AN INTRODUCTION TO (THE ART OF) STOCHASTIC CONTROL[1]

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1 Introduction

We start by giving some examples

1.1 Some examples

1. **Drift control** Suppose that

   $$dX_t = \sigma dB_t + \mu_t dt \text{ with } |\mu_t| \leq c \text{ for all } t:$$

   find an upper/lower bound for

   (i) $E\int_0^\infty e^{-\alpha t} X_t^2 dt$;

   (ii) $P_x(\tau_0 > t)$, where $\tau_0 = \inf\{t \geq 0 : X_t = 0)$.

2. **Tracking/coupling** Suppose that we have a fixed Brownian motion (BM) B on the filtration $(\mathcal{F}_t)$ and two processes, X and Y, satisfying:

   $$dX_t = \sigma_t^1 dB_t$$

   and

   $$dY_t = \sigma_t^2 dW_t.$$

   Choose W from amongst all the BMs on the filtration $(\mathcal{F}_t)$:

   (i) to minimise $E(X_T - Y_T)^2$;

   (ii) to minimise $P(\tau_0(X - Y) > T)$, where $\tau_0(X - Y) = \inf\{t \geq 0 : X_t - Y_t = 0)$.

3. **Investment/consumption** Suppose that we have $n+1$ assets: $S^0, \ldots, S^n$; and

   $$dS_t^0 = \mu_0 S_t^0 dt;$$

   while, for each $1 \leq i \leq n$,

   $$dS_t^i = S_t^i(\sum_j \sigma^{i,j} dB_t^j + \mu_i dt).$$

   Suppose that we may invest in these assets so that our wealth process $X^{\pi,c}$ satisfies (after consumption):

   $$dX_t^{\pi,c} = X^{\pi,c}(\sum_i \pi_t^i \frac{dS_t^i}{S_t^i} - c_t dt),$$

where π and c are constrained to be adapted and to satisfy $\sum_i \pi_t^i = 1$ and $c \geq 0$, for all t.

Given $p < 1$, find

$$\sup_{\pi,c} E_x\Big[\int_0^\infty (c_t X_t^{\pi,c})^p dt\Big].$$

4. **Good Lambda Inequalities** Suppose that X and Y are two increasing processes (e.g. $X_t = B_t^* \equiv \sup_{s\leq t}|B_s|$ and $Y_t \equiv t^{\frac{1}{2}}$): find the best constant, c, appearing in the inequality

$$P(X_\tau \geq x, Y_\tau < y) \leq cP(X_\tau > z) \text{ for all stopping times } \tau.$$

5. **Stopping Time Inequalities**

   (i) Find
   $$\sup_\tau E(B_\tau^* - k\tau)$$
   where τ runs through all stopping times.

   (ii) Find the best constant, c, appearing in the inequality
   $$E(B_\tau^*)^p \leq c||B_\tau^p||_q \text{ for all stopping times } \tau.$$

2 The Bellman principle and HJB equation

2.1 A Typical Problem

Problems in (continuous time) stochastic control usually involve a *controlled Ito process* in $\mathbb{R}^d$:

$$dX_t^u = b(u_t, X_t^u)dt + \sigma(u_t, X_t^u)dB_t,$$

where u_t may be chosen from a control set A so that u (the control) may be any adapted process taking values in A.

A typical problem would then be:

> find the *value function*
>
> $$v(x) = \sup_u E_x J(u, X^u), \tag{2.1}$$
>
> where the *performance functional* J is given by
>
> $$J(u, X^u) = \int_0^\infty e^{-\int_0^t \alpha(u_s, X_s^u)ds} f(u_t, X_t^u)dt.$$

Remark 1. *Notice that we can be charged for the control as, in general, f depends on u. Notice also that minimisation problems just correspond to replacing f by $-f$. The process $\phi_t^u \equiv \int_0^t \alpha(u_s, X_s^u)ds$ is referred to as the* discount process*.*

2.2 Bellman's Principle

Suppose that we follow some control u up to time t and then control optimally (using the control $\hat{u}$) thereafter. Call the resulting control $\bar{u}$; then

$$J(\bar{u}, X^{\bar{u}}) = \int_0^\infty e^{-\phi_s^{\bar{u}}} f(\bar{u}_s, X_s^{\bar{u}})ds.$$

Splitting up the range of integration we get

$$J(\bar{u}, X^{\bar{u}}) = \int_0^t e^{-\phi_s^u} f(u_s, X_s^u)ds + e^{-\phi_t^u} \int_t^\infty e^{-\phi_{s-t}^{\hat{u}}} f(\hat{u}_s, X_s^{\hat{u}})ds.$$

It follows that (since $\hat{u}$ is optimal)

$$V_t^u \equiv E[J(\bar{u}, X^{\bar{u}})|\mathcal{F}_t] = \int_0^t e^{-\phi_s^u} f(u_s, X_s^u)ds + e^{-\phi_t^u} v(X_t^u).$$

Now the longer we follow an arbitrary policy u the longer we fail to follow the optimal policy $\hat{u}$ and so the worse we expect to perform, whilst if $u = \hat{u}$ then we behave optimally throughout. We get from this:

> **Bellman's Principle** Under every u, V^u is a supermartingale while under the optimal control $\hat{u}$, $V^{\hat{u}}$ is a martingale.

