s^\theta, \quad \alpha(v) = av, \quad \beta(v) = bv, \quad \rho(v) = \rho \tag{5}$$

where $0 < \theta < 1$ and a is positive. (In fact, in the original SABR model takes $b = 0$.)

The first question for this model is to decide whether the resulting discounted stock price process is a true martingale. For the SABR model Andersen and Piterbarg [1] answer this question in the positive by deriving bounds on the moments of S, but here we give a direct proof. In keeping with the spirit of the rest of the paper this proof relies on stochastic calculus and a coupling argument.

Theorem 5.1 (Andersen-Piterbarg [1])**.** *Consider the* SABR *model with parameters as in (5). Then S_t is a true martingale.*

Proof. The model is

$$dS_t = S_t^\theta V_t(\rho dW_t + \rho^\perp dW_t^\perp) \qquad dV_t = aV_t dW_t + bV_t dt$$

where W_t is a shorthand for B_t^V. Consider $Z_t = S_t V_t^{-1/(1-\theta)} e^{-\delta t}$ where δ is chosen such that

$$M_t = e^{\delta t} V_t^{1/(1-\theta)} v^{-1/(1-\theta)}$$

is a martingale. In particular,

$$M_t = \exp\left(\frac{a}{(1-\theta)} W_t + \frac{(b-a^2/2)}{(1-\theta)} t + \delta t\right) = \exp\left(\frac{a}{(1-\theta)} W_t - \frac{a^2}{2(1-\theta)^2} t\right)$$

so that $\delta = -b/(1-\theta) - \theta a^2/2(1-\theta)^2$.

Then $\mathbb{E}[S_T] = \mathbb{E}[Z_T V_T^{1/(1-\theta)} e^{\delta T}] = v^{1/(1-\theta)} \mathbb{E}^*[Z_T]$ where $\mathbb{P}^*$ is defined via $d\mathbb{P}^*/d\mathbb{Q} = M_T$ on $\mathcal{F}_T$. Under $\mathbb{P}^*$ we have that $W_t^* = W_t - (a/(1-\theta))t$ is a Brownian motion.

Applying Itô's formula to Z we obtain

$$\begin{aligned} dZ_t &= Z_t^\theta e^{\delta(\theta-1)t} \rho^\perp dW^\perp + \left(Z_t^\theta e^{\delta(\theta-1)t} \rho - \frac{a}{(1-\theta)} Z_t\right) dW_t^* \\ &= K(Z_t, t) d\tilde{W}^*, \end{aligned}$$

where the $\mathbb{P}^*$-Brownian motion $\tilde{W}^*$ is the appropriate combination of the $\mathbb{P}^*$-Brownian motions W^* and $W^\perp$ and

$$K(z,t)^2 = z^{2\theta} e^{2\delta(\theta-1)} + \frac{a^2 z^2}{(1-\theta)^2} - \frac{2a\rho e^{\delta(\theta-1)t} z^{1+\theta}}{(1-\theta)}.$$

Then, for $t \leq T$, $K(z,t) \leq (\eta_0 + \eta_1 z) \equiv \bar{K}(z)$ for appropriate constants η_0 and η_1. In particular, if $\bar{Z}$ solves $d\bar{Z} = \bar{K}(\bar{Z}, t) d\tilde{W}^*$ subject to $\bar{Z}_0 = z$, then $\bar{Z}$ is a true martingale.

Finally, by a time-change argument (see Theorem 3 of Hajek [12]) we can write $Z_t = \bar{Z}_{C_t}$ for a time-change C_t with $C_t \leq t$ and then Z_t is also a true martingale. □

Example 5.2. Consider the SABR model with $\beta = 0$ and $\theta = 0$. (In this case we modify $\sigma(s)$ so that $\sigma(s) = I_{\{s>0\}}$ in order to preserve limited liability. Then $dX = (\rho dB^Y + \rho^\perp dB^\perp)$ and $dY = a dB^Y$ with $Y_0 = v$, and we have $\int_{0+} x dx < \infty$, so that X hits zero in finite time. It also follows that Γ explodes when Y first hits zero, and that $A_\infty = H_0^Y \equiv B^Y_{H^{B^Y}_{-v/a}}$. Moreover, until S first hits zero,

$$S_t = \frac{\rho(Y_{A_t} - v)}{a} + \rho^\perp B^\perp_{A_t}$$

and S hits zero in finite time if and only if

$$\inf_{t \le H^{B^Y}_{-v/a}} (\rho B_t^Y + \rho^\perp B_t^\perp) \le -s.$$

If $\rho = 1$ and $as > v$ then S does not hit zero.

