A short proof of an identity for a Brownian Bridge due to Donati-Martin, Matsumoto and Yor.

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Abstract

Let $(W_t)_{0\leq t\leq 1}$ be a Brownian Bridge. Then, as shown by Donati-Martin, Matsumoto and Yor, the following identity holds:

$$\mathbb{E}\left[\left(\int_0^1 e^{\alpha W_t} dt\right)^{-1}\right] = 1.$$

We give an elegant direct proof of this result, based on an identification between a Brownian bridge and a Brownian excursion due to Vervaat and Biane.

Keywords: Brownian bridge, Brownian excursion, Vervaat's decomposition.

1 The main result

Let W_t be a standard Brownian bridge on $[0, 1]$. Then:

Theorem 1.1 (Donati-Martin, Matsumoto and Yor [4])

$$\mathbb{E}\left[\left(\int_0^1 e^{\alpha W_t} dt\right)^{-1}\right] = 1. \tag{1}$$

In particular, note that the left-hand-side is independent of α. The aim of this note is to give a short proof of this result based on a pathwise relationship between a Brownian bridge and a Brownian excursion. This relationship was discovered by Vervaat [7], although the form we use is due to Biane [1].

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Let $U = \arg\min_{0\le t\le 1}\{W_t\}$ and let $E_t = W_{[t+U]} - W_U$, where $[s] = s\bmod 1$. Then (Vervaat [7, Theorem 1]) E is a (scaled) Brownian excursion on $[0,1]$. Furthermore (Biane [1, Théorème 1]) U is uniformly distributed on $(0,1)$, and U is independent of E.

Conversely, (Biane [1, Théorème 2]) given a scaled Brownian excursion E and an independent uniform random variable U, let $W_t = E_{[t+U]} - E_U$. Then W is a Brownian bridge.

Clearly E constructed from W is positive on $(0,1)$, and conversely W constructed from E returns to 0 at time 1, but in either case the Brownian properties are also inherited. Vervaat proves his result by considering the limit of simple symmetric random walks conditioned to be back at zero at time $2n$. Biane uses results of Bismut [2] which describe the Itô excursion process for Brownian motion to prove his results directly.

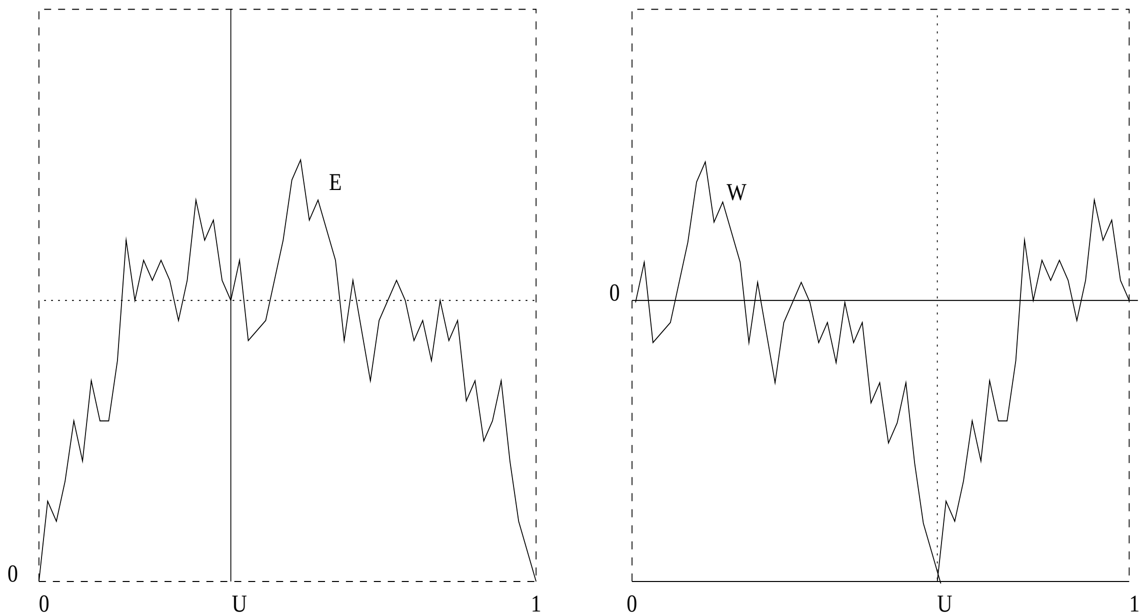

Figure 1: A representation of the mapping from the Brownian excursion to the Brownian bridge by swapping the order of the excursion before and after time U. The mapping is given by $W_t = E_{[t+U]} - E_U$. The mapping is reversible, by re-ordering the pre- and post minimum parts of the Brownian Bridge, so that $E_t = W_{[t+U]} - W_U$.

Now, consider $A_\alpha \equiv \int_0^1 e^{\alpha W_t}dt$. By the above relationship,

$$A_\alpha = \int_0^1 e^{\alpha(E_{[t+U]}-E_U)}dt = e^{-\alpha E_U}\int_0^1 e^{\alpha E_{[t+U]}}dt = e^{-\alpha E_U}\int_0^1 e^{\alpha E_t}dt.$$

Then, conditional on the excursion $(E_t)_{0\leq t\leq 1}$, but averaging over the uniform random variable U,

$$\mathbb{E}[(A_\alpha)^{-1}|(E_t)_{0\leq t\leq 1}] = \frac{\mathbb{E}[e^{\alpha E_U}|(E_t)_{0\leq t\leq 1}]}{\int_0^1 e^{\alpha E_t}dt} = \frac{\int_0^1 e^{\alpha E_t}dt}{\int_0^1 e^{\alpha E_t}dt} = 1.$$

It follows immediately that $\mathbb{E}[(A_\alpha)^{-1}] = 1$.

2 Remarks

There are now several proofs of (1) in the literature. A closely related proof based on the cyclic-exchangeability property of the Brownian bridge, is given by Chaumont et al [3]. Another proof by Donati-Martin et al [5] is based on conditioning the Brownian bridge at an intermediate time. Recently Lyasoff [6] has given a further proof based on consideration of a partial differential equation for the density of the joint law of B_1 and $\int_0^1 e^{\alpha B_t}dt$ for a Brownian motion B. Given the continued interest in exponential functionals of Brownian motion, including the identity (1), it seems appropriate that this short proof should appear in the literature.

References

[1] Ph. Biane; Relations entre pont et excursion du mouvement Brownien réel. *Annales de l'I. Henri Poincaré*, B, **22**, 1–7, 1986.

[2] J.M. Bismut; Last exit decompositions and regularity at the boundary of transition probability. *Zeit. für Wahr.* **69**, 65–99, 1985.

[3] L. Chaumont, D. G. Hobson and M. Yor; Some consequences of the cyclic exchangeability property for exponential functionals of Lévy processes. *Sém. de Prob.* XXXV, 334–347, 2001.

[4] C. Donati-Martin, H. Matsumoto, and M. Yor; The law of geometric Brownian motion and its integral, revisited; application to conditional moments. *Preprint.* 1999.

[5] C. Donati-Martin, H. Matsumoto, and M. Yor; On a striking identity about the exponential functional of the Brownian bridge. *Periodica Math. Hung.* **41**, 103–119, 2000.

[6] A. Lyasoff; On the distribution law of the integral of geometric Brownian motion. *Preprint*, Boston University, 2005. `http://andrew.lyasoff.com/igbm/paper_igbm_AL.pdf`

[7] W. Vervaat; A relation between Brownian bridge and Brownian excursion. *Ann. Probab.* **7**, 143–149, 1979.