2.3 A Converse

Suppose we are given a function $\tilde{v}$ and for each policy u we define the process $\tilde{V}^u$ by

$$\tilde{V}_t^u \equiv \int_0^t e^{-\phi_s^u} f(u_s, X_s^u)ds + e^{-\phi_t^u} \tilde{v}(X_t^u);$$

now consider the following four conditions (the first three of which are assumed to hold for all controls u and all initial conditions x):

(1) $\tilde{V}^u$ is a supermartingale;

(2) $E_x[e^{-\phi_t^u} \tilde{v}(X_t^u)] \to 0$ as $t \to \infty$;

(3) $E_x[\int_0^t e^{-\phi_s^u} f(u_s, X_s^u)ds] \to E_x[J(u, X^u)]$ as $t \to \infty$; and

(4) for all x there exists a $\hat{u}$ such that $\tilde{V}^{\hat{u}}$ is a martingale.

Theorem 2. *Suppose that conditions (1) to (3) hold then*

$$\tilde{v} \geq v.$$

If, in addition, (4) holds, then

$$\tilde{v} = v.$$

Proof: from (1) it follows that

$$\tilde{v}(x) = \tilde{V}_0^u \geq E_x \tilde{V}_t^u = E_x[\int_0^t e^{-\phi_s^u} f(u_s, X_s^u)ds] + E_x[e^{-\phi_t^u} \tilde{v}(X_t^u)]. \quad (2.2)$$

Now, taking limits as $t \to \infty$ in equation (2.2), (2) and (3) imply that

$$\tilde{v}(x) \geq E_x[J(u, X^u)],$$

and, since this holds for arbitrary u and x we see that

$$\tilde{v} \geq v.$$

If, in addition (4) holds, then we have equality in (2.2) when $u \equiv \tilde{u}$, so

$$\tilde{v}(x) = E_x[J(\hat{u}, X^{\hat{u}})]$$

and thus

$$\tilde{v}(x) \leq v(x) = \sup_u E_x[J(u, X^u)].$$

□

2.4 Extensions

- We may make the problem time-dependent by enlarging the statespace (so $(X_t^u)_{t\geq 0}$ becomes $((X_t^u, t))_{t\geq 0}$).
- We can now consider finite time-horizon problems by setting $f(u_t, X_t^u, t) = 0$ for $t \geq T$.
- We may 'stop' the problem on first exit from a domain D, setting v to a prescribed function, g, on ∂D.
- We may also incorporate optional stopping (v is then set to a prescribed function, g, at a stopping time of our choice).

2.5 The Hamilton-Jacobi-Bellman (HJB) equation

Suppose that the solution to problem (2.1) is v, and that v is C^2. Ito's formula tells us that (omitting the argument (u_t, X_t^u) wherever it should appear, and denoting $\sigma\sigma^T$ by a):

$$\begin{aligned} dV_t^u = e^{-\phi_t^u}\big([\frac{1}{2}\sum_{i,j} a_{i,j}\frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_i b_i \frac{\partial v}{\partial x_i} - \alpha v + f]dt \\ + \sum_{i,j} \sigma_{i,j}\frac{\partial v}{\partial x_i} dB_t^j\big), \end{aligned}$$

or, defining the differential operator L^u by

$$L^u : g \mapsto L^u g,$$

where

$$L^u g(x) = \frac{1}{2}\sum_{i,j} a_{i,j}(u,x)\frac{\partial^2 g}{\partial x_i \partial x_j}(x) \\ + \sum_i b_i \frac{\partial g}{\partial x_i}(x) \quad -\alpha(u,x)g(x),$$

$$dV_t^u = e^{-\phi_t^u}\big([L^u v + f] + \sum_{i,j}\sigma_{i,j}\frac{\partial^v}{\partial x_i}dB_t^j\big).$$

Now Bellman's principle tells us that V^u is a supermartingale and should be a martingale under the optimal control: it follows that for each u we want

$$L^u v + f \leq 0$$

and for $u = \hat{u}$ we want

$$L^{\hat{u}} v + f = 0,$$

or, more succinctly

$$\sup_{u \in A}[L^u v + f] = 0. \tag{2.3}$$

Equation (2.3) is known as the Hamilton-Jacobi-Bellman (HJB) equation.