6 Option price comparisons and convexity

In the standard Samuelson-Black-Scholes model an application of Jensen's inequality shows that if the payoff function of a European-style claim is convex in the underlying, then that property is inherited by the price of the option at earlier times. Avellaneda, Levy and Paras [2] and Lyons [27] showed that if volatility is known to lie within a band, (and if the payoff function is convex), then the prices calculated using the Black-Scholes model with volatilities corresponding to the ends of the bands provide bounds on the value of the option price.

These first comparison theorems inspired further study of the convexity and monotonicity properties of option prices in diffusion models. Bergman et al [5], El Karoui et al [10] and Hobson [18] each considered this problem using different approaches. Let $dS = \hat{\sigma}(S_t)dB_t$ and $dS = \tilde{\sigma}(S_t)dB_t$ be two competing candidate models for the discounted asset price under the risk-neutral measure. We distinguish the different models by considering the price process S under $\hat{\mathbb{Q}}$ and $\tilde{\mathbb{Q}}$. There are two types of comparisons which are important:

Option Price monotonicity: we say there is option price monotonicity if, whenever $\tilde{\sigma}(s) \le \hat{\sigma}(s)$ and Φ is convex it follows that $\tilde{\mathbb{E}}[\Phi(S_T)] \le \hat{\mathbb{E}}[\Phi(S_T)]$, so that option prices are monotonic in the diffusion co-efficient.

Super-replication property: Suppose that there exists a strategy $\hat{\theta} \equiv \hat{\theta}_u$ such that if S is governed by the dynamics $\hat{\sigma}$

$$\hat{\mathbb{E}}[\Phi(S_T)] + \int_0^T \hat{\theta}_u dS_u = \Phi(S_T) \qquad \hat{\mathbb{Q}} \text{ a.s.}$$

(so that $\hat{\theta}$ is a replicating strategy for the option payoff under $\hat{\mathbb{Q}}$, with associated replication price $\hat{\mathbb{E}}[\Phi(S_T)]$). Then the model has the super-replication property if when the dynamics of S are governed by $\tilde{\sigma}$

$$\hat{\mathbb{E}}[\Phi(S_T)] + \int_0^T \hat{\theta}_u dS_u \geq \Phi(S_T) \qquad \tilde{\mathbb{Q}} \text{ a.s..}$$

In this case an investor who believes in the model $dS = \hat{\sigma}(S_t)dB_t$, and who acts accordingly (in terms of pricing and hedging) will super-replicate the option payout, even if the true model is $dS = \tilde{\sigma}(S_t)dB_t$.

Bergman et al [5] used an analysis of the option pricing partial differential equation to prove the monotonicity property, whereas El Karoui et al [10] used stochastic flows and Hobson [18] used a coupling approach to prove the stronger super-replication property. In all three cases a key stepping stone was to prove that the price of the option at intermediate times is convex in the underlying.

6.1 Convexity

Suppose we now consider the question of whether a similar result holds true in the stochastic volatility context. (This variant on this question has already been considered by Ekström et al [8], see also Janson and Tysk [23]. They give examples to show that in a bi-variate diffusion model $(S^{(1)}, S^{(2)})$ it does not follow that if $\Phi(s^{(1)}, s^{(2)})$ is convex then $\mathbb{E}^{S_0^{(1)}=s^{(1)}, S_0^{(2)}=s^{(2)}}[\Phi(S_T^{(1)}, S_T^{(2)})]$ is convex. However, this does not quite cover the situation in stochastic volatility models since there $S^{(2)} \equiv V$ is autonomous, and Φ is a function of $S^{(1)}$ alone.)

