NONEXPANSIVE MAPS AND OPTION PRICING THEORY

Published in Kibernetica 34:6 (1998), 713-724

Vassili N. Kolokoltsov

Abstract. *The famous Black-Sholes (BS) and Cox-Ross-Rubinstein (CRR) formulas are basic results in the modern theory of option pricing in financial mathematics. They are usually deduced by means of stochastic analysis; various generalisations of these formulas were proposed using more sophisticated stochastic models for common stocks pricing evolution. In this paper we develop systematically a deterministic approach to the option pricing that leads to a different type of generalisations of BS and CRR formulas characterised by more rough assumptions on common stocks evolution (which are therefore easier to verify). On the other hand, this approach is more elementary, because it uses neither martingales nor stochastic equations.*

Key words. *Nonexpansive maps, Bellman operator, Cox-Ross-Rubinstein and Black-Sholes models, option pricing, surplus value.*

1. Generalised CRR model: a deterministic approach.

We start with an exposition of a deterministic approach to the analysis of the standard discrete Cox-Ross-Rubinstein model of financial market and its natural modification with more rough assumptions on the underlying common stocks prices evolution. This discussion leads naturally to three types of the prices of an option: hedge price, minimal price and mean price. Next, we develop this approach to cover more general models of options, in particularly those depending on several types of common stocks, and then consider these models in the continuous limit deriving the multidimensional versions of the Black-Sholes formula and more general equations proposed recently by T.Lyons in [L]. In the last section we discuss the connection with the theory of nonexpansive mappings.

A simplest model of financial market deals with only two securities: the risk-free bonds (or bank account) and common stocks. The prices of the units of these securities, $B = (B_k)$ and $S = (S_k)$ respectively, change in discrete moments of time $k = 0, 1, ...$ according to the recurrent equations $B_{k+1} = \rho B_k$, where $\rho \geq 1$ is a fixed number, and $S_{k+1} = \xi_{k+1} S_k$, where ξ_k is an (a priori unknown) sequence taking value in a fixed compact set $M \in \mathcal{R}$. We denote by u and d respectively the exact upper and lower bounds of M (u and d stand for up and down) and suppose that $0 < d < \rho < u$. We shall be interested especially in two cases:

(i) M consists of only two elements, its upper and lower bounds u and d,

(ii) M consists of the whole closed interval $[d, u]$.

No probability assumptions on the sequence ξ_k are specified. Case (i) corresponds to the CRR model and case (ii) stands for the situation when only minimal information on the future evolution of common stocks pricing is available, namely, the rough bounds on its growth per unit of time.

An investor is supposed to control the growth of his capital in the following way. Let X_{k-1} be his capital at the moment $k-1$. Then the investor chooses his portfolio defining the number γ_k of common stock units held in the moment $k-1$. Then one can write

$$X_{k-1} = \gamma_k S_{k-1} + (X_{k-1} - \gamma_k S_{k-1}),$$

where the sum in brackets corresponds to the part of the capital laid on the bank account (and which will thus increases deterministically). All operations are friction-free. The control parameter γ_k can take all real values, i.e. short selling and borrowing are allowed. In the moment k the value ξ_k becomes known and thus the capital becomes equal to

$$X_k = \gamma_k \xi_k S_{k-1} + (X_{k-1} - \gamma_k S_{k-1})\rho.$$

The strategy of the investor is by definition any sequence of numbers $\Gamma = (\gamma_1, ..., \gamma_n)$ such that each γ_j can be chosen using the whole previous information: the sequences $X_0, ..., X_{j-1}$ and $S_0, ...S_{j-1}$. It is supposed that the investor, selling an option by the price $C = X_0$ should organise the evolution of this capital (using the described procedure) in a way that would allow him to pay to the buyer in the prescribed moment n some premium $f(S_n)$ depending on the price S_n. The function f defines the type of the option under consideration. In the case of the standard European call option, which gives to the buyer the right to buy a unit of the common stocks in the prescribed moment of time n by the fixed price K, the function f has the form