2.6 A worked example

Problem Find

$$v(x) \equiv \inf_{u \in \mathcal{A}} E_x \int_0^\infty e^{-\alpha t} f(X_t^u)dt, \tag{2.4}$$

where α is a fixed positive constant, the process X^u satisfies

$$dX_t^u = dB_t + u_t dt,$$

$$\mathcal{A} = \{\text{adapted } u : |u_t| \leq 1 \text{ for all } t\}$$

and

$$f : x \mapsto x^2.$$

Solution The HJB equation is

$$\inf_{|u|\leq 1} \frac{1}{2}v''(x) + uv'(x) - \alpha v + x^2 = 0.$$

Guess, by symmetry, that v is symmetric and increasing for $x \geq 0$. It follows that the infimum is attained (for $x > 0$) at $u = -1$. It follows that we want to solve

$$\frac{1}{2}w''(x) - w'(x) - \alpha w + x^2 = 0. \tag{2.5}$$

The general solution is

$$w = Ae^{(\gamma+\frac{1}{2})x} + Be^{(-\gamma+\frac{1}{2})x} + \frac{1}{\alpha}x^2 - \frac{2}{\alpha^2}x + \frac{1}{\alpha^2} - \frac{2}{\alpha^3},$$

where

$$\gamma = \frac{\sqrt{1+2\alpha}}{2}.$$

We want a C^2 symmetric solution, so we want $w'(0) = 0$. We can also conclude that $A = 0$ (by comparison with the Brownian motion case where $u \equiv 0$ and by positivity). This results in the guess:

$$v = w \equiv -\frac{2}{\alpha^2(\gamma - \frac{1}{2})}e^{(-\gamma+\frac{1}{2})|x|} + \frac{1}{\alpha}x^2 - \frac{2}{\alpha^2}|x| + \frac{1}{\alpha^2} - \frac{2}{\alpha^3}.$$

It's easy to check that w is increasing on $\mathbb{R}^+$, since we've set $w'(0)$ to zero and w" is clearly positive. It follows that w satisfies the HJB equation. It is now relatively easy to check that w satisfies the conditions of Theorem 2, since properties (2) and (3) follow from the fact that $|X_t^u| \leq |B_t| + t$, whilst (1) and (4) follow from the fact that w satisfies the HJB equation. $\square$

Exercise 1: solve the problem when $f : x \mapsto x^4$.

3 Optimal Portfolio Allocation/Consumption

Recall the setup of example 3.

3.1 Case 1

Suppose that there is no risk-free asset (S^0) and $n = d = 1$ (so there is no allocation problem — $\pi \equiv 1$).

We get

$$L^c g(x) = \frac{1}{2}\sigma^2 x^2 \frac{d^2 g}{dx^2} + (\mu - c)x\frac{dg}{dx}, \tag{3.1}$$

with $\alpha = 0$ and $f(c,x) = (cx)^p$.

A quick check shows that everything scales in x, so we must have $v(x) = kx^p$ for a suitable k.

The HJB equation is

$$\sup_c [-\frac{1}{2}\sigma^2 p(1-p)k + (\mu - c)pk + c^p]x^p = 0, \tag{3.2}$$

or, setting $\delta = \frac{1}{2}\sigma^2 p(1-p) - \mu p$ and since $x > 0$

$$\sup_c [c^p - cpk - \delta k] = 0. \tag{3.3}$$

The supremum in (3.3) is attained at $\hat{c} = k^{-\frac{1}{1-p}}$ and so, substituting back in (3.3), we get

$$k = (\frac{1-p}{\delta})^{\frac{1-p}{p}},$$

provided

$$\mu < \frac{1}{2}\sigma^2(1-p). \tag{3.4}$$

Exercise 2: by considering consumption policies of the form $c_t \equiv c_0$ for suitable values of c_0, show that if (3.4) fails then $v \equiv \infty$.

Now, assuming that (3.4) holds, we can check conditions (1)-(4).

(3) holds by positivity and monotone convergence— notice this is always OK if $f \geq 0$. (2) follows from (3.4) and the fact that $X_t^c \leq X_t^0$, (1) and (4) follow from the HJB equation.

3.2 Case 2

Now assume that $n \geq d$ and there is a risk-free asset present.

We get

$$L^{\pi,c}g(x) = \frac{1}{2}\tilde{\pi}^T a\tilde{\pi}x^2\frac{d^2g}{dx^2} + (\tilde{\mu}^T\tilde{\pi} + \mu_0\pi_0 - c)x\frac{dg}{dx}, \tag{3.5}$$

where $a = \sigma\sigma^T$, $\mu^T = (\mu_0, \tilde{\mu})$ and $\pi = (\pi_0, \tilde{\pi})$.