Consider a stochastic volatility model. For a convex payoff function Φ define the corresponding European option price

$$\phi(s, v) = \mathbb{E}^{S_0=s, V_0=v}[\Phi(S_T)]$$

Proposition 6.1. *Suppose that the coefficients in (3) are such that there exists a strong solution to (4) and suppose that for all initial starting points* $\mathbb{E}[\Phi(S_T)] < \infty$.

Suppose either that $\rho \equiv 0$ *and that* S_t *is a true martingale, or that* $\sigma(s) = s$. *Then for each fixed* v, $\phi(s, v)$ *is convex in* s.

Proof. If $\rho(v) \equiv 0$ then $S_t = X_{A_t}$ where X and A are independent. Then, conditioning on the Brownian motion B^Y which generates A we can apply the result for the diffusion $dX = \sigma(X)dB^X$ (in particular the coupling proof in Hobson [18][Theorem 3.1] works for random expiry) to conclude that $\mathbb{E}[\Phi(X_{A_T})|A_T]$ is convex in s. The convexity property is maintained when we average over A_T.

If $\sigma(s) = s$ then $S_t = se^{B^X_{A_t} - A_t/2} =: sZ_t$ where Z_t is independent of s. Then for $\lambda \in (0,1)$, for $s = \lambda q + (1-\lambda)r$, and for any Z_T we have $\phi(sZ_T) \leq \lambda\phi(qZ_T) + (1-\lambda)\phi(rZ_T)$. The result follows on taking expectations. $\square$

Remark 6.2. The results in Proposition 6.1 can be found in Bergman et al [5] where a formal proof is given involving differentiation of the option pricing PDE and Henderson [13], who takes $\rho = 0$.

If the true martingale property fails in the uncorrelated case then the convexity property may fail also. To see this take $\sigma(x) = (x-1)^2 I_{\{x>1\}}$. Then S is a strict local martingale, (unless $S_0 \leq 1$, in which case S is constant) and even for the linear payoff $\Phi(s) = s$ we find that $\phi(s)$ is not convex.

If ϕ is non-zero and S is not log-linear then the convexity property may fail even when S is a true martingale as the following example shows.

Example 6.3. Consider the model

$$\sigma(s) = I_{\{s>0\}}, \qquad \alpha(v) = v, \qquad \beta(v) = -\delta v^2, \qquad \rho(v) = 1$$

with $\delta > 0$.

Fix $V_0 = 1$ and consider the call option with unit strike and maturity T; $\Phi(S_T) = (S_T - 1)^+$.

Write $B^X_t = B^Y_t = B_t$ and set $\underline{B}_t = \inf\{B_r; 0 \leq r \leq t\}$. Denote by $S^{(s)}_t \equiv X^{(s)}_{A_t}$ the solution to (3) with initial value $S_0 = X_0 = s$. Then $X^{(s)} = (s + B_t) I_{\{\underline{B}_t > -s\}}$. Similarly $Y_t = 1 + B_t - \delta t$ and $H^Y_0 < \infty$ almost surely. We have $\Gamma_{H^Y_0} = \infty$ so that $A_\infty = H^Y_0 \leq H^B_{-1}$.

If $s = 0$ then $S^{(0)}_t = X^{(0)}_{A_t} \equiv 0$ and $\mathbb{E}[\Phi(S^{(0)}_T)] = 0$.

If $s = 2$ then $S^{(2)}_t = X^{(2)}_{A_t} = (2 + B_{A_t}) I_{\{\underline{B}_{A_t} > -2\}}$, but since $A_t \leq A_\infty \leq H^B_{-1}$ we have $S^{(2)}_t \geq 1$. Further, by the uniform integrability of the stopping time H^Y_0, we have $\mathbb{E}[B_{H^Y_0}] = 0$ and $\mathbb{E}[H^Y_0] = 1/\delta$. Finally, for any T, $A_T \leq H^Y_0$, and $\mathbb{E}[X^{(2)}_{A_T}] = 2$. We conclude that $\mathbb{E}[\Phi(S^{(2)}_T)] = \mathbb{E}[(X^{(2)}_{A_T} - 1)^+] = 1$.

