Utility Indifference Pricing - An Overview

Vicky Henderson [0]

Princeton University

David Hobson [1]

University of Bath

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[0]ORFE and Bendheim Center for Finance, E-Quad, Princeton University, Princeton, NJ. 08544. USA. vhenders@princeton.edu

[1]Department of Mathematical Sciences, University of Bath, Bath. BA2 7AY. UK. dgh@maths.bath.ac.uk

1 Introduction

The idea of gamblers ranking risky lotteries by their expected utilities dates back to Bernoulli [4]. An individual's certainty equivalent amount is the certain amount of money that makes them indifferent between the return from the gamble and this amount, as described in Chapter 6 of Mas-Colell et al [59]. Certainty equivalent amounts and the principle of equi-marginal utility (see Jevons [49]) have been used by economists for many years.

More recently, these concepts have been adapted for derivative security pricing. Consider an investor receiving a particular derivative or contingent claim offering payoff C_T at future time $T > 0$. When there is a financial market, and the market is complete, the price the investor would pay can be found uniquely. Option pricing in complete markets uses the idea of replication whereby a portfolio in stocks and bonds re-creates the terminal payoff of the option thus removing all risk and uncertainty. The unique price of the option is given by the law of one price - the initial wealth necessary to fund the replicating portfolio. However, in reality, most situations are incomplete and complete models are only an approximation to this. Market frictions, for example transactions costs, non-traded assets and portfolio constraints, make perfect replication impossible. In such situations, many different option prices are consistent with no-arbitrage, each corresponding to a different martingale measure. There is no longer a unique price.

However, in this situation, even with an incomplete financial market, the investor can maximize expected utility of wealth and may be able to reduce the risk due to the uncertain payoff through dynamic trading. She would be willing to pay a certain amount today for the right to receive the claim such that she is no worse off in expected utility terms than she would have been without the claim. Hodges and Neuberger [44] were the first to adapt the static certainty equivalence concept to a dynamic setting like the one described. Similar ideas are also important in actuarial mathematics. There is a valuation method called the "premium principle of equivalent utility" which has desirable properties if utility is of the exponential form, see Gerber [30] for details. The method described above, termed utility indifference pricing, and the subject of this book, is one approach that can be taken in an incomplete market.

Other potential approaches which will be discussed in this overview include superreplication, the selection of one particular measure according to a minimal distance criteria (for example the minimal martingale measure or the minimal entropy measure) and convex risk measures.

The advantages of utility indifference pricing include its economic justification and incorporation of risk aversion. It leads to a price which is non-linear in the number of units of claim, which is in contrast to prices in complete markets and some of the alternatives mentioned above. The indifference price reduces to the complete market price which is a necessary feature of any good pricing mechanism. Indifference prices can also incorporate wealth dependence. This may be desirable as the price an investor is willing to pay could well depend on the current position of his derivative book. Although we concentrate on pricing issues here, utility indifference also gives an explicit identification of the hedge position. This is found naturally as part of the optimization problem.

Limitations of indifference pricing methodology include the fact that explicit calculations may be done in only a few concrete models, mainly for exponential utility. Exponential utility has the feature that the wealth or initial endowment of the investor factors out of the problem which makes the mathematics tractable but is also a strong assumption. Different investors with varying initial wealths are unlikely to assign the same value to a claim. Practically, users may not be satisfied with the concept of utility functions and unable to specify the required risk aversion coefficient.

In this overview, we will introduce utility indifference pricing and give a survey of the literature, both more theoretical and applications. Inevitably, such a survey reflects the authors' experience and interests.

2 Utility Functions

We begin by introducing utility functions which are central to indifference pricing. Define a utility function $U(x)$ as a twice continuously-differentiable function with the property that $U(x)$ is strictly increasing and strictly concave. Utility functions are increasing to reflect that investors prefer more wealth to less, and concave because investors are risk-averse, see Copeland and Weston [9] and Mas-Colell et al [59]. A utility function can be defined either over the positive real line or over the whole real line. An popular example of the former is the power utility $U(x) = \frac{x^{1-R}}{1-R}; R \neq 1, R > 0$ whilst the exponential utility $U(x) = -\frac{1}{\gamma}e^{-\gamma x}; \gamma > 0$ is an example of the latter. We will describe both of these in more detail below. The main distinction concerns whether wealth is restricted to be positive or allowed to become negative.

2.1 HARA Utilities

A useful quantity for the discussion of utilities is the coefficient of absolute risk aversion (due to Arrow [1] and Pratt [74]), given by

$$R_a(x) = -\frac{U''(x)}{U'(x)}.$$

A utility function is of the HARA class if $R_a(x)$ satisfies

$$R_a(x) = \frac{1}{A + Bx} \qquad x \in I_D \tag{1}$$

where I_D is the interval on which U is defined (or equivalently the interval on which the denominator is positive) and B is a non-negative constant. The constant A is such that $A > 0$ if $B = 0$, whereas A can take any value if B is positive. If $B > 0$ then $U(x) = -\infty$ for $x < -A/B$ and $I_D = (-A/B, \infty)$. Conversely if $B = 0$, then $I_D = \mathbb{R}$, and U is finite valued for all wealths. These definitions ensure U is concave and increasing. Our notation can easily be made consistent with that of Merton [62].

If $B > 0$ and $B \neq 1$, then integration leads to

$$U(x) = \frac{C}{B-1}(A + Bx)^{1-\frac{1}{B}} + D; \qquad C > 0, D \in \mathbb{R}, x > -A/B$$

where C and D are constants of integration. This is called the extended power utility function, see Huang and Litzenberger [45].

If $A = 0$, this becomes the well known narrow power utility function:

$$U(x) = \frac{CB^{-1/B}}{B-1}Bx^{1-\frac{1}{B}} + D; \qquad C > 0, D \in \mathbb{R}, x > 0$$

It is more usually written with $R = 1/B$ and $D = 0, C = B^{1/B}$, giving

$$U(x) = \frac{x^{1-R}}{1-R} \qquad R \neq 1.$$

The narrow power utility has constant relative risk aversion of R, where relative risk aversion $R_r(x)$ is defined to be $R_r(x) = xR_a(x)$.

Returning to further utility functions in the HARA class, for $B = 1$ we have

$$U(x) = C \ln(A + x) + E \qquad C > 0, E \in \mathbb{R}, x > -A,$$

the logarithmic utility function. Taking $A = 0, E = 0, C = 1$ gives the standard or narrow form.

Finally, for $B = 0$

$$U(x) = -\frac{F}{A} e^{-x/A} + G \qquad F > 0, A > 0, G \in \mathbb{R}, x \in \mathbb{R},$$

the exponential utility function. It is usual to take $G = 0, A = 1/\gamma, F = 1/\gamma^2$. For this utility function, $R_a(x) = \gamma$, a constant.

2.2 Non-HARA Utilities

We now briefly discuss some utility functions which do not fit into the HARA class. An important example is the quadratic utility which has received much attention historically in the literature. Taking $B = -1, A > 0$ in (1) gives

$$U(x) = x - \frac{1}{2A} x^2 \qquad x \in \mathbb{R}$$

This decreases over part of the range, violating the assumption that investors desire more wealth and so have an increasing utility function, but has excellent tractability properties.

When used to price options, some of the common utility functions discussed in the previous sections have shortcomings. For instance, we shall show that it is not possible to price a short call option with exponential or power type utilities. The power utility requires wealth be non-negative and the exponential, although allowing for negative wealth, gives problems at least in the case of lognormal models. To circumvent these problems, it is worth mentioning another moderately tractable family of utilities. These are functions of the form:

$$U(x) = \frac{1}{\kappa}\left(1 + \kappa x - \sqrt{1 + \kappa^2 x^2}\right); \qquad \kappa > 0 \tag{2}$$

This class penalises negative wealth less severely.

We will also briefly introduce some special utilities related to other methodologies for pricing in incomplete markets. Firstly, consider the concept of super-replication. A claim is super-replicated if the hedging portfolio is guaranteed to produce at least the payoff of the claim. The super-replication price is the smallest initial fortune with which it is possible to super-replicate the payoff of the claim with probability one. It is the supremum of the possible prices consistent with no arbitrage, and is therefore often unrealistically high. The super-replication price corresponds to a utility function of the form

$$U(x) = \begin{cases} -\infty & x < 0 \\ 0 & x \geq 0 \end{cases}$$

although this does not satisfy our formal definition.

A second approach is that of shortfall hedging which minimises expected loss, see Föllmer and Leukert [23]. This criteria corresponds to a utility function of the form:

$$U(x) = -x^-$$

2.3 Utilities and the Legendre-Fenchel Transform

Denote by I the inverse of the strictly decreasing mapping U' from $(-\infty, \infty)$ onto itself. If $B > 0$ we have

$$I(y) = \frac{1}{B}\left[\left(\frac{y}{C}\right)^{-B} - A\right] \tag{3}$$

whereas if $B = 0$,

$$I(y) = -A\ln(y/F).$$

The convex conjugate function $\tilde{U}$ of the utility function is the Legendre-Fenchel transform of the convex function $-U$, that is

$$\tilde{U}(y) := \sup_{x>0}\{-xy + U(x)\} \qquad\qquad y > 0. \tag{4}$$

For B positive with $B \neq 1$,

$$\tilde{U}(y) = \frac{C^{-B}}{B(B-1)}y^{1-B} + \frac{A}{B}y + D,$$

whereas for $B = 1$,

$$\tilde{U}(y) = -C\ln y + Ay + (E - C - C\ln C)$$

and for $B = 0$,

$$\tilde{U}(y) = Ay\ln y - (A\ln F + 1/A)y + G.$$

Note also that unifying these gives another characterization of HARA utilities, those with

$$\tilde{U}''(y) = \frac{H}{y^{1+B}}$$

for a positive constant H. Of course, both I and $\tilde{U}$ in (3) and (4) can be defined for any utility, not just the HARA class. Much of the general duality theory can be extended to arbitrary utility functions, at least as long as the utility satisfies the reasonable asymptotic elasticity property of Kramkov and Schachermayer [56].