$$f(S_n) = \max(S_n - K, 0). \tag{1}$$

Thus the income of the investor will be $X_n - f(S_n)$. The strategy $\gamma_1, ..., \gamma_n$ is called a hedge, if for any sequence $\xi_1, ..., \xi_n$ the investor is able to meet his obligations, i.e. $X_n - f(S_n) \geq 0$. The minimal value of the initial capital X_0 for which the hedge exists is called the hedging price C_h of an option. The hedging price C_h will be called correct (or fair), if moreover, $X_n - f(S_n) = 0$ for any hedge and any sequence ξ_j. The correctness of the price is equivalent to the impossibility of arbitrage, i.e. of a risk-free premium for the investor. It was in fact proven in [CRR] (using some additional probabilistic assumptions on the sequence ξ_j) that for case (i) the hedging price C_h exists and is correct. On the other hand, it is known that when the set M consists of more than two points, the hedging price will not be correct anymore. We shall show now using exclusively deterministic arguments that both for cases (i) and (ii) the hedge exists and is the same for both cases whenever the function f is nondecreasing and convex (possibly not strictly).

When calculating prices, one usually introduces the relative capital Y_k defined by the equation $Y_k = X_k/B_k$. Since the sequence B_k is positive and deterministic, the problem of the maximisation of the value $X_n - f(S_n)$ is equivalent to the maximisation of $Y_n - f(S_n)/B_n$. Consider first the last step of the game. If the relative capital of the investor at moment $n-1$ is equal to $Y_{n-1} = X_{n-1}/B_{n-1}$, then his relative capital at the next moment will be

$$Y_n(\gamma_n, \xi_n) - \frac{f(\xi_n S_{n-1})}{B_n} = Y_{n-1} + \gamma_n \frac{S_{n-1}}{B_n}(\xi_n - \rho) - \frac{1}{B_n} f(\xi_n S_{n-1}).$$

Therefore, it is clear that the guaranteed income (in terms of relative capital) in the last step can be written as

$$Y_{n-1} - \frac{1}{B_{n-1}}(\mathcal{B}f)(S_{n-1}),$$

where the Bellman operator $\mathcal{B}$ is defined by the formula

$$(\mathcal{B}f)(z) = \frac{1}{\rho} \min_{\gamma} \max_{\xi \in M} [f(\xi z) - \gamma z(\xi - \rho)]. \tag{2}$$

We suppose further the function f to be nondecreasing and convex (perhaps, not strictly), having in mind the main example, which corresponds to the standard European call option and where this assumption is satisfied. Then the maximum in (2) is evidently attained on the end points of M and thus

$$(Bf)(z) = \frac{1}{\rho} \min_{\gamma} \max\,[f(dz) - \gamma z(d-\rho),\ f(uz) - \gamma z(u-\rho)]. \tag{3}$$

One sees directly that for $\gamma \geq \gamma^h$ (resp. $\gamma \leq \gamma^h$), the first term (resp. the second) under max in (3) is maximal, where

$$\gamma^h = \gamma^h(z, [f]) = \frac{f(uz) - f(dz)}{z(u-d)}. \tag{4}$$

It implies that the minimum in (3) is given by $\gamma = \gamma^h$, which yields

$$(\mathcal{B}f)(z) = \frac{1}{\rho} \left[\frac{\rho - d}{u - d} f(uz) + \frac{u - \rho}{u - d} f(dz) \right]. \tag{5}$$

The mapping $\mathcal{B}$ is a linear operator on the space of continuous functions on the positive line that preserves the set of nondecreasing convex functions. Using this property and induction in k one gets that the guaranteed relative income of the investor to the moment of time n is given by the formula $Y_0 - B_0^{-1}(\mathcal{B}^n f)(S_0)$ and thus his guaranteed income is equal to

$$\rho^n(X_0 - (\mathcal{B}^n f)(S_0)). \tag{6}$$

The hedge strategy (the use of which guarantees him this guaranteed income) is $\Gamma^h = (\gamma_1^h, ..., \gamma_n^h)$, where each γ_j^h is calculated step by step using formula (4). The minimal value of X_0 for which this income is not negative (and which by definition is the hedge price C_h of the corresponding option contract) is therefore given by the formula