In order to show that $\phi(s,1)$ is not convex it is sufficient to show that $\phi(1,1) > 1/2$. If $s = 1$ then $X^{(1)}_t = (1 + B_t) I_{\{\underline{B}_t > -1\}} = (Y_t + \delta t) I_{\{H^B_{-1} > t\}}$ and $S^{(1)}_\infty = X^{(1)}_{A_\infty} = \delta H^Y_0$. Then $\mathbb{E}[\Phi(S^{(1)}_\infty)] = \mathbb{E}[(\delta H^Y_0 - 1)^+]$. Now $\delta H^Y_0 = \inf\{\delta u \geq 0 : B_u = \delta u - 1\} = \inf\{r \geq 0 : \tilde{B}_r = \sqrt{\delta}(r-1)\}$ where $\tilde{B}_r = \sqrt{\delta} B_{r/\delta}$. Standard calculations using the reflection principle and a change of measure show that if $I_\delta = \mathbb{E}[(\delta H^Y_0 - 1)^+]$ then

$$I_\delta = \int_0^\infty \frac{y}{\sqrt{2\pi}} e^{-y\sqrt{\delta} - y^2/2} \frac{2 \sinh y\sqrt{\delta}}{\sqrt{\delta}} dy.$$

In particular, if $I_0 := \lim_{\delta \downarrow 0} I_\delta$ then $I_0 = 1$.

Now choose δ so small that $I_\delta > I_0 - 1/4$ and T so large that $\mathbb{E}[\Phi(S_T^{(1)})] > \mathbb{E}[\Phi(S_\infty^{(1)})] - 1/4$. Then $\mathbb{E}[\Phi(S_T^{(1)})] > 1/2$.

6.2 Option Price Comparisons

Notwithstanding the lack of option price convexity, it is still possible to obtain comparison theorems between pairs of candidate stochastic volatility models. The following result extends the main result of Henderson et al [15] to a wider class of stochastic volatility models, including models outside the log-linear case. The proof uses the construction in Theorem 3.1, whereas [15] used a comparison based on an analysis of the option pricing partial differential equation. One corollary of Theorem 6.4 is that in a stochastic volatility model the vega of an option is positive, provided the option has convex payoff.

Theorem 6.4. *Consider a pair of stochastic volatility models indexed by $i = 0, 1$ which differ only in the form of the drift on volatility, or in the initial value of volatility:*

$$\begin{aligned} i = 0,1 \qquad & S_0 = s, V_0 = v^{(i)} \\ dS_t = \sigma(S_t)V_t dB_t^S, \qquad & dV_t = \alpha(V_t)dB_t^V + \beta^{(i)}(V_t)dt, \\ & dB_t^S dB^V = \rho(V_t)dt. \end{aligned}$$

Denote by $(S^{(0)}, V^{(0)})$ and $(S^{(1)}, V^{(1)})$ the solutions under the two different models. Suppose that $S^{(i)}$ is a true martingale in each case.

Suppose that for each model the corresponding time-changed stochastic differential equation

$$\begin{aligned} i = 0,1 \qquad & X_0 = s, Y_0 = v^{(i)} \\ dX_t = \sigma(S_t)dB_t^X, \qquad & dY_t = (\alpha(Y_t)/Y_t)dB_t^Y + (\beta^{(i)}(Y_t)/Y_t^2)dt, \\ & dB_t^X dB_t^Y = \rho(Y_t)dt, \end{aligned}$$

has a strong solution.

Suppose that $\beta^{(0)}(y) \leq \beta^{(1)}(y)$ for all y, and $v^{(0)} \leq v^{(1)}$. Then for any convex Φ,

$$\mathbb{E}[\Phi(S_T^{(0)})] \leq \mathbb{E}[\Phi(S_T^{(1)})].$$

Proof. We extend the superscript notation representing the pair of models to cover all processes of interest. The exception is the process X_t, the construction of which is the same in both models. Then the stock price processes $S_t^{(i)} = X_{A_t^{(i)}}$ differ only in the time change.