3 Utility Indifference Prices - Definitions

The utility indifference buy (or bid) price p^b is the price at which the investor is indifferent (in the sense that his expected utility under optimal trading is unchanged) between paying nothing and not having the claim C_T and paying p^b now to receive the claim C_T at time T. Consider the problem with $k > 0$ units of the claim. Assume the investor initially has wealth

x and zero endowment of the claim. The definitions extend to cover non-zero endowments also. Define

$$V(x,k) = \sup_{X_T \in \mathcal{A}(x)} \mathbb{E}U(X_T + kC_T) \tag{5}$$

where the supremum is taken over all wealths X_T which can be generated from initial fortune x. The utility indifference buy price $p^b(k)$ is the solution to

$$V(x - p^b(k), k) = V(x, 0). \tag{6}$$

That is, the investor is willing to pay at most the amount $p^b(k)$ today for k units of the claim C_T at time T. Similarly, the utility indifference sell (or ask) price $p^s(k)$ is the smallest amount the investor is willing to accept in order to sell k units of C_T. That is, $p^s(k)$ solves

$$V(x + p^s(k), -k) = V(x, 0). \tag{7}$$

Formally, the definitions of these two quantities can be related by $p^b(k) = -p^s(-k)$, and with this in mind, we let $p(k)$ denote the solution to (6) for all $k \in \mathbb{R}$.

Utility indifference prices are also known as reservation prices, for example, see Munk [66]. Also used in the finance literature is the terminology "private valuation", which emphasizes the proposed price is for an individual with particular risk preferences, see Teplá [82] and Detemple and Sundaresan [18].

Utility indifference prices have a number of appealing properties. We will comment further on these properties later in specific model frameworks.

(i) *Non-linear pricing*

Firstly, in contrast to the Black and Scholes price (and many alternative pricing methodologies in incomplete markets), utility indifference prices are non-linear in the number of options, k. The investor is not willing to pay twice as much for twice as many options, but requires a reduction in this price to take on the additional risk. Alternatively, a seller requires more than twice the price for taking on twice the risk. This property can be seen from the value function (5) since U is a concave function. Put differently, the amount an agent is prepared to pay for a claim C_T depends on his prior exposure to non-replicable risk. We will assume throughout that this prior exposure is zero, and some of our conclusions (such as concavity below) depend crucially on this assumption.

(ii) *Recovery of complete market price*

If the market is complete or if the claim C_T is replicable, the utility indifference price $p(k)$ is equivalent to the complete market price for k units. To show this, let R_T denote the time T value of one unit of currency invested at time 0. If $X_T \in \mathcal{A}(x)$, then we can write $X_T = xR_T + \tilde{X}_T$ for some $\tilde{X}_T \in \mathcal{A}(0)$, where $\mathcal{A}(0)$ is the set of claims which can be replicated with zero initial wealth. (We assume that $\tilde{X}_T \in \mathcal{A}(0)$ if and only if $\tilde{X}_T + xR_T \in \mathcal{A}(x)$.) Since C_T is replicable, from an initial fortune p^{BS}, write $C_T = p^{BS}R_T + \tilde{X}_T^C$ where $\tilde{X}_T^C \in \mathcal{A}(0)$. The superscript BS is intended to denote the Black Scholes or complete market price. Then for $X_T \in \mathcal{A}(x)$,

$$X_T + kC_T = (x + kp^{BS})R_T + \tilde{X}_T + k\tilde{X}_T^C = (x + kp^{BS})R_T + \tilde{X}'_T$$

where $\tilde{X}'_T \in \mathcal{A}(0)$. Thus $X_T + kC_T \in \mathcal{A}(x + kp^{BS})$ and so

$$V(x,k) = \sup_{X_T \in \mathcal{A}(x)} \mathbb{E}U(X_T + kC_T) = \sup_{X_T \in \mathcal{A}(x + kp^{BS})} \mathbb{E}U(X_T) = V(x + kp^{BS}, 0)$$

and thus $p(k) = kp^{BS}$. That is, the indifference price for k units is simply k times the compete market price p^{BS}.

(iii) *Monotonicity*

Let p^i be the utility indifference price for one unit of payoff C_T^i and let $C_T^1 \leq C_T^2$. Then $p^1 \leq p^2$. This follows directly from (5) and the price definitions.

Using properties (ii) and (iii), we have

$$p^{sub}(k) \leq p(k) \leq p^{sup}(k)$$

where p^{sup}, p^{sub} are the super and sub-replicating prices for k units of claim, respectively.

(iv) *Concavity*

Let p_λ be the utility indifference price for the claim $\lambda C_T^1 + (1-\lambda)C_T^2$ where $\lambda \in [0,1]$. Then

$$p_\lambda \geq \lambda p^1 + (1-\lambda)p^2.$$

This can be shown as follows. Let X_T^i be the optimal target wealth for an individual with initial wealth $x - p^i$ due to receive the claim C_T^i. Then, by definition

$$V(x,0) = V(x-p^i, 1, C_T^i) = \sup_{X_T \in \mathcal{A}(x-p^i)} \mathbb{E}U(X_T + C_T^i) = \mathbb{E}U(X_T^i + C_T^i)$$

where the dependence on the claim C_T^i has been made explicit in the notation. Define $\bar{X}_T = \lambda X_T^1 + (1-\lambda)X_T^2$. Then $\bar{X}_T \in \bar{\mathcal{A}} = \mathcal{A}(x - \lambda p^1 - (1-\lambda)p^2)$. Then

$$\begin{aligned}
V(x-\lambda p^1-(1-\lambda)p^2, 1, \lambda C_T^1 + (1-\lambda)C_T^2) &= \sup_{X_T \in \bar{\mathcal{A}}} \mathbb{E}U(X_T + \lambda C_T^1 + (1-\lambda)C_T^2) \\
&\geq \mathbb{E}U(\bar{X}_T + \lambda C_T^1 + (1-\lambda)C_T^2) \\
&= \mathbb{E}U(\lambda(X_T^1 + C_T^1) + (1-\lambda)(X_T^2 + C_T^2)) \\
&\geq \lambda \mathbb{E}U(X_T^1 + C_T^2) + (1-\lambda)\mathbb{E}U(X_T^2 + C_T^2) \\
&= V(x,0) \\
&= V(x-p_\lambda, 1, \lambda C_T^1 + (1-\lambda)C_T^2)
\end{aligned}$$

Note that if we consider sell prices rather than buy prices then p_λ is convex rather than concave.

As can be seen from (6), to compute the utility indifference price of a claim, two stochastic control problems must be solved. The first is the optimal investment problem when the investor has a zero position in the claim. These optimal investment problems date back to Merton [60], [61]. Merton used dynamic programming to solve for an investor's optimal portfolio in a complete market where asset prices follow Markovian diffusions. This approach leads to Hamilton Jacobi Bellman (HJB) equations and a PDE for the value function representing the investor's maximum utility. Merton was able to solve such PDE's analytically in a number of now well known special cases.

The second is the optimal investment problem when the investor has bought or sold the claim. This optimization involves the option payoff, and problems are usually formulated as one of stochastic optimal control and again solved in the Markovian case using HJB equations. We will see an example of this in Section 5.4. An alternative solution approach is to convert this primal problem into the dual problem which involves minimizing over martingale measures, see Section 5.5. Under this approach, the problems are no longer restricted to be Markovian in nature.

As well as the utility indifference price of the option, the hedging strategy of the investor is crucial. Since the market is incomplete, the hedge will not be perfect. The investor's hedge arises from the optimization problem (5) where the optimization takes place over admissible strategies. The hedge typically involves a Merton term which would be the appropriate hedge for the no-option problem and an additional term which accounts for the option position.

The remaining concept to introduce is that of the marginal price. The marginal price is the utility indifference price for an infinitesimal quantity. As we will see later, marginal prices are linear pricing rules and amount to choosing a particular martingale measure. Marginal prices are commonly used in economics, and have been proposed in an option pricing context in various forms by Davis [11], [12], Karatzas and Kou [53] and Kallsen [52].

4 Discrete Time Approach to Utility Indifference Pricing

The problem of pricing European options on non-traded assets in a binomial model was first tackled in Smith and Nau [79] in the context of real options and by Detemple and Sundaresan [18] as part of a study of the effect of portfolio constraints, see Section 6.2. Both papers consider options on a non-traded asset where a second, correlated asset is available for trading. Smith and Nau [79] treat European options where the investor has exponential utility. Detemple and Sundaresan [18] represent price movements in a trinomial model which they solve numerically for the utility indifference price in the case of power utility. They also consider American style options but in a simpler framework with no traded correlated asset. In this case, the investor cannot short-sell the underlying asset and indifference prices are found numerically.

More recently, Musiela and Zariphopoulou [71] (and Chapter 1 of this book) revisit the problem and place it in a mathematical setting. They derive the European option's indifference value in the European case with exponential utility. Other utilities and the alternative dual approach are used in Chapter 2.

We can illustrate briefly the main ideas in a simple one-period binomial model where current time is denoted 0 and the terminal date is time 1. This exposition follows that of Musiela and Zariphopoulou [71]. The market consists of a riskless asset, a traded asset with price P_0 today and a non-traded asset with price Y_0 today. Assume the riskless asset pays no interest for simplicity. The traded price P_0 may move up to $P_0\psi^u$ or down to $P_0\psi^d$ where the random variable ψ takes either value ψ^u, ψ^d and $0 < \psi^d < 1 < \psi^u$. The non-traded price satisfies $Y_1 = Y_0\phi$ where $\phi = \phi^d, \phi^u$ and $\phi^d < \phi^u$. There are four states of nature corresponding to outcomes of the pair of random variables: (ψ^u, ϕ^u), (ψ^u, ϕ^d), (ψ^d, ϕ^u), (ψ^d, ϕ^d). Wealth X_1 at time 1 is given by $X_1 = \beta + \alpha P_1 = x + \alpha(P_1 - P_0)$ where α is the number of shares of stock held, β is the money in the riskless asset and x is initial wealth. The investor is pricing k units of a claim with payoff C_1 and has exponential utility. The value function in (5) becomes

$$V(x,k) = \sup_{\alpha} \mathbb{E}\left[-\frac{1}{\gamma}e^{-\gamma(X_1+kC_1)}\right]$$

and the utility indifference buy price defined in (6) solves

$$V(x,0) = V(x - p^b(k), k).$$

The price $p^b(k)$ is given by

$$p^b(k) = \mathbb{E}^{\mathbb{Q}^{\langle 0 \rangle}} \left(\frac{1}{\gamma} \log \mathbb{E}^{\mathbb{Q}^{\langle 0 \rangle}} \left(e^{\gamma k C_T} | P_1 \right) \right) \tag{8}$$

where $\mathbb{Q}^{\langle 0 \rangle}$ is the measure under which the traded asset P is a martingale and the conditional distribution of the non-traded asset given the traded one is preserved with respect to the real world measure $\mathbb{P}$. This is the minimal martingale measure of Föllmer and Schweizer [25]. In fact, in this simple setting, all minimal distance measures are identical, so $\mathbb{Q}^{\langle 0 \rangle}$ is also the minimal entropy measure of Frittelli [28], see the discussion in Section 7.2.