The HJB equation is

$$\sup_{\pi,c}[L^{\pi,c}v + (cx)^p] = 0.$$

Scaling again forces a solution of the form $v(x) = kx^p$, and so we get

$$\sup_{\pi,c}[-\frac{1}{2}\tilde{\pi}^T a\tilde{\pi}p(1-p)k + (\tilde{\mu}^T\tilde{\pi} + \mu_0\pi_0 - c)pk + c^p] = 0$$

Notice that the maximization in c is independent of π. It follows that it's essentially the same problem as Case 1, unless a is not of full rank. If not then $\sup_\pi = \infty$ unless

$$Ker(a) \equiv Ker(\sigma) \perp (\tilde{\mu} - \mu_0 1). \tag{3.6}$$

In financial terms, there is an arbitrage unless (3.6) holds. Assuming that (3.6) does hold, we may assume wlog that a is of full rank). Now we need

$$\mu_0 + \frac{1}{2}\frac{(\tilde{\mu} - \mu_0 1)^T a^{-1}(\tilde{\mu} - \mu_0 1)}{1-p} < 0.$$

3.3 Case 3

We revert to a single asset but now we wish to find

$$\sup_c E_x[\int_0^T (c_sX_s)^p ds + \lambda(X_T^c)^p]$$

Note that this is (a shift of) the Lagrangian for the constrained problem

$$\sup_c E_x[\int_0^T (c_sX_s)^p ds] \text{ subject to } E_x[(X_T^c)^p] = b.$$

Once more by scaling, we can see that the solution must be of the form

$$v(x,T) = k_\lambda(T)x^p.$$

The HJB equation is

$$\sup_c \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 v}{\partial x^2} + (\mu - c)x\frac{\partial v}{\partial x} - \frac{\partial v}{\partial t} + (cx)^p = 0,$$

(WHY?), with the boundary condition

$$v(x,0) = \lambda x^p.$$

Substituting the 'solution' $v(x,t) = k_\lambda(t)x^p$, we get

$$\sup_c[(1-p)rk_\lambda - pck_\lambda - k'_\lambda] = 0,$$

with $(1-p)r = \mu p - \frac{1}{2}\sigma^2 p(1-p)$.

As before $\hat{c} = k^{-\frac{1}{1-p}}$. Amazingly, if we set $\xi = k_\lambda^{\frac{1}{1-p}}$ we get linear (affine) differential equation for ξ:

$$\xi' = 1 + r\xi.$$

Finally, we can solve and substitute to obtain v and $E_x[(X_T^c)^p]$. Finally, by varying λ we can solve the constrained problem (for a range of b.
QUESTION: If we use the corresponding optimal control policy, what will the value of X_T be?

4 Tracking Problems

Exercise 3: Let A be the set of $d \times d$ symmetric real matrices a, satisfying the inequalities:

$$\mu|x|^2 \le x^T a x \le \nu|x|^2,$$

where $0 < \mu < \nu$. This is the collection of real symmetric matrices with each eigenvalue in the interval $[\mu,\nu]$. Let $\mathcal{A} = \{$ adapted $\sigma : \quad \sigma_t\sigma_t^T \in A$ for each $t\}$.

Suppose that $0 < \epsilon < R$, the domain D is given by $D = \{x \in \mathbb{R}^d : \epsilon < |x| < R\}$ and τ_D is the first exit time of D.

Find

$$\sup_{\sigma\in\mathcal{A}} E_x g(|X^\sigma_{\tau_D}|),$$

where

$$dX_t^\sigma = \sigma_t dB_t$$

and

$$g(\epsilon) = 1 - g(R) = 1.$$

4.1 Tracking Problem 1

Recall Problem 2(i) from the introduction:

suppose that we have a fixed Brownian motion (BM) B on the filtration $(\mathcal{F}_t)$ and two processes, X and Y, satisfying:

$$dX_t = \sigma_1(X_t)dB_t,\ X_0 = x$$

and

$$dY_t^W = \sigma_2(Y_t)dW_t,\ Y_0 = y.$$

Choose W from $\mathcal{V}$, the set of all the BMs on the filtration $(\mathcal{F}_t)$, to minimise $E(X_T - Y_T)^2$.

Before we continue, let us add the assumption that

> σ_2 is Lipschitz, and σ_1 is Hölder continuous with Hölder parameter $\alpha > 0$.

Under this assumption,

(i) For any W, Y is a strong solution and is adapted to the filtration of W;

(ii) X is unique in law but need not be adapted to the filtration of B.