$$C_h = (\mathcal{B}^n f)(S_0). \tag{7}$$

Using (4) one easily finds for C_h the following CRR formula [CRR]:

$$C_h = \rho^{-n} \sum_{k=0}^{n} C_n^k \left(\frac{\rho - d}{u - d} \right)^k \left(\frac{u - \rho}{u - d} \right)^{n-k} f(u^k d^{n-k} S_0), \tag{8}$$

where C_n^k are standard binomial coefficients. When f is defined by (1), this yields

$$C_h = S_0 \mathcal{P}_\mu \left(\frac{u}{\rho} \frac{\rho - d}{u - d} \right) - K \rho^{-n} \mathcal{P}_\mu \left(\frac{\rho - d}{u - d} \right),$$

where the function $\mathcal{P}_k$ is defined by the formula

$$\mathcal{P}_k(q) = \sum_{j=k}^{n} C_n^j q^j (1-q)^{n-j},$$

the integer μ is the minimal integer k such that $u^k d^{n-k} S_0 > K$, and it is supposed that $\mu \leq n$.

If the investor uses his hedge strategy $\Gamma^h = (\gamma_1^h, ..., \gamma_n^h)$, then the two terms under max in expression (3) are equal (for each step $j = 1, ..., n$). Therefore, in the case (i) (when the set M consists of only two elements), if $X_0 = C_h$, the resulting income (6) does not depend on the sequence $\xi_1, ..., \xi_n$ and vanishes always, whenever the investor uses his hedge strategy, i.e. the prize C_h is correct in that case (Cox-Ross-Rubinstein theorem).

In general case it is not so anymore. Let us give first the exact formula for the maximum of the possible income of the investor in the general case supposing that he uses his hedge strategy. Copying the previous arguments one sees that this maximal income is given by the formula

$$\rho^n (X_0 - (\mathcal{B}_{min}^n f)(S_0)), \tag{9}$$

where

$$(\mathcal{B}_{min} f)(z) = \frac{1}{\rho} \min_{\xi \in M} [f(\xi z) - \gamma z (\xi - \rho)]|_{\gamma = \gamma^h}. \tag{10}$$

Thus, in the case of general M, the income of the investor playing with his hedge strategy will consists of the sum of the guaranteed income (6) and some unpredictable surplus (risk-free premium), which does not exceed the difference between expressions (13) and (10). Hence, a reasonable price for the option should belong to the interval $[C_{min}, C_h]$ with C_h given by (7) and

$$C_{min} = (\mathcal{B}_{min}^n f)(S_0). \tag{11}$$

Since the value $\mathcal{B}_{min}^n$ is essentially more difficult to calculate than $\mathcal{B}^n$, it may be useful to have some simple reasonable estimate for it. Taking $\xi = \rho$ in (10) yields $(\mathcal{B}_{min}^n f)(z) \leq \rho^{-1} f(\rho z)$ and therefore by induction

$$(\mathcal{B}_{min}^n f)(z) \leq \rho^{-n} f(\rho^n z). \tag{12}$$

Looking at the evolution of the capital X_k as at the game of the investor with the nature (γ_k and ξ_k are their respective controls) one can say that (for the hedge strategy of the investor) the nature plays against the investor, when its controls ξ_k lie near the boundary $[d, u]$ of the set M (then the investor gets his minimal guaranteed income (6)) and conversely, it plays for the investor, when its controls ξ_k are in the middle of M, say, near ρ. If it is possible to estimate roughly the probability p that ξ_k would be near the boundaries of M, one can estimate the mean income of the investor (who uses his hedge strategy) by

$$\rho^n(X_0 - ((\mathcal{B}_{mean})^n f)(S_0)),$$

where

$$(\mathcal{B}_{mean} f)(z) = p(\mathcal{B}f)(z) + (1-p)\frac{1}{\rho}f(\rho z)$$

$$= \frac{1}{\rho}\left[p\frac{u-\rho}{u-d}f(dz) + (1-p)f(\rho z) + p\frac{\rho-d}{u-d}f(uz)\right], \tag{13}$$