The price in (8) can be shown to satisfy properties (i)-(iv) in Section 3, see Chapter 1 of this book. The above formulation shows the utility indifference price (in a one period model with exponential utility) can be written as a new non-linear, risk adjusted payoff, and then expectations are taken with respect to $\mathbb{Q}^{\langle 0 \rangle}$ of this new payoff. This is in contrast to the usual linear pricing structures found in complete markets and in other approaches to incomplete markets pricing. A similar representation appears in Smith and Nau [79].

5 Utility indifference pricing in continuous time.

Consider a model on a stochastic basis $(\Omega, (\mathcal{F}_t)_{0 \le t \le T}, \mathbb{P})$. For simplicity we assume that the model supports a single traded asset with price process P_t and a second auxiliary process Y_t, which may correspond to a related but non-traded stock or a diffusion process which drives the dynamics of P_t. For example Y_t may represent the volatility of P. Suppose that P_t and Y_t are governed by the SDEs

$$\frac{dP_t}{P_t} = \sigma_t(dB_t + \lambda_t dt) + r_t dt \tag{9}$$

$$dY_t = a_t dW_t + b_t dt \tag{10}$$

Here B and W are Brownian motions with correlation ρ_t which together generate the filtration $\mathcal{F}_t$ and σ, λ, r, a, b and ρ are adapted to $\mathcal{F}_t$. The problem is to price a (typically non-negative) contingent claim $C_T \in m\mathcal{F}_T$, where T is the horizon time.

We will concentrate on this bi-variate model which is rich enough to contain interesting examples and to illustrate the central concepts of the theory, but simple enough for explicit solutions to sometimes exist. It is possible to extend the analysis to higher dimensions, but as we shall see it is already difficult to find solutions to the utility indifference pricing problem even in the two-dimensional special case. Throughout this overview we will assume that even though the asset Y_t is not directly traded, its value is an observable quantity. When Y_t is a hidden Markov process and its value needs to be estimated from the information contained in the filtration generated by the traded asset P_t the issues become much more delicate, see Chapter 4.

It is convenient to write the Brownian motion W as a composition of two independent Brownian motions B_t and $B_t^\perp$ so that $dW_t = \rho_t dB_t + \rho_t^\perp dB_t^\perp$ where $\rho_t^\perp$ is the positive solution to $\rho_t^2 + (\rho_t^\perp)^2 = 1$. Note also that we have chosen to parameterise the traded asset P via its volatility σ_t and Sharpe ratio λ_t rather than volatility and drift. Of course, there is a simple relationship between the two parameterisations whereby the drift is given by $r_t + \lambda_t \sigma_t$, but as we shall see the Sharpe ratio plays a fundamental role in the characterisation of the solution to the utility indifference pricing problem. Moreover, it is the Sharpe ratio which determines whether an investment is a good deal.

There are two canonical situations which fit into our general framework:

Example 5.1 Non-traded assets problem
Let Y represent the value of a security which is not traded, or on which trading is difficult or impossible for an agent because of liquidity or legal restrictions. Let P represent the value of a related asset such as the market index. The problem is to calculate a utility indifference price for a claim $C_T = C(Y_T)$ on the non-traded asset. Davis [12] calls this example a model with basis risk. Examples might include a real option, or an executive stock option where the executive is forbidden from trading on the underlying stock, see Section 6.

We shall identify a special case of this problem as the constant parameter case. By this we mean that σ, λ, r and ρ are constants and Y_t is an autonomous diffusion. The analysis of the problem does not depend on the precise specification of the dynamics of this process, but if Y_t is to represent a stock price process it is most natural to take $a_t = \eta Y_t$ and $b_t = Y_t(r + \eta\xi)$ where ξ is the Sharpe ratio of the non-traded asset.

This specification is common in the finance literature. Duffie and Richardson [19] considered the problem of determining the optimal hedge in this model under the assumption of a quadratic utility. The problem of finding a utility indifference price under exponential utility was studied by Tepla [82] who considered the case where Y_t is Brownian motion. In the specific case where the claim is units of the non-traded asset $C_T = Y_T$ she derived an explicit formula for the utility indifference price. The exponential Brownian case was solved explicitly by Henderson and Hobson [39] and Henderson [33], see also Section 5.3. They gave a general representation of the price of a claim which is a function of the non-traded asset $C_T = C(Y_T)$, see (16) below. Subject to a transformation of variables this formula includes the Tepla result as a special case. Finally, Musiela and Zariphopoulou [70] observed that the same analysis carries over to arbitrary diffusion processes Y_t.

As we shall see exponential utility and the non-traded assets model is one of the few examples for which an explicit form for the utility indifference price is known.

Example 5.2 Stochastic volatility models
The second important situation is when Y governs the volatility of the asset, so that $\sigma_t = \sigma(Y_t, t)$. In this setting the fundamental problem is to price a derivative, such as a call option, on the traded asset P, but it is also possible to consider options on volatility itself.

We shall be interested in the situation where the Sharpe ratio depends on Y_t and Y_t is an autonomous diffusion (popular models include the Ornstein-Uhlenbeck process of Stein and Stein [80], and the square-root or Bessel process proposed by Hull and White [48] and investigated by Heston [41]). In this case some progress can be made towards characterising the solution, but unlike in the non-traded assets model there is no explicit representation of the utility indifference option price, even for common classes of utility functions.

Note that if $\sigma(Y_t) = Y_t$ then given observations on the asset price process it is possible to determine the quadratic variation and hence Y_t^2. If Y_t is modelled as a non-negative process (such as in the Bessel process model) this means that Y_t is adapted to the filtration generated by the price process $\mathcal{F}_t^P$. With sufficient additional regularity conditions the filtration $\mathcal{F}_t$ can be identified with the filtration generated by the price process $\mathcal{F}_t^P$. In this overview we will limit the analysis to the case where this identification is valid, not least because in the general case additional complications over filtering arise, see Rheinländer [75].

5.1 Martingale Measures and State-Price Densities

One common approach to option pricing in incomplete markets in the mathematical financial literature is to fix a measure $\mathbb{Q}$ under which the discounted traded assets are martingales and to calculate option prices via expectation under this measure. This is related to the notion of a state-price-density from economics. The advantage of using a state-price-density ζ_T is that prices can be calculated as expectations under the physical measure: $p = \mathbb{E}[\zeta_T C_T]$.

In an incomplete market there is more than one martingale measure, or equivalently there are infinitely many state-price-densities. In the model given by (9) and (10) it is straightforward to characterise the equivalent martingale measures, see Frey [26]. They are given by

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_T} = Z_T$$

where Z is a uniformly integrable martingale of the form

$$Z_s = \exp\left\{-\int_0^s \lambda_t dB_t - \frac{1}{2}\int_0^s \lambda_t^2 dt - \int_0^s \chi_t dB_t^\perp - \frac{1}{2}\int_0^s \chi_t^2 dt\right\}. \qquad (11)$$

Here λ_t is the Sharpe ratio of the traded asset, but χ_t is undetermined, save for the fact that for $\mathbb{Q}$ to be a true probability measure it is necessary to have $\mathbb{E}[Z_T] = 1$. An example of a candidate martingale Z_t is given by the choice $\chi_t \equiv 0$ which leads to the minimal martingale measure of Föllmer and Schweizer [25].

The state-price-densities ζ_T take the form

$$\zeta_s = \exp\left\{-\int_0^s r_t dt\right\} Z_s$$

where Z_s is given by (11) and have the property that $\zeta_t P_t$ is a $\mathbb{P}$-local martingale. If the interest rate is deterministic then the state-price density and the density of the martingale measure differ only by a positive constant. Otherwise they differ by a stochastic discount factor.

The martingale measures $\mathbb{Q}^{\langle\chi\rangle}$, associated martingales $Z^{\langle\chi\rangle}$ and state-price densities $\zeta_T^{\langle\chi\rangle}$ can all be parameterised by the process χ_t which governs the change of drift on the non-traded Brownian motion $B^\perp$, and we shall use the superscript $\langle\chi\rangle$ to denote this dependence.

5.2 Numéraires

In a complete market there is just one martingale measure or state-price density. All options can be replicated and the unique fair price for the option is given by the replication price. As we saw in Section 3 the replication price is also the utility indifference bid and ask price.

The price calculated in this way does not depend on the choice of numéraire, see Geman et al [29]. As a result we are free to choose any numéraire which is convenient for the calculations: for example in an exchange or Margrabe [58] option the analysis is greatly simplified if one of the assets in the exchange is used as numéraire.

Although in a complete market the fair price does not depend on the choice of numéraire, the definition of martingale measure does depend on the choice of numéraire, see Branger [5]. We have the relationship

$$\zeta_T = \frac{1}{N_T} \frac{d\mathbb{Q}^N}{d\mathbb{P}} \bigg|_{\mathcal{F}_T},$$

where N_t is the numéraire, and $\mathbb{Q}^N$ is a martingale measure for this numéraire. The formulae of the previous section are quoted with respect to the bank account numéraire. Again, note that the state-price-density has the advantage of being numéraire independent.

For utility indifference pricing the situation is somewhat different. In an incomplete market there is risk, and an agent needs to specify the units in which these risks are to be measured, as well as the concave utility function. If the numéraire is to be changed then the utility needs to be modified, sometimes in a non-trivial and unnatural way, in order that the analysis remains consistent.

We shall fix cash at time T as the units in which utility is measured.

5.3 The primal approach

Recall that the utility indifference price of the claim C_T is given as the solution to

$$V(x - p(k), k) = V(x, 0) \tag{12}$$

where

$$V(x, k) = \sup \mathbb{E}\left[U\left(X_T^{x,\theta} + kC_T\right)\right].$$

Here the notation $X_T^{x,\theta}$ denotes the terminal fortune of an investor with initial wealth x who follows a trading strategy which consists of holding θ_t units of the traded asset.