It is clear that $Y_t^{(0)} \leq Y_t^{(1)}$ and hence $\Gamma_t^{(0)} \geq \Gamma_t^{(1)}$ and $A_t^{(0)} \leq A_t^{(1)}$. Then, recall the notation of Theorem 3.1, conditional on $\mathcal{G}_{A_T^{(0)}}$ and using Jensen's inequality and the martingale assumption,

$$\mathbb{E}\left[\mathbb{E}[\Phi(X_{A_T^{(1)}})|\mathcal{G}_{A_T^{(0)}}\right] \geq \mathbb{E}[\Phi(\mathbb{E}[X_{A_T^{(1)}}|\mathcal{G}_{A_T^{(0)}}])] = \mathbb{E}[\Phi(X_{A_T^{(0)}})]$$

so that $\mathbb{E}[\Phi(S_T^{(1)})] \geq \mathbb{E}[\Phi(S_T^{(0)})]$. □

Remark 6.5. As discussed in Henderson et al [15], there are two distinct usages of Theorem 6.4. In the first case we imagine comparing two different models (or the same model with different initial values of volatility) under a fixed pricing measure $\mathbb{Q}$. In the second case we consider a stochastic volatility model under the physical measure $\mathbb{P}$, and two different choices of martingale measure $\mathbb{Q}^{(0)}$ and $\mathbb{Q}^{(1)}$. Typically the models for the price and volatility process under these two measures will differ only in the drift on volatility, thus placing us immediately in the setting of Theorem 6.4. See [15] for more details.

7 Conclusions

In this paper we use a stochastic time-change to construct the solutions to stochastic volatility models, and then use these solutions to deduce properties of the underlying model. The advantage of the time-change construction is that it gives insight into the sample-path behaviour of the model. Previously the literature has focused on the (true) martingale property of the (discounted) asset price; in addition in this article we study the uniform integrability properties, and the potential for asset prices to hit zero in finite time. These properties have implications for put and call prices, especially in the limits of large maturity and extreme strike.

The time-changed volatility process Y has the same scale function as the volatility process V itself, so that answers to questions about whether this process tends to infinity or zero are unchanged. However, the effect of the time-change is that volatility may explode, even when Y is non-explosive. The behaviour of Y governs the properties of the discounted asset price process and partly determines whether the discounted asset price converges to a positive value, or whether it hits zero in finite time. Neither of these behaviours is consistent with a constant volatility, exponential Brownian motion model.

A Identities for Bessel processes

Proposition A.1. *Let R_s be a Bessel process with dimension ϕ and such that $R_0 = r > 0$. Define $\Gamma_t \equiv \Gamma_t^{(\eta)} = \int_0^t R_s^{-\eta} ds$.*
(i) Suppose $0 \leq \phi < 2$. Then $H_0^R < \infty$ almost surely, and $\Gamma_{H_0^R}^{(\eta)} = \infty$ if and only if $\eta \geq 2$.
(ii) Suppose $\phi > 2$. Then $H_0^R = \infty$ almost surely, and $\Gamma_\infty^{(\eta)} = \infty$ if and only if $\eta \leq 2$.
(iii) Suppose $\phi = 2$. Then $H_0^R = \infty$ almost surely, and $\Gamma_\infty^{(\eta)} = \infty$ for all η.

Proof. As a general principle, if Z is a one-dimensional diffusion in natural scale then the additive functional $C_t = \int_0^t c(Z_u)du$ does not explode if and only if c is locally integrable with respect to the speed measure of Z. By appealing to well known properties of financial models we can prove this result directly for the Bessel process.

(i) Suppose $\phi < 2$. Let $P_s = R_s^{2-\phi}$ and $p = r^{2-\phi}$. Then P is in natural scale and

$$P_t = r^{2-\phi} + \int_0^t (2-\phi) P_s^{(1-\phi)/(2-\phi)} dW_s = p + B_{D_t}$$

for an appropriate time-change D_s. By considering the case $\eta = 0$ of the following argument it follows that $H_0^R = H_0^P \equiv \inf\{u : R_u = 0\}$ is finite almost surely.