which gives for the mean price the following approximation

$$C_{mean} = ((\mathcal{B}_{mean})^n f)(S_0). \tag{14}$$

Denoting by C_k^{ij} the coefficients in the polynomial development

$$(\epsilon_1 + \epsilon_2 + \epsilon_3)^k = \sum_{i+j\le k} C_k^{ij} \epsilon_1^{k-i-j}\epsilon_2^i\epsilon_3^j,$$

and using induction, one gets for (14) the following representation:

$$((\mathcal{B}_{mean})^n f)(S_0) = \frac{1}{\rho^n}\sum_{i+j\le n} C_n^{ij}\left(p\frac{u-\rho}{u-d}\right)^{n-i-j}(1-p)^i\left(p\frac{\rho-d}{u-d}\right)^j f(d^{n-i-j}\rho^i u^j S_0). \tag{15}$$

For the function f of form (1), it can be rewritten as

$$\frac{1}{\rho^n}\sum_{i,j\in P} C_n^{ij}\left(p\frac{u-\rho}{u-d}\right)^{n-i-j}(1-p)^i\left(p\frac{\rho-d}{u-d}\right)^j (d^{n-i-j}\rho^i u^j S_0 - K),$$

where the set P is given by the formula

$$P = \{i \ge 0, j \ge 0 : i+j \le n \ \& \ i\log\frac{\rho}{d} + j\log\frac{u}{d} > \log K - \log S_0 - n\log d\}.$$

2. Option contracts on several common stocks.

Suppose now there is a number, say I, of common stocks whose prices $S_k^i, i \in I, k = 0, 1, ...,$ satisfy the recurrent equations $S_k^i = \xi_k^i S_{k-1}^i$, where ξ_j^i take values in compact sets M_i with bounds d_i and u_i respectively. The investor controls his capital by choosing in each moment of time $k-1$ his portfolio consisting of γ_k^i units of common stocks of the type i, the rest of the capital being laid on the risk-free bank account. His capital at the next time k becomes therefore

$$X_k = \gamma_k^1\xi_k^1 S_{k-1}^1 + ... + \gamma_k^I\xi_k^I S_{k-1}^I + \rho(X_{k-1} - \gamma_k^1 S_{k-1}^1 - ... - \gamma_k^I S_{k-1}^I).$$

The premium to the buyer of the option at a fixed time n will be now $f(S_n^1,...,S_n^I)$, where f is a given nondecreasing convex continuous function on the positive octant $\mathcal{R}_+^n$. For instance, the analog of the standard European option is given by the function

$$f(z_1,...,z_I) = \max(\max(0, z_1 - K_1), ..., \max(0, z_I - K_I)), \tag{15}$$

which describes the option contract that permits to the buyer to purchase one unit of the common stocks belonging to any type $1,...,I$ by his choice.

To simplify formulas, we reduce ourselves to the case of two types of common stocks, i.e. to the case $I = 2$. Similarly to the case $I = 1$ one obtains a similar formula to the guaranteed relative income of the investor in the last step of the game starting from the relative capital Y_{n-1} at the time $n-1$, namely

$$Y_{n-1} - \frac{1}{B_{n-1}}(\mathcal{B}f)(S_{n-1}^1, S_{n-1}^2),$$

where the Bellman operator $\mathcal{B}$ has the form

$$(\mathcal{B}f)(z_1,z_2) = \frac{1}{\rho}\min_{\gamma^1,\gamma^2}\max_{\xi^1\in M_1,\xi^2\in M_2}[f(\xi^1 z_1, \xi^2 z_2) - \gamma^1 z_1(\xi^1 - \rho) - \gamma^2 z_2(\xi^2 - \rho)]. \tag{16}$$

In order to give an explicit formula for this operator (similar to (5)), one should make additional assumptions on the function f. We say that a nondecreasing function f on $\mathcal{R}_+^2$ is nice, if the expression

$$f(d_1 z_1, u_2 z_2) + f(u_1 z_1, d_2 z_2) - f(d_1 z_1, d_2 z_2) - f(u_1 z_1, u_2 z_2)$$

is nonnegative everywhere. One easily sees for instance, that any function of the form $f(z_1,z_2) = \max(f_1(z_1), f_2(z_2))$ is nice for any nondecreasing functions f_1, f_2 and any numbers $d_i < u_i$, $i = 1,2$, and in particular, function (15) is nice. Clear the nice functions constitute a linear space and the set of continuous nondecreasing convex nice functions is a convex subset in this space, which we denote NS (nice set). Furthermore, let

$$\kappa = \frac{(u_1 u_2 - d_1 d_2) - \rho(u_1 - d_1 + u_2 - d_2)}{(u_1 - d_1)(u_2 - d_2)}. \tag{17}$$