Remark 3. *for any $W \in \mathcal{V}$, there is a predictable H and a $\tilde{B} \in V$ such that*

- $W_\cdot = \int_0^\cdot cos H_s dB_s + \int_0^\cdot sin H_s d\tilde{B}_s$
- *X is unique in law but need not be a strong solution and hence may not be adapted to the filtration of B.*

Let's generalise the problem (without making it any harder): fix $T > 0$ and suppose that Φ is C^2, convex and of polynomial growth:

$$\text{find } \psi(x, y, T) \equiv \inf_{W\in\mathcal{V}} E_{x,y}\Phi(X_T - Y_T^W). \tag{4.1}$$

Theorem 4. *The infimum in (4.1) is attained by setting $W = B$.*

Corollary 5. *If σ_1 is such that no strong solution for X exists then*

$$\eta \stackrel{def}{=} \inf_{y\in\mathbb{R},\ W\in\mathcal{V},\ \sigma_2 Lipschitz} E_{x,y}[X_T - Y^{W,\sigma_2}]^2 > 0,$$

and so we cannot approximate X by a sequence of adapted strong solutions in L^2 with Lipschitz coefficients.

Sketch proof: suppose we could, i.e we have a sequence (y_n, W^n, σ_n) s.t.

$$E[X_T - Y_T^n]^2 \to 0,$$

then by Theorem 4, Z^n given by $Z_t^n = y_n + \int_0^t \sigma_n(Z_s^n) dB_s$ will do at least as well and so $E[Z_T^n - X_T]^2 \to 0$, which implies, by Doob's maximal inequality, that

$$E \sup_{0 \le t \le T} (Z_t^n - X_t)^2 \to 0.$$

But each Z^n is adapted so X is adapted, which is a contradiction. □

Sketch proof of Theorem 4: the candidate optimal policy is $\hat{W} \equiv B$, so define

$$w(x, y, t) = \mathbb{E}_{x,y} \Phi(X_t - Y_t^B). \tag{4.2}$$

Now look at the HJB equation (with

$$W. = \int_0^{\cdot} C_s dB_s + \int_0^{\cdot} S_s d\tilde{B}_s$$

with C and S adapted and $C^2 + S^2 = 1$).

Assuming that v is $C^{2,1}$:

$$dv(X_t, Y_t^W, T - t) = (\frac{1}{2}\sigma_1^2 v_{xx} + C_t \sigma_1 \sigma_2 v_{xy} + \frac{1}{2}\sigma_2^2 v_{yy} - v_t)dt + dM_t^C, \tag{4.3}$$

where M^C is a martingale; then it follows that the HJB equation is:

$$\inf_{c \in [-1,1]} (\frac{1}{2}\sigma_1^2 v_{xx} + c\sigma_1\sigma_2 v_{xy} + \frac{1}{2}\sigma_2^2 v_{yy} - v_t) = 0.$$

If we apply this to w we see that we want optimal $c \equiv 1$ (corresponding to $W = B$), so it's (nearly) necessary and sufficient that $w_{xy} \le 0$. Again, assuming that w is $C^{2,1}$, we need only show that

$$I_R \equiv \int \int_R w_{xy} dx dy \le 0$$

for any rectangle $R = [x, x'] \times [y, y']$.

Now

$$\begin{aligned} I_R &= w(x', y', t) - w(x, y', t) - w(x', y, t) + w(x, y, t) \\ &= E[\Phi(X_t(x') - Y_t^B(y')) - \Phi(X_t(x') - Y_t^B(y)) \\ &\quad - \Phi(X_t(x) - Y_t^B(y')) + \Phi(X_t(x) - Y_t^B(y))] \\ &= -E[\int\int_{R'} \Phi''(u - v)dudv], \end{aligned}$$

where $R' = [X_t(x), X_t(x')] \times [Y_t^B(y), Y_t^B(y')]$. So, since $\Phi'' \geq 0$ we are done provided that $Y_t^B(y) \leq Y_t^B(y')$ and $X_t(x) \leq X_t(x')$ whenever $x < x'$ and $y < y'$. This follows for Y from the skip-free property for one-dimensional strong solutions. For X we need to take the two solutions and paste them when they collide to get the required property. □

One small problem: *IS w a $C^{2,1}$ function*?

Trick: fix a finite square domain, freeze (X, Y) on exit from this domain, restrict controls to the interval

$$[-1 + \epsilon, 1 - \epsilon]$$

then (PDEs result) the corresponding w is $C^{2,1}$ so is the value function for the revised problem. Now let $\epsilon \to 0$ and the domain $\uparrow \mathbb{R}^2$.