At this stage we are not very explicit about set of attainable terminal wealths, except to say that $X_T^{x,\theta}$ is the terminal value of the wealth process satisfying $X_0^{x,\theta} = x$, the self-financing condition

$$dX_t^{x,\theta} = \theta_t dP_t + r_t(X_t^{x,\theta} - \theta_t P_t)dt \tag{13}$$

and sufficient regularity conditions to exclude doubling strategies.

In order to calculate the utility indifference price it is necessary to solve a pair of optimisation problems. As in the binomial setting there are two approaches to each problem, via primal and dual methods. We begin with a discussion of the primal approach for which it is necessary to assume that we are in a Markovian setting, and to consider the dynamic version of the optimisation problem at an intermediate time t.

Define

$$V(x, 0) = V(x, p, y, t) = \sup_\theta \mathbb{E}_t \left[U(X_T^{x,\theta}) \Big| X_t = x, P_t = p, Y_t = y\right].$$

Using the observation that $V(x, p, y, t)$ is a martingale under the optimal strategy θ, and a supermartingale otherwise, we have that V solves an equation of the form

$$\sup_\theta \mathcal{L}^\theta V = 0 \qquad V(x, p, y, T) = U(x)$$

where

$$\mathcal{L}^\theta f = \frac{1}{2}\theta_t^2\sigma_t^2p^2f_{xx} + \frac{1}{2}\sigma_t^2p^2f_{pp} + \frac{1}{2}a_t^2f_{yy} + \theta_t\sigma_t^2p^2f_{xp} + \theta_t\sigma_ta_t\rho_tpf_{xy} + \sigma_ta_t\rho_tpf_{py} + (\theta_t\sigma_t\lambda_tp + r_tx)f_x + (\sigma_t\lambda_tp + r_tp)f_p + b_tf_y + \dot{f}$$

Here a subscript t refers to an adapted process whereas other subscripts refer to partial derivatives. Given that $\mathcal{L}^\theta$ is quadratic in θ the minimisation in θ is trivial and the problem can be reduced to solving a non-linear Hamilton-Jacobi-Bellman equation in four variables.

If P_t is an exponential Brownian motion with constant parameters (or more precisely if none of the parameters σ, λ, r, a, b or ρ depends on the price level), then the traded asset value scales out of the problem and the dimension can be reduced by one. Further, in the non-traded asset model where σ, λ, r and ρ are independent of Y_t, the solution of the Merton problem is independent of Y_t and again the number of state-variables can be reduced by one. Finally for HARA utility functions it is possible to conjecture the dependence of the value function on wealth and again to reduce the number of dimensions. For example, for exponential utility wealth factors out of the problem and it is possible to consider $V \equiv -(1/\gamma)e^{-\gamma x}\overline{V}(p, y, t)$, where $\overline{V}(p, y, T)$=1.

5.4 The Non-traded assets model

Suppose we are in the non-traded asset problem with constant interest rates such that P_t follows a constant parameter Black-Scholes model. Suppose that Y_t is also a representation of a share price so that it is again natural to think of Y_t as following an exponential Brownian motion:

$$\frac{dY_t}{Y_t} = \eta dW_t + (r + \eta\xi)dt.$$

It follows that with exponential utility

$$V(x, p, y, t) = -\frac{1}{\gamma}e^{-\gamma x e^{r(T-t)} - \lambda^2(T-t)/2}, \tag{14}$$

whereas for power utility

$$V(x, p, y, t) = \frac{1}{1-R}x^{1-R}e^{(1-R)\lambda^2(T-t)/2R}e^{r(T-t)(1-R)}, \tag{15}$$

see Merton [60, 61].

Now consider the problem of evaluating the left-hand side of (12) under the assumption that $C_T = C(Y_T)$. At this stage we only sketch the details of the argument because we are going to give a fuller discussion via the dual approach in later sections. The only change from the analysis of the previous section is that the boundary condition becomes $V(x, p, y, T) = U(x + kC(y))$. Again the above simplifications can be used to reduce the dimension of the problem (see Henderson and Hobson [39]) except that in this case it is not possible to remove the dependence on y.

Suppose that the agent has exponential utility. Then the non-linear HJB equation can be linearised using the Hopf-Cole transformation (this idea was introduced to mathematical finance by Zariphopoulou [83] with the terminology *distortion*). It is now possible to write down the solution to this equation and the value function to the problem with the option (at $t = 0$) is given by (Henderson and Hobson [39])

$$V(x,k) = -\frac{1}{\gamma} e^{-\gamma x e^{rT} - \lambda^2 T/2} \left(\mathbb{E}^{\mathbb{Q}^{\langle 0 \rangle}} \left[e^{-k\gamma(1-\rho^2)C(Y_T)} \right] \right)^{1/(1-\rho^2)}$$

Here $\mathbb{Q}^{\langle 0 \rangle}$ is the minimal martingale measure: the measure under which the discounted traded asset is a martingale, but the law of the orthogonal martingale measure is unchanged. In the non-traded assets model $\mathbb{Q}^{\langle 0 \rangle}$ is also the minimal distance measure for any choice of distance metric, including the minimal entropy measure, see Section 7.2.

It follows that the price can be expressed as

$$p(k) = -\frac{e^{-rT}}{\gamma(1-\rho^2)} \ln \mathbb{E}^{\mathbb{Q}^{\langle 0 \rangle}} \left[e^{-k\gamma(1-\rho^2)C(Y_T)} \right]. \tag{16}$$

Observe that this price is independent of the initial wealth of the agent, and that it is a non-linear concave function of k. When $k > 0$, and C is non-negative, the bid price is well defined, but for $k < 0$ it may be that the price is infinite. (This is true if the claim is units of the asset Y, or calls on Y_T.) Thus one of the disadvantages of exponential utility is that the ask price for many important examples of contingent claims is infinite. This is one of the motivations for considering the utility U defined in (2).

Suppose now that the agent has power utility. In this case there is no known solution to the the HJB equation, although it can be solved numerically. Instead Henderson and Hobson [38] and Henderson [33] consider expansions in the number of claims k. The former paper considers claims which are units of the non-traded asset, whereas the latter considers more general European claims. Henderson [33] finds that the utility indifference price is given by

$$p(k) = k e^{-rT} \mathbb{E}^{\mathbb{Q}^{\langle 0 \rangle}} [C(Y_T)] - \frac{k^2}{2} \frac{R}{x} \eta^2 (1-\rho^2) \mathbb{E}^{\mathbb{Q}^{\langle 0 \rangle}} \int_0^T e^{-rt} \frac{Y_t^2 (C_t^Y)^2}{(X_t^{0,*}/x)} dt + o(k^2) \tag{17}$$

where $C_t = e^{-r(T-t)} \mathbb{E}_t^{\mathbb{Q}^{\langle 0 \rangle}} [C(Y_T)]$, $C_t^Y = \partial C_t / \partial Y$ and $X_t^{0,*}$ is the wealth process consistent with the optimal solution of the Merton problem in the absence of the claim. Note that by scaling $X_t^{0,*}/x$ is independent of x, so that the integral in the above expression is independent of initial wealth.

The idea of the proofs in both [38] and [33] is that the value function in the presence of the claim can be approximated from below by considering a cleverly chosen, but sub-optimal, wealth process X_t, and from above by considering a well-chosen state-price density ζ_T (see (20) below). Given upper and lower bounds on the value function it is possible to deduce bounds on the option price, which agree to $o(k^2)$.

Given this expansion for the utility-indifference price of the claim it is possible to investigate the comparative statics of the price with respect to parameters such as initial wealth. Consider (17). As wealth increases, the second term becomes less important and the bid price rises. Note that the first term in the expansion is independent of the relative risk aversion co-efficient R and of the initial wealth of the agent x, and is the discounted expected payoff under the Föllmer-Schweizer minimal martingale measure, and the second term is negative which is consistent with a price function which is concave in k. Further the first term is linear in the claim, (in the sense that $\mathbb{E}^{\mathbb{Q}^{\langle 0\rangle}}[C_1(Y_T)+C_2(Y_T)] = \mathbb{E}^{\mathbb{Q}^{\langle 0\rangle}}[C_1(Y_T)]+\mathbb{E}^{\mathbb{Q}^{\langle 0\rangle}}[C_2(Y_T)]$), but the second term is non-linear, recall the first property in Section 3.

Again, if the claim is unbounded the ask price can be infinite, so that the above expansion is only valid in general for positive claims and positive k.

5.5 The Dual Approach

The primal approach involves finding

$$\sup_{\theta} \mathbb{E}[U(X_T^{x,\theta} + kC_T)]. \tag{18}$$

The dual approach involves solving (18) via translating the problem into one of minimisation over state-price densities or martingale measures, see Karatzas et al [54] and Cvitanic et al [10].

In a complete market it is possible to write the set of attainable terminal wealths generated from an initial fortune x and a self-financing strategy as the set of random variables which satisfy $\mathbb{E}[\zeta_T X_T] \leq x$. In an incomplete market this condition becomes that $\mathbb{E}[\zeta_T X_T] \leq x$ for all state-price-densities. This allows us to take a Lagrangian approach to solving (18). For all state-price densities ζ_T, terminal wealths X_T satisfying the budget constraint and non-negative Lagrange multipliers μ

$$\begin{aligned} &\mathbb{E}[U(X_T + kC_T) - \mu(\zeta_T X_T - x)] \\ &\quad = \mathbb{E}[U(X_T + kC_T) - \mu\zeta_T(X_T + kC_T) + \mu(x + \zeta_T kC_T)] \\ &\quad \leq \mu x + \mu k\mathbb{E}[\zeta_T C_T] + \mathbb{E}[\tilde{U}(\mu\zeta_T)] \end{aligned} \tag{19}$$

where $\tilde{U}$ was the Legendre-Frenchel transform of $-U$ introduced earlier in (4). Optimising over wealths on the one hand, and Lagrange multipliers and state-price densities on the other we have

$$\sup_{X_T} \mathbb{E}[U(X_T + kC_T)] \leq \inf_{\mu} \inf_{\zeta_T} \left\{ \mu x + \mu k\mathbb{E}[\zeta_T C_T] + \mathbb{E}[\tilde{U}(\mu\zeta_T)] \right\}, \tag{20}$$

and provided certain regularity conditions are met, see for example Owen [73], there is equality in this expression.

