Optimal Stopping and Technical Analysis

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Technical Analysis: based on idea that history of price process (alone) enables (good) predictions about its future movement.

Example is identification of *resistance and support levels*.

- *Resistance level* is a critical level which lq'price struggles to exceed" (e.g. because of large number of sellers at that price level). Price can 'break through' eventually. Breakthrough identified by hitting higher level or time spent above 'resistance level'.
- *Support level* very similar but 'breakthrough' means price deteriorating.

Our model for the stock price process:

- Two regimes: positive and negative;
- Two levels L and H with $L < H$ (think of support and resistance levels both being $\frac{L+H}{2}$ and breakthrough occurs if reach L (downwards) or H(upwards));
- current regime denoted by a flag process F_t, values in $\{-,+\}$.
- when process in negative regime ($F_t = -$), dynamics correspond to the infintesimal generator:

$$L^- : g \mapsto \frac{1}{2}\sigma_-^2 g'' + \mu_- g',$$

similar for positive regime. So

$$S_t = S_0 + \int_0^t \sigma_{F_s}(S_s)dB_s + \int_0^t \mu_{F_s}(S_s)ds.$$

Assume that σ_+, σ_-, μ_+ and μ_- are Lipschitz (each f) and σ_+, σ_- are uniformly elliptic (at least on any compact not containing 0). We assume *price process is absorbed at 0*.

Finally, we assume that risk-free rate is r and that

$$\mu_- \leq r \leq \mu_+,$$

so that discounted stock price is "a submg in positive regime" and "a supermg in negative regime" and inequality is somewhere strict .

F_t-dynamics are simple:

- if $F_{t-} = +$ and $S_{t-} = L$ then $F_t = -$;
- conversely, if $F_{t-} = -$ and $S_{t-} = H$ then $F_t = -$.

Easy to see under these assumptions that

- $X_t \stackrel{def}{=} (S_t, F_t)$ is Feller with statespace $E = [0, H) \times \{-\} \cup (L, \infty) \times \{+\}$.
  - corresponding topology: so e.g. neighbourhoods of $(H, +)$ are of the form $(x, H) \times \{-\} \cup [H, y) \times \{+\})$
- X has infinitesimal generator $\mathcal{G}$ given by

$$\mathcal{G} : g \mapsto L^f g$$

for functions g which are C^2 in x and cts on E (so $g(L-, +) = g(L, -)$ and $g(H-, -) = g(H, +)$).

- Assume trader holds one unit of stock and seeks to optimize expected discounted proceeds at time of sale τ: $e^{-r\tau}S_\tau$. They must take profit if the price reaches $M > H$.

- So we seek V, payoff to optimal stopping problem:

$$V(x,f) \stackrel{def}{=} \sup_{optional \tau \leq \tau_{[M,\infty)}} \mathbb{E}_{x,f}[e^{-r\tau}S_\tau],$$

where (for A closed subset of $\mathbb{R}_+$) τ_A is the first hitting time of A by S.

- Will discuss buying problem later.

Positive Regime

Claim (1)

Optimal to continue in positive regime unless at (or above) M.

Remark

Use repeatedly that

- V *is continuous;*
- $V(x, f) \geq x$;
- *defining* $D \stackrel{def}{=} \inf\{t : V(X_t) = S_t\}$, τ_D *is the a.s. smallest optimal stopping time and* $e^{-r(t \wedge \tau_D)} V(X_{t \wedge \tau_D})$ *is a martingale..*

Proof of Claim (1).

Suppose not, so that there is an $x \in (L, M)$ with $V(x, +) = x$. Let $\sigma = \inf\{t : S_t = L \text{ or } M\}$. Then using locally submg property of gains process $G_t \stackrel{def}{==} e^{-rt} S_t$ in +ve regime, we see that

$$x = G_0 < \mathbb{E}_{x,+}[G_\sigma]$$

so it is *strictly* suboptimal to stop at x. $\diamondsuit$

Negative Regime Define $D_- = \{x : V(x,-) = x\}$.

Claim (2)

There is an $m \in [0, H)$ *such that*

$$D_- = [0, m].$$

Proof.

Show D_- is a closed subinterval of $[0, H)$ containing 0.

- By cty, D is closed.
- Since $V(H-,-) = V(H,+) > H$ (by Claim (1)), so $(H,+) \notin D$.

Follows that D_- is closed in normal topology.

- Since S is absorbed at 0, $V(0,-) = 0$ so $0 \in D_-$.
- $C_- \stackrel{def}{=} [0, H) \setminus D_-$ is open. Hence a union of (disjoint) open intervals. Suppose D_- is not an interval, then there exist $y, z \in D_-$ with $(y, z) \subseteq C_-$.
- Take an $x \in (y, z)$ and let $\sigma = \min(\tau_y, \tau^z)$, then this is optimal for the problem started at $(x,-)$.
- So

$$V(x,-) = \mathbb{E}_{x,-}[G_\sigma].$$

But on $[0, \sigma]$ G "is a supermg", so $G_0 = x \geq \mathbb{E}_{x,-}[G_\sigma] = V(x,-)$ and so $x \in D_-$. $\diamondsuit$

So optimal policy is to wait until hit level M unless price goes too low in -ve regime (below m) in which case give up.

How do we identify threshold m?

Claim (Smooth Pasting)

Either $m = 0$ or else $V(\cdot, -)$ is C^1 at m so that $V(\cdot, -)$ is the (unique) solution to the free-boundary problem:

$$\begin{aligned} L^- V - rV =& 0 : \ \ on\ (m, H) \\ V(x, -) =& x : \ \ on\ (0, m] \\ V(H-, -) =& V(H, +) \\ V'(m, -) =& 1 \end{aligned} \tag{1}$$

Proof.