4.2 Tracking Problem 2: coupling

Now we seek (with the same X and Y as above) to find

$$v(x, y, t) = \inf_{W \in V} P_{x,y}(\tau_0(X - Y^W) > t). \tag{4.4}$$

Note that we have the boundary condition $v(x, y, 0) = 1_{x \neq y}$ and we stop the problem on the diagonal $x = y$ so that

$$v(x, x, t) = 0.$$

As in the previous problem, the HJB equation is

$$\inf_{c \in [-1,1]} (\frac{1}{2}\sigma_1^2 v_{xx} + c\sigma_1\sigma_2 v_{xy} + \frac{1}{2}\sigma_2^2 v_{yy} - v_t) = 0.$$

Now coupling ideas suggest that mirror coupling might be best: i.e. to choose $W = -B$ (at least assuming that σ_1 and σ_2 have the same sign). Choosing $W = -B$ corresponds to $c \equiv -1$ in the HJB equation. So, set

$$\psi(x, y, t) = P_{x,y}(\tau_0(X - Y^{-B}) > t).$$

Then, we want to show that the inf in the HJB equation is attained at -1 which means we want $\psi_{xy} \geq 0$.

Example Suppose that $\sigma_1 \equiv 1$, $\sigma_2 \equiv \sigma > 0$.

Now

$$X_t - Y_t^{-B} = (x - y) + (1 + \sigma)B_t.$$

So τ_0 is the first hitting time by $-B$ of the point $\frac{x-y}{1+\sigma}$. It follows that (assuming w.l.o.g. that $x > y$)

$$\psi(x, y, t) = P(|B_t| < \frac{x-y}{1+\sigma}) = \frac{2}{\sqrt{2\pi}} \int_0^{\frac{x-y}{(1+\sigma)\sqrt{t}}} e^{-\frac{u^2}{2}} du.$$

Now

$$\psi_{xy} = k(x - y) \exp\left(-\frac{(x-y)^2}{2(1+\sigma)^2 t}\right),$$

for some constant $k > 0$ (and ψ is $C^{2,1}$). This establishes (1) and (4) and the optimality of mirror coupling.

4.3 Tracking Problem 3: 'staying small'

We have the drift control setup of initial problem 1:

$$dX^u_t = \sigma dB_t + u_t dt$$

with $u \in [-a, a]$. We seek to find

$$v(x,t) = \inf_u E_x f(X^u_t),$$

where f is C^2, symmetric, increasing and bounded.

We still guess that the optimal control is that v is an increasing function of $|x|$ and hence to set $\hat{u}_t = -a.sign(X_t)$.

The HJB equation is

$$\inf_{-a \le u \le a} \frac{1}{2}\sigma^2 v_{xx} + u v_x - v_t = 0,$$

with boundary condition $v(x,0) = f(x)$. Following our guess, let

$$w(x,t) = E_x f(X^{\hat{u}}_t)$$

Standard arguments show that w is $C^{2,1}$ except on $\{0\} \times \mathbb{R}_+$, where $w_x = 0$. It follows that

$$\frac{1}{2}\sigma^2 w_{xx} - a w_x - w_t = 0, \tag{4.5}$$

on $x > 0$ and so, from Tanaka's generalisation of Ito's formula, that w satisfies the HJB equation (and is optimal) provided that w is increasing on $\mathbb{R}_+$.

How do we prove this?

Well, look at w_x. It's (fairly) clear, by differentiating (4.5), that w_x satisfies

$$\frac{1}{2}\sigma^2 (w_x)_{xx} - a(w_x)x - (w_x)_t = 0,$$

on $\mathbb{R}_+ \times \mathbb{R}_+$, with the boundary conditions $w_x = 0$ on $\{0\} \times \mathbb{R}_+$, $w_x = f'(x)$ on $\mathbb{R}_+ \times \{0\}$ and $w_x \to 0$ as $x \uparrow \infty$ (this follows from the fact that f is bounded and continuous).

It follows from the strong minimum principle for parabolic operators that w_x has no negative minima on $D \equiv (\mathbb{R}_+ \times \mathbb{R}_+)^o$. But w_x is non-negative on the 'boundary' of D so it follows that w_x is non-negative on $\mathbb{R}_+ \times \mathbb{R}_+$.

Now we can approximate arbitrary increasing f by C^2 bounded, increasing f and so it follows that $\hat{u}$ achieves the stochastic minimum of the X^u_t.

5 Optimal Stopping

5.1 The HJB equation

The Problem: we seek optional τ (i.e. τ is a stopping time) to achieve

$$\sup_{\tau} E_x[\int_0^{\tau} f(X_t)dt + g(X_\tau)],$$

where

$$dX_t = \sigma(X_t)dB_t + b(X_t)dt.$$

To see what we should do, Krylov's trick is to allow randomised stopping at a rate r_t with $0 \leq r_t \leq n$. Thus

$$v^{(n)}(x) = \sup_{\text{predictable } r} E_x[\int_0^{\infty} e^{-\phi_t^r}\big(f(X_t) + r_t g(X_t)\big)dt],$$

where $\phi_t = \int_0^t r_s ds$.