The dual problem is to find the infimum on the right-hand-side of (20). By considering the derivation of the dual problem it is clear that if we can find suitable random variables $X_T^{k,*}$ and $\zeta_T^{k,*}$, and a constant $\mu^{k,*}$ such that $U'(X_T^{k,*} + kC_T) = \mu^{k,*}\zeta_T^{k,*}$ then there should be equality in (19) and hence $X_T^{k,*}$ is the optimal primal variable, and $\mu^{k,*}$ and $\zeta_T^{k,*}$ are the optimal dual variables.

Consider first the case where $k = 0$. In solving the Merton problem it is possible to ignore the presence of the process Y_t and reduce the optimisation problem to a complete market problem involving Y_t alone. In a complete market there is a unique state-price density and finding the infimum over ζ_T is trivial. In this case the primal problem of minimising over random variables X_T is reduced to the problem of minimising over a real-valued quantity. This illustrates the power of the dual method: for many problems the dual problem is a considerable simplification.

Let $\mu^{0,*}$ and $\zeta_T^{0,*}$ be the solutions to the dual problem when there is zero endowment of the contingent claim (and the agent has initial wealth x). We suppose that such solutions exist. Then

$$\begin{aligned} V(x - k\mathbb{E}[\zeta_T^{0,*} C_T], k) &= \inf_{\mu} \inf_{\zeta_T} \left\{ \mu(x - k\mathbb{E}[\zeta_T^{0,*} C_T]) + \mu k \mathbb{E}[\zeta_T C_T] + \mathbb{E}[\tilde{U}(\mu\zeta_T)] \right\} \\ &\leq \mu^{0,*} x + \mathbb{E}[\tilde{U}(\mu^{0,*}\zeta_T^{0,*})] \\ &= V(x, 0) = V(x - p(k), k). \end{aligned}$$

It follows that $p(k) \leq k\mathbb{E}[\zeta_T^{0,*} C_T]$ and we have a simple upper bound on the utility indifference of the bid price of a claim in terms of an expectation related to the solution of a Merton problem. We return to this idea later in Section 7.3.

Now consider the right-hand-side of (20). In the case of deterministic interest rates and exponential utility the minimisation over ζ_T, or equivalently Z_T, reduces to

$$\inf_{Z_T} \left\{ \mathbb{E}[Z_T \ln Z_T] + \gamma k \mathbb{E}[Z_T C_T] \right\}.$$

Furthermore, given the simple dependence of exponential utility on initial wealth, it is possible (see Rouge and El Karoui [76], Delbaen et al [17], Frittelli [28] and Becherer [3]) to deduce an expression for the form of the utility-indifference price

$$p(k) = \frac{e^{-rT}}{\gamma} \left(\inf_{Z_T} \left\{ \mathbb{E}[Z_T \ln Z_T] + k\gamma \mathbb{E}[Z_T C_T] \right\} - \inf_{Z_T} \left\{ \mathbb{E}[Z_T \ln Z_T] \right\} \right). \tag{21}$$

Note that the second minimisation in (21) involves finding the minimal entropy measure.

5.6 Solving the Merton problem via duality.

The goal in this section is to solve the optimisation problem for a class of problems of the forms given in Examples 5.1 and 5.2 under the assumption of a utility function of HARA type. As we observed in the previous section the key is to find a solution to $X_T^* = I(\mu^* \zeta_T^*)$.

Suppose that U satisfies a power law, $U(x) = x^{1-R}/(1-R)$. Then $I(y) = y^{-1/R}$ is again of power form. Using the substitution $\pi_t = \theta_t P_t / X_t$, the gains from trade from a self-financing strategy can be written

$$\frac{dX_t}{X_t} = \pi_t \left(\frac{dP_t}{P_t} - r_t dt \right) + r_t dt \tag{22}$$

where π_t is the proportion of wealth invested in the risky traded asset. This has solution

$$X_T = x \exp \left\{ \int_0^T \pi_t \sigma_t (dB_t + \lambda_t dt) + \int_0^T \left(r_t - \frac{1}{2} \pi_t^2 \sigma_t^2 \right) dt \right\}.$$

Hence, if $X_T^* = I(\mu^* \zeta_T^*)$, where π^* denotes the optimal strategy and χ^* the market price of risk for the Brownian motion $B^\perp$, then we must have

$$\begin{aligned} &\ln x + \int_0^T \pi_t^* \sigma_t (dB_t + \lambda_t dt) + \int_0^T \left(r_t - \frac{1}{2} (\pi_t^*)^2 \sigma_t^2 \right) dt \\ &= -\frac{1}{R} \ln \mu^* + \frac{1}{R} \int_0^T r_t dt + \frac{1}{R} \int_0^T \lambda_t dB_t + \frac{1}{2R} \int_0^T \lambda_t^2 dt + \frac{1}{R} \int_0^T \chi_t^* dB_t^\perp + \frac{1}{2R} \int_0^T (\chi_t^*)^2 dt. \end{aligned}$$

After some algebra and the substitution $\phi^*_t = \lambda_t - R\pi^*_t\sigma_t$ this equation can be reduced to

$$\frac{1}{2}\left(1-\frac{1}{R}\right)\int_0^T \lambda_t^2 dt - (1-R)\int_0^T r_t dt = c + M_T + \frac{1}{2R}[M]_T + M_T^{\perp} + \frac{1}{2}[M^{\perp}]_T \tag{23}$$

where c is a constant depending on μ^*, x and R,

$$M_t = \int_0^t \phi^*_u \left(dB_u + \lambda_t\left(1-\frac{1}{R}\right)dt\right), \qquad M_t^{\perp} = \int_0^t \chi^*_u dB_t^{\perp}$$

and where $[\cdot]$ denotes quadratic variation. If interest rates are deterministic then this simplifies to a representation

$$\frac{1}{2}\left(1-\frac{1}{R}\right)\int_0^T \lambda_t^2 dt = \alpha + M_T + \frac{1}{2R}[M]_T + M_T^{\perp} + \frac{1}{2}[M^{\perp}]_T \tag{24}$$

where α is the constant $c + (1-R)\int_0^T r_t dt$. Note that in the case $R = 1$ (logarithmic utility) there is a trivial solution for which both sides of (23) are zero. In this case we can find a solution to $X_T = I(\mu\zeta_T)$, for which the state-price-density is the minimal state-price-density in the sense of Föllmer and Schweizer [25]. Note also that (23) is an identification of random variables and not processes, and is a representation of a $\mathcal{F}_T$ random variable in terms of a pair of Brownian martingales *and* their quadratic variations.

Let us now consider the case of exponential utility, under the assumption of constant interest rates. Formally, taking $R = \infty$ in Equation (24) gives

$$\frac{1}{2}\int_0^T \lambda_t^2 dt = \alpha + \int_0^T \phi^*_t(dB_t + \lambda_t dt) + \int_0^T \chi^*_u dB_u^{\perp} + \frac{1}{2}\int_0^T (\chi^*_u)^2 du \tag{25}$$

This equation can be derived directly from the relationship $X_T = I(\mu\zeta_T)$, using (13) rather than (22), but we want to give a direct argument which shows that (25) leads to the solution of the Merton problem.

For some state-price density $\zeta_T^{\langle\chi\rangle} = e^{-rT}Z_T^{\langle\chi\rangle}$ we have

$$\mathbb{E}[\tilde{U}(\mu\zeta_T^{\langle\chi\rangle})] = \mathbb{E}\left[\frac{\mu\zeta_T^{\langle\chi\rangle}}{\gamma}\left(\ln\zeta_T^{\langle\chi\rangle} - (1-\ln\mu)\right)\right] = \frac{\mu e^{-rT}}{\gamma}\left(\mathbb{E}^{\mathbb{Q}^{\langle\chi\rangle}}[\ln Z_T^{\langle\chi\rangle}] - (1-\ln\mu + rT)\right).$$

Here $\mathbb{Q}^{\langle\chi\rangle}$ is the measure given by $(d\mathbb{Q}^{\langle\chi\rangle}/d\mathbb{P})|_{\mathcal{F}_T} = Z_T^{\langle\chi\rangle}$. Further,

$$\begin{aligned}\ln Z_T^{\langle\chi\rangle} &= -\int_0^T \lambda_t dB_t^{\mathbb{Q}^{\langle\chi\rangle}} + \frac{1}{2}\int_0^T \lambda_t^2 dt - \int_0^T \lambda_t^{\perp} dB_t^{\perp,\mathbb{Q}^{\langle\chi\rangle}} + \int_0^T \frac{1}{2}\int_0^T \chi_t^2 dt \\ &= \alpha + \int_0^T (\eta_t - \lambda_t) dB_t^{\mathbb{Q}^{\langle\chi\rangle}} + \int_0^T (\chi_t - \lambda_t^{\perp}) dB_t^{\perp,\mathbb{Q}^{\langle\chi\rangle}} + \frac{1}{2}\int_0^T (\chi^*_t - \chi_t)^2 dt \end{aligned} \tag{26}$$

where $B^{\mathbb{Q}^\chi}$ and $B^{\perp,\mathbb{Q}^\chi}$ are $\mathbb{Q}^\chi$-Brownian motions. Taking expectations under $\mathbb{E}^{\mathbb{Q}^\chi}$, and assuming that the local martingales in (26) are true martingales, we have

$$\mathbb{E}^{\mathbb{Q}^{\langle\chi\rangle}}[\ln Z_T] = \alpha + \mathbb{E}^{\mathbb{Q}^{\langle\chi\rangle}}\left[\frac{1}{2}\int_0^T (\chi^*_t - \chi_t)^2\right] \tag{27}$$

which is minimised by the choice $\chi_t = \chi_t^*$. Hence, subject to certain technical conditions, if we can solve (25) then we have found the measure which minimises $\mathbb{E}[\tilde{U}(\mu\zeta_T^{\langle\chi\rangle})]$, and moreover,

$$\inf_{\zeta_T} \left\{ \mu x + \mathbb{E}[\tilde{U}(\mu\zeta_T)] \right\} = \mu x + \frac{\mu e^{-rT}}{\gamma} \left(\alpha - (1 - \ln \mu + rT) \right).$$

A further minimisation over μ gives

$$V(x,0) = \inf_{\mu} \inf_{\zeta_T} \left\{ \mu x + \mathbb{E}[\tilde{U}(\mu\zeta_T)] \right\} = -\frac{1}{\gamma} \exp\left(-\gamma x e^{rT} - \alpha \right). \tag{28}$$

Hence finding the solution to the dual Merton problem under exponential utility reduces to solving (25).