• Standard except for smooth pasting/C^1 condition at m.
• Since gains function is in domain of martingale generator, $\mathcal{G}$, of X, V is *also in domain* (Norgilas and Jacka).
• Ito-Tanaka formula and fact that $E^{-rt}V(X_t)$ is a supermg and a mg on $(m,H)\times\{-\}$ now gives us that $V(\cdot,-)$ is C^2 on (m,H) and $\Delta V'(m)\leq 0$ (if $m>0$).

• Conversely, fact that $V(x,-)\geq x$ shows that $\Delta V'(m,-)\geq 0$.

• [Uniqueness] Argument based on strong maximum principle shows any solution, ϕ, to (1) must have $\phi(x)-x$ increasing , and hence non-negative, on $[m,H]$. Snell's characterisation of V then tells us that $\phi=V$ so m must be *unique* free-boundary. $\diamondsuit$

Provided with seller's payoff V, can now define the buyer's problem—find optimal time to buy to maximise expected profit.

- Problem is to find W given by

$$W(x,f) = \sup_{optional \tau \leq \tau_{[M,\infty)}} \mathbb{E}_{x,f}[e^{-r\tau}(V(X_\tau) - S_\tau].$$

Remark

Could consider revised problem where not restricted to buying before $\tau_{[M,\infty)}$. Does not change form of solution much.

Define gains function $F : (x,f) \mapsto V(x,f) - x$ and then set $D' \stackrel{def}{=} \{(x,f) : W(x,f) = F(x,f)\}$.

Claim

$$D' = \{(x,+) : x \in [R,M]\}$$

for some $R \in (L,M)$.

Remark

In case where not restricted to buying before $\tau_{[M,\infty)}$: change interval to $[R, R']$ with $R' < M$.

Proof.

• Gains process $G'_t = e^{-rt}F(X_t)$ is a submg until first exit by X from $[0, H) \times \{-\}$ so clearly not optimal to stop in -ve regime.

• Similar argument to proof of Claim (2) shows $\{x : (x, f) \in D'\}$ must be a closed interval I (in the normal topology on $\mathbb{R}$).

• Finally, $(M, +) \in D'$ so result follows. $\diamondsuit$

Remark

Same techniques will show smooth pasting at R (and R').

Example

We take $L = 1$, $H = 3$ and $M = 4$;

- with $\sigma_+(x) = \sigma_-(x) = .2x$
- $\mu_+(x) = .05x$, $\mu_-(x) = -.01x$ and $r = .02$.

- The (selling) solution is $V(x,+) = \frac{63}{31}x^{\frac{1}{2}} - \frac{32}{31}x^{-2}$ and

$$V(x,-) = \begin{cases} \frac{3}{5m}x^2 + \frac{2m^{\frac{3}{2}}}{5}x^{-\frac{1}{2}} & x \in (m,3) \\ x & x \in [0,m] \end{cases}$$

where m is root (≈ 1.94237) of $\frac{27}{5m} + \frac{2m^{\frac{3}{2}}}{5\sqrt{3}} = \frac{63\sqrt{3} - \frac{32}{9}}{31}$.

Example (cont.)

It then follows that $W(x,-)=\frac{5-6R^{\frac{1}{2}}+R^3}{5}x^2$ and

$$W(x,+)=\begin{cases}(\frac{63}{31}-\frac{6R^{\frac{1}{2}}}{5})x^{\frac{1}{2}}-(\frac{32}{31}-\frac{R^3}{5})x^{-2} & x\in(1,R)\\ V(x,+)-x & x\in[R,4]\end{cases}$$

with R is the root (≈ 2.06888) of $R^3-6R^{\frac{1}{2}}=\frac{5}{9}\frac{63\sqrt{3}-\frac{32}{9}-93}{31}$

- Deliberately chose price model equivalent to risk-neutral one. Follows this model at least weakly consistent with a market.
- What happens under this model? Stocks are sold in the negative regime (depressing price) and are bought in the positive regime (increasing price) so not unbelievable.
- What about resistance and support levels? If $\mu_+ >> 0$ in neighborhood of $\frac{L+H}{2}$ and $\mu_- << 0$ in a neighborhood of $\frac{L+H}{2}$ then get something like resistance and support. Can take limits and get skew diffusion with skewness at L and H.
- Can consider concave utility with $L^+ u - ru$ positive on some interval (L, A) only. Get similar results (joint work with Henderson and Liu).
- Nothing special about constant r.
- Can look for endogenous specification of price process...

Example (Second)

We take $L = 1$, $H = 2$ and $M = 8$;

- with $\sigma_+(x) = \sqrt{.06}x$, $\sigma_-(x) = .1x$
- $\mu_+(x) = .04x$, $\mu_-(x) = .005x$ and $r = .02$.

- The (selling) solution is $V(x,+) = \frac{(2\gamma+1)m^3}{8\gamma}x^{-1} + \frac{42m^{-1}-m^3}{8\gamma}x^{\frac{2}{3}}$

and

$$V(x,-) = \begin{cases} \frac{3m^{-1}}{4}x^2 + \frac{m^3}{4}x^{-2} & x \in (m,2) \\ x & x \in [0,m] \end{cases}$$

where m is $\approx .09524$ and $\gamma = 2^{\frac{5}{3}} - 1$.