Then the HJB equation is

$$\sup_{r\in[0,n]} [Lv^{(n)} + f - rv^{(n)}] + rg = 0,$$

where $L = \frac{1}{2}\sum\limits_{i,j} a_{i,j}\frac{\partial^2}{\partial x_i \partial x_j} + \sum\limits_i b_i \frac{\partial v}{\partial x_i}$.

Clearly the supremum in the HJB equation is attained at $r = 0$ if $v^{(n)} > g$ and at n if $v^{(n)} \leq g$, so the HJB equation is

$$Lv^{(n)} + f + n(g - v^{(n)})^+ = 0.$$

Thus we 'want to stop 'when $v^{(n)} \leq g$ and continue otherwise, when $Lv^{(n)} + f = 0$.

Formally, if we let $n \to \infty$ then we see that we get

$$Lv + f \leq 0$$

$$v \geq g$$

and

$$Lv^{(n)} + f = 0$$

whenever $v > g$.

5.2 The Snell envelope

Given a process X satisfying weak integrability conditions, define

$$V_t = \operatorname*{ess\ sup}_{\text{optional } \tau \geq t} E[X_\tau | \mathcal{F}_t].$$

Notice that by setting $\tau = t$ we see that $V_t \geq X_t$ a.s.

V is called the Snell envelope for X. Under weak conditions V is the minimal supermartingale dominating X.

Theorem 6. *Suppose that*

$$V_t = \operatorname*{\mathit{ess\ sup}}_{\text{\textit{bounded optional}}\ \tau \geq t} E[X_\tau | \mathcal{F}_t]$$

(for example, if X is bounded below) and suppose that $\tilde{V}$ is a process satisfying the following three conditions:

(5) $\tilde{V} \geq X$*;*

(6) $\tilde{V}$ *is a supermartingale;*

(7) for every t, there is an optional $\tau \geq t$ such that

$$E[X_\tau | \mathcal{F}_t] = \tilde{V}_t,$$

then

$$\tilde{V} = V.$$

Proof: take a bounded optional $\tau \geq t$, then (5), (6) and the optional sampling theorem tell us that

$$\tilde{V}_t \geq E[\tilde{V}_\tau | \mathcal{F}_t] \geq E[X_\tau | \mathcal{F}_t] \text{ a.s.},$$

and since τ is arbitrary we see that

$$\tilde{V}_t \geq V_t.$$

Conversely, (7) tells us that

$$\tilde{V}_t \leq V_t \text{ a.s.}$$

□

5.3 Good lambda inequalities

Recall that the problem is to find the best constant, c, appearing in the inequality

$$P(X_\tau \geq x, Y_\tau < y) \leq cP(X_\tau > z) \text{ for all stopping times } \tau,$$

where X and Y are increasing processes. We shall assume (for convenience) that X is continuous and $x > z$.

Let us set (for each $t \geq 0$):

$$\begin{aligned} S_t = & \quad \inf\{s \geq 0 : \; X_s > t\} \\ T_t = & \quad \inf\{s \geq 0 : \; X_s \geq t\} \\ W^\lambda \equiv W_t = & \quad 1_{(X_t \geq x,\, Y_t < y)} - \lambda 1_{(X_t > z)} \\ = & \quad 1_{(t \geq T_x,\, Y_t < y)} - \lambda 1_{(t > S_z)}, \end{aligned}$$

and suppose that we wish to optimally stop W_t.

This is as simple a non-trivial problem as is possible!

Notice that

$$W_t = \begin{cases} 0 & \text{on } [0, S_z] \\ -\lambda & \text{on }]S_z, T_x[\\ 1_{(Y_t < y)} - \lambda & \text{on } [T_x, \infty[. \end{cases}$$

Now Y is increasing, so W is decreasing on $[T_x, \infty[$ and so we must have $V = W$ on $[T_x, \infty[$. Conversely, W is increasing on $]S_z, T_x]$, so we must have '$V_t = E[V_{T_x} | \mathcal{F}_t]$ on $]S_z, T_x]$', i.e.

$$V_t = E[W_{T_x} | \mathcal{F}_t] = P(Y_{T_x} < y | \mathcal{F}_t) - \lambda \text{ on }]S_z, T_x].$$

Finally, W is constant on $[0, S_z]$ so we must have

$$V_t = E[V_{S_z} | \mathcal{F}_t] \text{ on } [0, S_z].$$

From this it's clear that the only time we have a choice is at time S_z when we must choose to stop immediately or continue until time T_x. In other words, the optimal stopping time τ must be of the form

$$\tau = \tau_A \equiv S_z 1_{A^c} + T_x 1_A$$

for some event $A \in \mathcal{F}_{S_z}$.