We have that $\Gamma_t = \int_0^t P_s^{-\eta/(2-\phi)} ds$ and by the occupation time formula (Revuz and Yor [29][Corollary VI./1.6]) with $L_t(x)$ denoting the local time of $p + B$ at x by time t,

$$\begin{aligned} \Gamma_{H_0^R} &= \int_{\mathbb{R}_+} q^{-\eta/(2-\phi)} (2-\phi)^{-2} q^{-2(1-\phi)/(2-\phi)} L_{H_0^{p+B}}(q) dq \\ &= (2-\phi)^{-2} \int_{\mathbb{R}_+} q^{(2\phi-\eta-2)/(2-\phi)} L_{H_0^{p+B}}(q) dq. \end{aligned} \tag{6}$$

Let $\tilde{\Gamma}_t = \int_0^t R_s^{-\eta} I_{\{R_s < r\}} ds$; then

$$\tilde{\Gamma}_{H_0^R} = (2-\phi)^{-2} \int_0^p q^{(2\phi-\eta-2)/(2-\phi)} L_{H_0^{p+B}}(q) dq. \tag{7}$$

Since R will only spend a finite amount of time above its initial point r, $\Gamma_{H_0^R}$ will be finite if and only if $\tilde{\Gamma}_{H_0^R}$ is finite, and since for $q < r^{2-\phi}$ we have $\mathbb{E}[L_{H_0^R}(q)] = q$, it follows that $\mathbb{E}[\tilde{\Gamma}_{H_0^R}] < \infty$ if and only if $\eta < 2$.

It remains to show that if $\eta \geq 2$ then $\tilde{\Gamma}_{H_0^R} = \infty$ (and not just $\mathbb{E}[\tilde{\Gamma}_{H_0^R}] = \infty$). Consider Y given by $Y_0 = y$

$$dY_t = (2 - \phi) Y_t^\theta dW_t.$$

Then Y is a CEV process, and it is well known that Y hits zero in finite time if and only if $\theta < 1$. (Alternatively this result follows from Theorem 51.2 in Rogers and Williams [30].) Using the fact that Y is a local martingale we have $Y_t = y + B_{A_t}$ and, if we set $\Gamma^Y \equiv A^{-1}$ then

$$\Gamma_t^Y = \int_0^t \frac{ds}{(2-\phi)^2 (y + B_s)^{2\theta}}$$

so that

$$\begin{aligned} H_0^Y \equiv \Gamma_{H_0^{y+B}}^Y &= \int_0^{H_0^{y+B}} \frac{ds}{(2-\phi)^2 (y+B_s)^{2\theta}} \\ &= (2-\phi)^{-2} \int_{\mathbb{R}_+} b^{-2\theta} L_{H_0^{y+B}}(b) db, \end{aligned}$$

where now L denotes the local time of $y + B$. By the above remark about the CEV process this quantity is infinite if and only if $\theta \geq 1$.

Comparing with (6) we see that if $-2\theta = (2\phi - \eta - 2)/(2 - \phi)$ or equivalently $\theta = (2 + \eta - 2\phi)/2(2 - \phi)$ we can identify $\Gamma_{H_0^R}$ with $\Gamma_{H_0^Y}^Y$ and then $\Gamma_{H_0^R}$ is infinite if and only if $\theta \geq 1$, or equivalently $\eta \geq 1$.

(ii) Suppose $\phi > 2$. In this case, from standard results about Bessel processes we know that H_0^R is infinite, almost surely, and that R does not explode, but drifts to plus infinity.

As before we have that $P_t = p + B_{D_t}$ is in natural scale, and since R_t diverges we have $P_t \to 0$ and

$$\lim_{t \uparrow \infty} \Gamma_t^{(\eta)} = (2-\phi)^2 \int_{\mathbb{R}_+} q^{(2+\eta-2\phi)/(\phi-2)} L_{H_0^{p+B}}(q) dq.$$

By the same argument as before this quantity is almost surely finite if and only if

$$1 + \frac{(2 + \eta - 2\phi)}{(\phi - 2)} \leq -1$$

or equivalently $\eta \leq 2$.