This direct argument can be adapted to cover power utilities. In this case we find

$$V(x,0) = \inf_{\mu} \inf_{\zeta_T} \left\{ \mu x + \mathbb{E}[\tilde{U}(\mu\zeta_T)] \right\} = \frac{x^{1-R}}{1-R} e^{(1-R)rT - \alpha} \tag{29}$$

5.7 Explicit solutions of the Merton problem via duality under constant interest rates

It remains to solve (24) or (25). Consider first the non-traded assets model in which r and λ are constants. Then (24) has a trivial solution

$$\alpha = \frac{1}{2}\left(1 - \frac{1}{R}\right) \lambda^2 T.$$

(The solution for exponential utility follows on taking $R = \infty$.) If we substitute this value into (28) and (29) then we recover the value functions (14) and (15) for $t = 0$.

Now consider the stochastic volatility model, under an assumption of a constant correlation between the Brownian motions B and W. Suppose that Y_t is an autonomous diffusion and that the Sharpe ratio is a function of this process: $\lambda_t = \lambda(Y_t)$. Then with the reparameterisation $\phi_t^* = \rho\psi_t$, $\chi_t^* = \rho^{\perp}\psi_t$ and $\Lambda = (\rho^{\perp})^2 + (\rho^2/R) = 1 + \rho^2(R^{-1} - 1)$, (24) becomes

$$\frac{1}{2}\left(1 - \frac{1}{R}\right) \int_0^T \lambda(Y_t)^2 dt = \alpha + \int_0^T \psi_t dW_t + \left(1 - \frac{1}{R}\right) \int_0^T \rho\psi_t dt + \frac{\Lambda}{2} \int_0^T \psi_t^2 dt. \tag{30}$$

Let $\hat{\mathbb{P}}$ be the measure under which $\hat{W}$ given by $d\hat{W} = dW + (1 - R^{-1})\rho dt$ is a Brownian motion. Then, see Kobylanski [55] or Hobson [43], multiplying by $-\Lambda$, exponentiating, and taking expectations under $\hat{\mathbb{P}}$ we have

$$\alpha = -\frac{1}{\Lambda} \ln \left(\mathbb{E}^{\hat{\mathbb{P}}} \left[\exp\left\{ -\frac{\Lambda}{2} \left(1 - \frac{1}{R}\right) \int_0^T \lambda(Y_t)^2 dt \right\} \right] \right). \tag{31}$$

Thus we have found a solution to the Merton problem. Provided that the expectation in (31) can be calculated, it is possible to represent the solution to the Merton problem in a simple form. Diffusions for which (31) can be solved include the case where Y_t is a Bessel process, and $\lambda(Y_t)$ is affine, see Grasselli and Hurd [32].

In the above analysis we concentrated on finding the value of α which is required to solve for the value function. The full solution of (30) includes an expression for ψ_t, which in turn gives expressions for ϕ_t^* and hence the optimal trading strategy. Given ψ_t it is also possible to calculate the market price of risk χ^*. In the stochastic volatility case it turns out that the market price of risk is a time-inhomogeneous function of Y_t, see Hobson [43].

5.8 Solving the dual problem with the claim.

Now we return to the problem with the claim. In order to characterise the solution we need to solve $X_T^* + kC_T = I(\mu^* \zeta_T^*)$.

Suppose interest rates are deterministic and the utility function is exponential. Suppose we have a solution to

$$\frac{1}{2}\int_0^T \lambda_t^2 dt + \gamma k C_T = \alpha + \int_0^T \phi_t^*(dB_t + \lambda_t dt) + \int_0^T \chi_u^* dB_u^\perp + \frac{1}{2}\int_0^T (\chi_u^*)^2 du. \tag{32}$$

Then, by a direct repeat of the argument leading to (27) we have

$$\mathbb{E}^{\mathbb{Q}^\chi}[\gamma k C_T] + \mathbb{E}^{\mathbb{Q}^\chi}[\ln Z_T^\chi] = \alpha + \mathbb{E}^{\mathbb{Q}^\chi}\left[\frac{1}{2}\int_0^T (\chi_t^* - \chi_t)^2\right] \tag{33}$$

which is minimised by the choice $\chi_t = \chi_t^*$, and then

$$\mu x + \mu k \mathbb{E}[\zeta_T^\chi C_T] + \mathbb{E}[\tilde{U}(\mu \zeta_T^\chi)] \geq \mu x + \frac{\mu e^{-rT}}{\gamma}(\alpha - (1 - \ln \mu + rT)).$$

with equality for $\chi = \chi^*$. It follows that

$$V(x,k) = -\frac{1}{\gamma}\exp\left(-\gamma x e^{rT} - \alpha\right). \tag{34}$$

It remains to solve (32). Consider the non-traded assets example with constant parameters. There are two cases which can be solved.

Case 1. Suppose that the claim C_T can be replicated for an initial price p^{BS}, so that

$$C_T = p^{BS}e^{rT} + \int_0^T \theta_t(dP_t/P_t - rdt) = p^{BS}e^{rT} + \int_0^T \theta_t \sigma(dB_t + \lambda dt).$$

Then there is a trivial solution of (32) for which $\chi^* \equiv 0$, $\phi_t^* = \sigma k \gamma \theta_t$ and $\alpha = \gamma k e^{rT} p^{BS} + \lambda^2 T/2$. Note that $p^{BS} = e^{-rT}\mathbb{E}^{\mathbb{Q}}[C_T]$ where $\mathbb{Q}$ is any martingale measure.

Case 2. Suppose that $C_T = C(Y_T)$. We look for a solution under $\mathbb{Q}^{\langle 0\rangle}$, the martingale measure under which $B^{\langle 0\rangle}$ given by $B_t^{\langle 0\rangle} = B_t + \lambda t$ and $B^\perp$ are Brownian motions. Set $\beta = \alpha - \lambda^2 T/2$ and $W_t^{\langle 0\rangle} = \rho B_t^{\langle 0\rangle} + \rho_t^\perp B_t^\perp$. Then (32) becomes

$$\gamma k C(Y_T) = \beta + \int_0^T \eta_t dB_t^{\langle 0\rangle} + \int_0^T \chi_u dB_u^\perp + \frac{1}{2}\int_0^T \chi_u^2 du. \tag{35}$$

We look for a solution of the form $\eta_t = \rho\psi_t$, $\chi_t = \rho^\perp \psi_t$, whence

$$\gamma k C(Y_T) = \beta + \int_0^T \psi_t dW^{\langle 0\rangle} + \frac{(1-\rho^2)}{2}\int_0^T \psi_u^2 du.$$

Using the same transformations as for the Merton problem we deduce

$$\beta = -\frac{1}{(1-\rho^2)}\ln \mathbb{E}^{\mathbb{Q}^{\langle 0\rangle}}\left[e^{-(1-\rho^2)\gamma k C_T}\right]$$

from which it is possible to deduce an expression for α and hence $V(x,k)$.

It is possible to combine these two cases to solve the utility indifference pricing problem for a payoff which is a linear combination of a claim on P_T and a claim on Y_T. Suppose $C_T = C^1(P_T) + C^2(Y_T)$. Then

$$V(x,k) = -\frac{1}{\gamma} e^{-\gamma x e^{rT} - \gamma k \mathbb{E}^{\mathbb{Q}^{\langle 0 \rangle}}[C^1(P_T)] - \lambda^2 T/2} \left(\mathbb{E}^{\mathbb{Q}^{\langle 0 \rangle}} \left[e^{-(1-\rho^2)\gamma k C_T} \right] \right)^{1/(1-\rho^2)}.$$

Finally we use the definition $V(x-p(k),k) = V(x,0)$ to obtain the following extension of the utility indifference price given in (16):

$$p(k) = k e^{-rT} \mathbb{E}^{\mathbb{Q}^{\langle 0 \rangle}} \left[C^1(P_T) \right] - \frac{e^{-rT}}{\gamma(1-\rho^2)} \ln \mathbb{E}^{\mathbb{Q}^{\langle 0 \rangle}} \left[e^{-k\gamma(1-\rho^2)C(Y_T)} \right]. \tag{36}$$

Here we have written the first expectation as being under the minimal martingale measure, but of course this component of the price is identical under all choices of martingale measure.

It is an open problem to price a general claim of the form $C_T = C(P_T, Y_T)$, or even to extend the study of claims on Y_T alone to the case of non-constant correlation. The fundamental problem is to solve (32) or one of its equivalent variations. Mania et al [57] translate (32) into a backward SDE, and Tehranchi [81] has an interesting approach via the Hölder inequality. It is possible to develop some intuition for the form of the general solution by considering the discrete time case, see Smith and Nau [79] and Musiela and Zariphopoulou [71], but it is not clear whether this can lead to a closed form expression for the price such as in (36).

Thus, even in the non-traded assets model with exponential utility, it is difficult to obtain an explicit expression for the utility-indifference price of a contingent claim. In other contexts the problem is even more difficult, although the general dual expression (21) remains valid. In the stochastic volatility context Sircar and Zariphopoulou [78] give a general characterisation of the price of an option on P_T and some expansions. In the same class of models, but in the slightly artificial case of an option on the volatility process, Tehranchi [81] and Grasselli and Hurd [32] show how to calculate an exact option price.

6 Applications, Extensions and a Literature Review

Utility indifference pricing can be a useful concept of value in any incomplete situation. It has already been developed in the context of transactions costs (see below) and for non-traded assets. Møller [63] uses utility indifference in an insurance context. A number of applications are treated in Part 4 of this book. In particular, Chapter 6 deals with indifference pricing of defaultable claims, which is a very new application of these techniques. Chapters 4 and 7 treat weather and energy applications of indifference pricing, see also Davis [13] who applied the concept of marginal price to weather. We discuss some of these areas in this section and give references where further details can be found.

6.1 Transactions Costs

Option pricing with transactions costs represents the first area in which utility indifference pricing was used. Hodges and Neuberger [44] consider an investor endowed with stock, bond, and an option and derive the investor's optimal investment in stock and bond such that his expected utility is maximized in the presence of transactions costs. Under exponential utility, they solve numerically for the optimal hedge in a binomial setting.

The idea was also used in papers of Davis and Norman [14], Davis et al [15], Davis and Zariphopoulou [16] and Constantinides and Zariphopoulou [8], again in the context of transactions costs. The latter derive a closed form upper bound to the utility indifference price of a call option. Monoyios [64], [65] considers marginal pricing of options under transactions costs and computes the option prices numerically.