Now

$$E[W_{\tau_A}|\mathcal{F}_{S_z}] = (P(Y_{T_x} < y|\mathcal{F}_{S_z}) - \lambda)1_A,$$

so, denoting by $\mathcal{E}_\lambda$ the $\mathcal{F}_{S_z}$-measurable random variable

$$P(Y_{T_x} < y|\mathcal{F}_{S_z}) - \lambda,$$

it's clear that an optimal choice of A is $(\mathcal{E}_\lambda > 0)$.

Thus, our candidate for V is $\tilde{V}$ given by :

$$\tilde{V} = \begin{cases} E[\mathcal{E}_\lambda^+|\mathcal{F}_t] & \text{on } [0, S_z] \ , \\ P(Y_{T_x} < y|\mathcal{F}_t) - \lambda & \text{on }]S_z, T_x[\ , \\ 1_{(Y_t<y)} - \lambda & \text{on } [T_x, \infty[\ . \end{cases}$$

Check: (5) $\tilde{V} \geq W$ is obvious; (6) $\tilde{V}$ is a supermartingale and is bounded; (7) we have explicitly exhibited τ_t such that $V_t = E[W_{\tau_t}|\mathcal{F}_t]$, therefore $\tilde{V} = V$.

Application:

$$\inf\{c: \ P(X_T \geq x, Y_T < y) \leq cP(X_T > z) \text{ for all optional } T\}$$
$$= \inf\{\lambda: \ V_0^\lambda \leq 0\}.$$

But $V_0^\lambda = E[\mathcal{E}_\lambda^+]$, so

$$V_0^\lambda \leq 0 \Leftrightarrow P(\mathcal{E}_\lambda > 0) = 0$$
$$\Leftrightarrow$$
$$\lambda \geq \text{ess } \sup P(Y_{T_x} < y|\mathcal{F}_{S_z}).$$

Thus the best constant is

$$c(x, y, z) = \text{ess } \sup P(Y_{T_x} < y|\mathcal{F}_{S_z}).$$

Theorem 7. *Suppose that*

$$c^*(\beta, \delta) \stackrel{def}{=} \sup_{\lambda>0} c(\beta\lambda, \delta\lambda, \lambda)$$

satisfies

$$k_p \stackrel{def}{=} \inf_{\beta>1,\delta>0} \beta^p c^*(\beta, \delta) < 1,$$

then there exists a C_p *(*$= \inf_{\beta>1,\delta} \frac{\beta^p}{\delta^p(1-\beta^p c^*(\beta,\delta))}$*) such that*

$$E[X_T^p] \leq C_p E[Y_T^p] \textit{ for all optional } T.$$

Corollary 8. *For every $p > 0$ there is a C_p (which is $O(p^{\frac{1}{2}})$) such that for all optional T:*

$$E[(B^*_T)^p] \leq C_p E[T^{\frac{p}{2}}] \leq C_p^2 E[(B^*_T)^p].$$

Proof: first take $X = B^*$ and $Y_t = t^{\frac{1}{2}}$, then

$$\begin{aligned} c(x,y,z) &= \text{ess sup}\, P(Y_{T_x} < y | \mathcal{F}_{S_z}) = \text{ess sup}\, P(T_x < y^2 | \mathcal{F}_{S_z}) \\ &\leq \text{ess sup}\, P(B^*_{y^2} \geq x | \mathcal{F}_{S_z}) \leq P(B^*_{y^2} \geq x - z), \end{aligned}$$

the last inequality following from the fact that B^* is a continuous sub-additive functional of B. So

$$c(x,y,z) \leq 4P(B_1 \geq \frac{x-z}{y}) = \frac{4}{\sqrt{2\pi}} \int_{\frac{x-z}{y}}^{\infty} e^{-u^2/2} du$$

$$\leq k \frac{y}{x-z} \exp(-\frac{(x-z)^2}{2y^2}),$$

for a suitable choice of k. Thus

$$c^*(\beta, \delta) \leq k \frac{\delta}{\beta - 1} \exp(-\frac{(\beta-1)^2}{2\delta^2}),$$

and it's easy to check that

$$\inf_{\beta > 1, \delta} \frac{\beta^p}{\delta^p(1 - \beta^p c^*(\beta, \delta))} = O(p^{\frac{1}{2}}).$$

A similar argument works if we reverse the roles of B^* and $T^{\frac{1}{2}}$. $\square$

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