(iii) If $\phi = 2$ and $p = \ln r$ then $P_t = \ln R_t = p + B_{D_t}$ is in natural scale and

$$\Gamma_t^{(\eta)} = \int_{\mathbb{R}} e^{-\eta q} e^{2q} L_{D_t}(q) dq.$$

Since P_t is recurrent, this integral is seen to diverge. □

Corollary A.2. *Let R_s be a Bessel process of dimension ϕ, with $R_0 = r > 0$. Define $\Gamma_t = \Gamma_t^{(\eta)} = \int_0^t R_s^{-\eta} ds$.*

Suppose that $\phi < 2$ and $\eta < 2$ so that H_0^R and $\Gamma_{H_0^R}$ are both finite. Then $\Gamma_{H_0^R+}$ is finite if and only if $\phi > \eta$.

Proof. For any $t > H_0^R$ the local martingale $P \equiv R^{2-\phi}$ will have generated a positive local time at zero, so that $\Gamma_{H_0^R+} < \infty$ if and only if

$$\int_{0+} q^{(2\phi-\eta-2)/(2-\phi)} dq < \infty$$

This condition is equivalent to $\phi > \eta$. □

B Conditions for a UI martingale.

Let N_t be a continuous non-negative local martingale on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. For a stopping time τ define the stopped martingale $\hat{N}_t = N_{t\wedge\tau}$. We want to decide when $\hat{N}$ is uniformly integrable.

Define η_n and $\hat{\mathbb{P}}^n$ via $\eta_n = \inf\{u \geq 0 : \hat{N}_u \geq n\} \leq \infty$ and

$$\frac{d\hat{\mathbb{P}}^n}{d\mathbb{P}} = \hat{N}_{t\wedge\eta_n} \quad \text{on } \mathcal{F}_{t\wedge\tau}.$$

Since $\hat{N}_{t\wedge\eta_n}$ is bounded it follows that $\hat{\mathbb{P}}^n$ is a well-defined probability measure.

Clearly $\hat{N}$ is uniformly integrable if and only if $\mathbb{E}[\hat{N}_\infty] \equiv \mathbb{E}[N_\tau] = 1$.

The following lemma follows from Lemma A.1 in Hobson and Rogers [19]

Lemma B.1. *The following are equivalent:*
$\hat{N}$ is uniformly integrable
$\hat{\mathbb{P}}^n(\eta_n < \infty) \to 0$.

Proof. Since $\hat{N}_{t\wedge\eta_n}$ is a bounded martingale we have

$$\begin{aligned} 1 = \mathbb{E}[\hat{N}_{\eta_n}] &= \mathbb{E}[\hat{N}_{\eta_n}; \eta_n = \infty] + \mathbb{E}[\hat{N}_{\eta_n}; \eta_n < \infty] \\ &= \mathbb{E}[N_\tau; \eta_n = \infty] + \hat{\mathbb{P}}(\eta_n < \infty) \end{aligned} \tag{8}$$

If $\hat{\mathbb{P}}^n(\eta_n < \infty) \to 0$ then $\mathbb{E}[N_\tau; \eta_n = \infty] \to 1$, and since this is a lower bound on $\mathbb{E}[N_\tau]$ we conclude that $\mathbb{E}[N_\tau] = 1$ and $\hat{N}$ is uniformly integrable.

Now suppose that $\mathbb{E}[N_\tau] = \mathbb{E}[\hat{N}_\infty] = 1$. Then, using Doob's submartingale inequality for $\hat{N}$ we get $\mathbb{P}(\eta_n < \infty) \to 0$, so that $\mathbb{E}[N_\tau; \eta_n < \infty] \downarrow 0$ and $\mathbb{E}[N_\tau; \eta_n = \infty] \uparrow \mathbb{E}[N_\tau] = 1$. Finally from (8) $\mathbb{P}^n(\eta_n < \infty) \to 0$ as required.

□

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