6.2 Portfolio Constraints

Munk [66] uses utility indifference pricing in the context of portfolio constraints. His constraints are on the total portfolio amounts, that is, the cash amount in risky stock and the riskless bond. The investor invests in stock and bond, and consumes at rate c in order to maximize the expected remaining lifetime utility from consumption

$$\sup \mathbb{E} \int_t^\infty e^{-\beta(s-t)} U(c_s) ds$$

where utility is power. The investor can also buy or sell units of a European style claim. In the special case where the constraint is a non-negative wealth constraint, Merton [60] solves the no-option optimal control problem. The option problem is solved numerically by Munk. A second case where the investor faces borrowing constraints is studied numerically.

It is also possible to recast the non-traded assets model as a portfolio constraint problem, where the constraint is that the investor cannot trade on a particular asset. There are a number of papers written in this vein. First, Kahl et al [51] consider a manager with non-traded stock who can invest in a correlated market asset and bank account, and consume. The manager maximizes expected power utility from consumption and terminal wealth. Kahl et al [51] use utility indifference to value the restricted stock to the manager and solve via numerical methods. Second, the paper of Teplá [82] considers an investor with exponential utility who holds some quantity of a non-traded asset in her portfolio. The non-traded payoff is assumed normally distributed, whilst the traded assets are lognormal. Finally, Detemple and Sundaresan [18] treat short sales constraints for American claims.

6.3 Unhedgeable Income Streams

Rather than valuing options or non-traded stock, utility indifference can also be used to value streams of income or payments over time where these streams cannot be hedged. In particular, there is a large finance literature on the effect of labour income on asset allocation decisions, see the text of Campbell and Viceira [6]. Labour income is non-traded since individuals cannot trade claims to their future wages.

Henderson [36] studies the effect of stochastic income on the optimal investment decision of an investor who has exponential utility. The income state variable is correlated with a risky traded asset and the investor maximizes expected utility of terminal wealth where wealth is generated by trading in the market and by receiving the stochastic income. Explicit solutions are found and the effects of income on the optimal portfolio are studied.

Munk [67] is a recent paper in a long line looking at an optimal investment and consumption problem with stochastic income and power utility over an infinite horizon. He carries out numerical computations to obtain the value of the non-traded income and optimal portfolio.

6.4 Indifference Pricing for American Options

Davis and Zariphopoulou [16] introduce the notion of American-style utility indifference prices in the context of transactions costs. Treating the problem of options on non-traded assets, Musiela and Zariphopoulou [69] examine the resulting American utility indifference prices with exponential utility. In the non-traded asset framework discussed in Section 5.4, the value function is shown to be a combination of the classical HJB equation and an obstacle problem because discretionary stopping is allowed. The buyer's indifference price solves a quasi-linear variational inequality and in the case where the stock dynamics are lognormal (and the option is only on the non-traded asset), the price can be written as the solution to an optimal stopping problem. The optimal stopping problem involves a non-linear criteria which is the same as that appearing in the European case earlier. Assuming interest rates are zero, the buyer's price for the American option with payoff $C(Y_\tau)$ can be expressed as

$$p_{Am}(k) = \sup_{\tau<T} -\frac{1}{\gamma(1-\rho^2)} \ln \mathbb{E}^{\mathbb{Q}^{(0)}} \left[e^{-\gamma(1-\rho^2)kC(Y_\tau)} \right]$$

Of course, there is no closed form solution to the American option indifference pricing problem, however, Oberman and Zariphopoulou [72] perform some numerical computations for this problem.

Henderson [37] obtains indifference prices in closed form for perpetual American options on a non-traded asset under exponential utility. The work was motivated by real options in corporate finance. Real options are options which arise in association with a firm's decision making. For example, the firm has the option to invest at some time in the future and this is thought of as a call option on the value of the project with strike equal to the investment cost. A key feature is that the the underlying project value is often not a traded asset and thus the market is incomplete.

In the framework of Section 5.4 and taking $r = 0$, with call option payoff at exercise time τ of $(Y_\tau - K)^+$, where K is constant, the value function is given by

$$G(x,y) = \sup_{\tau} \sup_{\theta_u, u\leq\tau} \mathbb{E}\left[U_\tau(X_\tau^{x,\theta} + (Y_\tau - K)^+) \right]$$

with the appropriate time-consistent utility function U_τ given by

$$U_\tau(x) = -\frac{1}{\gamma} e^{-\gamma x} e^{\frac{1}{2}\lambda^2\tau}.$$

A non-linear Bellman equation is solved to obtain the value function, and the utility indifference value for the call is given by

$$\begin{aligned} p_{perp}(1) &= \sup_{0\leq\tau<\infty} -\frac{1}{\gamma(1-\rho^2)} \ln \mathbb{E}^{\mathbb{Q}^{(0)}} \left(e^{-\gamma(1-\rho^2)(Y_\tau-K)^+} \right) \\ &= -\frac{1}{\gamma(1-\rho^2)} \ln \left(1 - (1 - e^{-\gamma(1-\rho^2)(\tilde{Y}-K)}) \left(\frac{y}{\tilde{Y}} \right)^{\beta_1^{(\rho,\gamma)}} \right) \end{aligned}$$

where

$$\beta_1^{(\rho,\gamma)} = 1 - \frac{2(\xi - \lambda\rho)}{\eta}$$

and the exercise trigger level $\tilde{Y}$ solves

$$\tilde{Y} - K = \frac{1}{\gamma(1-\rho^2)} \ln \left[1 + \frac{\gamma \tilde{Y}(1-\rho^2)}{\beta_1^{(\rho,\gamma)}} \right]$$

Provided $\beta_1^{(\rho,\gamma)} > 0$,

$$\tau = \inf \left\{ u : Y_u \geq \tilde{Y} \right\}$$

otherwise, smooth pasting fails and there is no solution. Kadam et al [50] treat the one asset version of this model, where no trading takes place in a second asset.

6.5 Executive Stock Options

Indifference pricing has been used in valuation of executive stock options. These are typically call options on company stock granted to managers as part of their compensation package, see Murphy [68] for an overview of the area and its many important features. The manager cannot typically trade in the company stock for insider trading reasons. Usually these options have a vesting period inside which the manager cannot sell the options, and after which, the options can be sold. However, it is common for models to be constructed assuming the options are either European or American for simplicity.

Detemple and Sundaresan [18] apply their binomial model to value executive stock options, using power utility for the manager's preferences. Henderson [34] applies the continuous time model with exponential utility to value the options, and also considers the optimal form of the compensation for the company. Her model is one of the only ones to consider the effect of both market and firm-specific risks on executive stock options. The paper of Kadam et al [50] approximates a long dated executive stock option with a perpetual American option.

6.6 Hedging with Additional Instruments

In portfolio optimization problems, it is typically assumed that investors can trade only in stocks and bonds. However, it is also reasonable to assume trades in derivative securities may be undertaken, particularly if static rather than dynamic positions are considered. In an incomplete market situation, the problem of choosing the best static hedge in a derivative can be solved via utility indifference techniques. This problem is treated in Chapter 5 where the optimal static position in a derivative with bounded payoff is found under the assumption that the investor has exponential utility.

7 Related approaches

7.1 Super-replication pricing and shortfall hedging

As we mentioned in Section 2.2 on non-HARA utility functions, super-replication pricing and pricing via shortfall hedging can both be interpreted in terms of utility indifference pricing via a degenerate utility function.

The super-replication price can be thought of as an extreme hedging criteria in which the agent is not willing to accept any risk. (The super-replication price is essentially a sell price for a claim, and the buy price is given by the sub-replication price.) In the non-traded assets model the super-replication price of a call option on Y is infinite ([46]) whilst in a stochastic volatility model the super-replication price of a call on X is the cost of buying one unit of the underlying ([27]).

A key alternative characterisation of the super-replication price is given in El Karoui and Quenez [20], see also [21, 22], as

$$\sup_{\zeta_T} \mathbb{E}[\zeta_T C_T]$$

where the supremum is taken over the set of state-price densities. Thus the super-replication price is the sell price under the worst case state-price density.

In shortfall hedging the goal is to minimise the expected losses. This problem has been considered by Föllmer and Leukert [23].

7.2 Minimal Distance Martingale Measures

As discussed in Section 5.1 one approach to option pricing in incomplete markets which has been popular in mathematical finance is to fix a martingale measure $\mathbb{Q}$ and to use this measure for pricing. Examples included the mimical martingale measure [25], the variance optimal measure [77] or the minimal entropy measure [76, 28]. The equivalent notion in finance is to choose the market prices of risk on the non-traded assets or Brownian motions.

The issue therefore is to determine a suitable criterion for choosing a martingale measure or market price of risk, or equivalently of choosing the state-price density. One approach is to choose the state-price density which is smallest in an appropriate sense. Given a convex function $f : \mathbb{R}^+ \mapsto \mathbb{R}$ the idea is to minimise $\mathbb{E}[f(\zeta_T)]$ over choices of state-price-density. As we have seen, when $f(z) = \tilde{U}(\mu z)$ this minimisation problem is a key component of the dual approach.

When interest rates are deterministic and f is homogeneous, this minimisation problem is equivalent to finding the minimal distance martingale measure, the (local) martingale measure $\mathbb{Q}$ which minimises

$$\mathbb{E}[f(Z_T)] \tag{37}$$

where $Z_T = d\mathbb{Q}/d\mathbb{P}$. The problem of finding minimal distance measures has been studied by many authors, but see especially Goll and Rüschendorf [31] who give various characterisations which determine the optimal $\mathbb{Q}$ in terms of f. Each of the examples listed above can be related to a minimal distance measure, for example the minimal entropy measure corresponds to the choice $f(z) = z(\ln z - 1)$.

Once a minimal distance martingale measure $\mathbb{Q}^f$ has been identified it can be used for pricing in the sense that we can define the option price to be

$$\mathbb{E}[\zeta_T^f C_T]$$

where ζ_T^f is the state-price-density associated with the pricing measure $\mathbb{Q}^f$. The resulting prices are linear in the number of units of claim sold, and as we shall see in the next section they are related to the marginal price of the claim for a utility maximising agent. If the plan is to price options in this way it is important to understand how the prices of options depend on the choice of distance metric. The beginnings of such a study are to be found in Henderson [35] and Henderson et al [40].

Note that in the constant parameter non-traded assets example a simple conditioning argument gives that the minimal martingale measure minimises (37) for all convex f. In this simple model all minimal distance measures including the minimal entropy measure are equal to the Föllmer-Schweizer minimal martingale measure.

7.3 Marginal pricing

Recall that in Section 5.5 on the dual approach we showed that, for positive claims C_T and positive quantities k,

$$\frac{p(k)}{k} \leq \mathbb{E}[\zeta_T^{0,*} C_T],$$

where $\zeta_T^{0,*}$ is the state-price density which arises from the optimal solution of the Merton problem. Furthermore, $p(k)$ is convex in k so that the marginal utility-indifference bid price exists, where we define the marginal bid price via

$$D_+ p := \lim_{k \downarrow 0} \frac{p(k)}{k}.$$

Note that we can also define the marginal utility-indifference ask price

$$D_- p := \lim_{k \uparrow 0} \frac{p(k)}{k}.$$

A discussion on the necessary and sufficient conditions for marginal prices to exist in a general semi-martingale model can be found in Hugonnier et al [47].

By definition we have that $V(x - p(k), k) = V(x, 0)$, and assuming sufficient smoothness, see Davis [11] and Karatzas and Kou [53], we can differentiate to find

$$-D_+ p \left. \frac{\partial V}{\partial x} \right|_{k=0+} + \left. \frac{\partial V}{\partial k} \right|_{k=0+} = 0.$$

Let $X_T^{k,*}$ be the optimal target wealth for an agent due to receive k units of the claim C_T. Then $X_T^{0,*}$ is the solution of the associated Merton problem with zero endowment of the claim, and $U'(X_T^{0,*}) = \mu^* \zeta_T^{0,*}$. Further, since $\mathbb{E}[\zeta_T^{0,*} X_T] = x$ for any terminal wealth process which can be financed from initial wealth x, it is reasonable to assume that $\mathbb{E}[\zeta_T^{0,*}(\partial X_T^{k,*}/\partial k)] = 0$.

Now consider $V(x, k) = \mathbb{E}[U(X_T^{k,*} + kC_T)]$. We have

$$\begin{aligned} \left. \frac{\partial V}{\partial k} \right|_{k=0+} &= \left. \mathbb{E}\left[U'\left(X_T^{k,*} + kC_T \right) C_T \right] \right|_{k=0+} + \left. \mathbb{E}\left[U'\left(X_T^{k,*} + kC_T \right) \frac{\partial X_T^{k,*}}{\partial k} \right] \right|_{k=0+} \\ &= \mu^* \mathbb{E}[\zeta_T^{0,*} C_T] \end{aligned}$$

and we have the representation

$$D_+ p = \frac{\mu^* \mathbb{E}[\zeta_T^{0,*} C_T]}{\partial V / \partial x} \tag{38}$$

This representation is due to Davis [11], at least in the case where D_+p and D_-p both exist and are equal and Davis calls it the fair price. However, see Karatzas and Kou [53] and Hobson [42], it is easy to see from the dual representation that $\partial V/\partial x = \mu^{0,*}$ so that (38) simplifies to

$$D_+p = \mathbb{E}[\zeta_T^{0,*} C_T]. \tag{39}$$

A key observation is that the quantity $\zeta_T^{0,*}$ is the state-price density which arises in the dual formulation of the solution to the Merton problem. Hence $\zeta_T^{0,*}$ is related to a minimal distance state-price density, and there is a correspondence between marginal utility-indifference prices and prices derived under a minimal distance martingale measure.

Consider the explicit expressions given for the utility-indifference price for a positive claim $C_T = C(Y_T)$ in the non-traded asset example. We see from (16) and (17) that for both exponential and power utilities

$$D_+p = e^{-rT}\mathbb{E}^{\mathbb{Q}^{\langle 0\rangle}}[C_T] = \mathbb{E}[\zeta_T^{\langle 0\rangle} C_T] \tag{40}$$

where $\mathbb{Q}^{\langle 0\rangle}$ is the minimal martingale measure. Karatzas and Kou [53] show the more general result that in the non-traded assets model the marginal bid price is independent of the choice of utility function and of initial wealth, and given by (40).

Now consider the marginal ask price. For bounded claims the marginal ask price is also given by $e^{-rT}\mathbb{E}^{\mathbb{Q}^{\langle 0\rangle}}[C_T]$ (see [42]), but this is no longer necessarily the case for unbounded claims. For example (see Henderson [33]) for HARA utility functions

$$e^{-rT}\mathbb{E}^{\mathbb{Q}^{\langle 0\rangle}}[C_T] = D_+p \neq D_-p = \infty.$$

Thus there are significant gains to be made from differentiating between the marginal bid and ask prices.

7.4 Convex risk measures

Recall the properties of utility-indifference prices we discussed in Section 3: they are non-linear; replicable claims are priced at their Black-Scholes values; prices are monotonic in the claim; bid prices are concave and sell prices are convex in the claim. Further, for exponential utility they are independent of the initial wealth.

Coherent risk measures (introduced by Artzner et al [2]) and convex risk measures (Föllmer and Schied [24]) were introduced in an attempt to axiomatise measures of risk and to generalise the above properties. In order to be consistent with the rest of this overview we talk about coherent pricing measures for claims rather than measures of risks. Again, as in the discussion of super-replication prices, the correspondence is between the price under a risk measure and the utility-indifference sell price of a claim.

Let $C \in \mathcal{C}$ be a contingent claim. Then $\phi : \mathcal{C} \mapsto \mathbb{R}$ is a coherent pricing measure if it has the properties

Subadditivity	$\phi(C_1 + C_2) \leq \phi(C_1) + \phi(C_2)$
Positive homogeneity	for $\lambda \geq 0, \phi(\lambda C) = \lambda\phi(C)$
Monotonicity	$C_1 \leq C_2 \implies \phi(C_1) \leq \phi(C_2)$
Translation invariance	$\phi(C + m) = \phi(C) + m$

For a convex risk measure the subadditivity and positive homogeneity conditions are replaced with a convexity condition: for $\mu \in (0,1)$,

$$\phi(\mu C_1 + (1-\mu)C_2) \leq \mu\phi(C_1) + (1-\mu)\phi(C_2).$$

The idea is that ϕ represents the amount of compensation which an agent would demand in order to agree to sell the claim C (or the size of the reserves he should hold if he has outstanding obligations amounting to C).

The key result of [2] is that there is a representation of a coherent pricing measure of the form

$$\phi(C) = \sup_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}^{\mathbb{Q}}[C],$$

where $\mathcal{Q}$ is a set of measures. For example the super-replication price is obtained by taking the set $\mathcal{Q}$ to be the set of all martingale measures. Note that a coherent risk measure is linear in the number of claims.

Convex risk measures allow for situations in which the ask price of a claim depends on the number of units sold. Again (see Chapter 3 of this book for a more complete discussion) there is a representation of a convex pricing measure of the form

$$\phi(C) = \sup_{\mathbb{Q}\in\mathcal{P}} \left\{\mathbb{E}^{\mathbb{Q}}[C] - \alpha(\mathbb{Q})\right\}, \tag{41}$$

where now $\mathcal{P}$ is the set of all probability measures, and α is a penalty function. For example, to recover the super-replication price we may take $\alpha(\mathbb{Q}) = 0$ if $\mathbb{Q}$ is a martingale measure, and $\alpha(\mathbb{Q}) = \infty$ otherwise.

A key motivation for the consideration of convex risk measures is that they are a generalisation of the utility-indifference price under exponential utility and zero interest rates. Comparing the expression (21) with $k = -1$ to (41) we find that the penalty function is of the form

$$\alpha(\mathbb{Q}) = \frac{1}{\gamma}\left(\mathbb{E}[Z_T \ln Z_T] - \mathbb{E}[Z_T^E \ln Z_T^E]\right)$$

if $\mathbb{Q}$ is a martingale measure, and $\alpha(\mathbb{Q}) = \infty$ otherwise. Here Z_T^E is the minimal entropy measure. Note that like the exponential utility price, but unlike utility-indifference prices in general, the price under a convex risk measure is independent of the initial wealth of the agent.

8 Conclusion

In a complete frictionless market all risks can be hedged away and a contingent claim has a unique, preference-independent price which is consistent with no-arbitrage. In an incomplete market it is not possible to hedge away all risk and there is a range of prices which are consistent with no-arbitrage. In order to specify a particular price for a claim it is necessary to make assumptions on the risk preferences of agents.

Utility-indifference pricing provides a mechanism for deriving the price of a contingent claim in an incomplete market. It is a dynamic extension of the static concept of certainty equivalence from economics. The utility-indifference pricing methodology acknowledges risk, and builds risk preferences into the pricing algorithm via a concave utility function.

The utility-indifference price is based on a comparison between optimal behaviours under the alternative scenarios of buying the claim now and receiving the payoff later, and not buying the claim. As such it is possible to incorporate a host of features into the model: the risk aversion of the agent, his initial wealth and his prior exposure to non-replicable risk at the moment he is offered the claim. All of these features will affect the claim price.

The resulting price has many desirable features. The price lies between the upper and lower limits consistent with no-arbitrage (and these limits can be attained by an appropriate choice of utility function), and it reduces to the complete market price for replicable claims. Moreover, implicit in the calculation of the utility-indifference price is the construction of an optimal hedge.

The difficulties with the utility-indifference paradigm are that firstly, in order to initiate the process it is necessary to determine the agent's utility function, a notoriously difficult task, and secondly, even when these preferences have been specified it is very difficult to solve the optimisation problems which arise. Analytic formula for contingent claims can be derived in only a few special cases. Such formula are easiest to derive under exponential, logarithmic or quadratic utility, but financial economists prefer to use power utility. (Estimates of the co-efficient of relative risk aversion R vary in the range 4 to 10, see Cochrane [7], whereas exponential, logarithmic and quadratic utility correspond to $R = \infty$, $R = 1$ and $R = -1$ respectively.)

The use of duality theory has led to several recent advances in the characterisation of optimal behaviour of utility maximising agents, but there is a need for more explicit solutions to be derived, and for the parallels with other incomplete-market pricing methodologies which we discussed in this overview to be developed and extended. Together with the translation of the utility-indifference approach to new problems and frameworks, these are the major themes of this book, which contains some of the latest developments in the subject.

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