Lemma. *If $f \in NS$ and $\kappa \geq 0$, then*

$$(\mathcal{B}f)(z_1,z_2) = \frac{1}{\rho}\left[\frac{\rho - d_1}{u_1 - d_1}f(u_1 z_1, d_2 z_2) + \frac{\rho - d_2}{u_2 - d_2}f(d_1 z_1, u_2 z_2) + \kappa f(d_1 z_1, d_2 z_2)\right] \tag{18}$$

and the γ^{h1}, γ^{h2} giving minimum in (20) are equal to

$$\gamma^{h1} = \frac{f(u_1 z_1, d_2 z_2) - f(d_1 z_1, d_2 z_2)}{z_1(u_1 - d_1)}, \quad \gamma^{h2} = \frac{f(d_1 z_1, u_2 z_2) - f(d_1 z_1, d_2 z_2)}{z_2(u_2 - d_2)}.$$

If $\kappa \le 0$ (and again $f \in NS$), then

$$(\mathcal{B}f)(z_1, z_2) = \frac{1}{\rho}\left[\frac{u_1 - \rho}{u_1 - d_1} f(d_1 z_1, u_2 z_2) + \frac{u_2 - \rho}{u_2 - d_2} f(u_1 z_1, d_2 z_2) + |\kappa| f(u_1 z_1, u_2 z_2)\right],$$

$$\gamma^{h1} = \frac{f(u_1 z_1, u_2 z_2) - f(d_1 z_1, u_2 z_2)}{z_1(u_1 - d_1)}, \quad \gamma^{h2} = \frac{f(u_1 z_1, u_2 z_2) - f(u_1 z_1, d_2 z_2)}{z_2(u_2 - d_2)}.$$

The proof of this lemma uses only elementary manipulations. It follows that the operator $\mathcal{B}$ preserves NS and by the same induction as in the previous section one proves that if the premium is defined by a function $f \in NS$, then the hedge price for the option contract exists and is equal to

$$C_h = (\mathcal{B}^n f)(S_0^1, S_0^2). \tag{19}$$

One can write down a more explicit expression (analogous to (8)). For instance, for the simplest case $\kappa = 0$,

$$C_h = \frac{1}{\rho^n} \sum_{k=0}^{n} C_n^k \left(\frac{\rho - d_1}{u_1 - d_1}\right)^k \left(\frac{\rho - d_2}{u_2 - d_2}\right)^{n-k} f(d_1^{n-k} u_1^k z_1, d_2^k u_2^{n-k} z_2). \tag{20}$$

For the most important particular case, when the function f is of form (15) with $I = 2$ formula (20) can be written in terms of the function $\mathcal{P}_k$ defined above (after formula (8)). The answer depends on the position of the integers μ and ν on the real line, where μ (resp. ν) is the minimal (resp. maximal) integer k such that $u_1^k d_1^{n-k} S_0^1 > K_1$ (resp. $u_2^{n-k} d_2^k S_0^2 > K_2$). For instance, if $0 < \nu < \mu < n$, then

$$C_h = S_0^1 \mathcal{P}_\mu\left(\frac{u_1(\rho - d_1)}{\rho(u_1 - d_1)}\right) - K_1 \rho^{-n} \mathcal{P}_\mu\left(\frac{\rho - d_1}{u_1 - d_1}\right)$$

$$+ S_0^2 \mathcal{P}_{n-\nu}\left(\frac{u_2(\rho - d_2)}{\rho(u_2 - d_2)}\right) - K_2 \rho^{-n} \mathcal{P}_{n-\nu}\left(\frac{\rho - d_2}{u_2 - d_2}\right),$$

and if $0 < \mu < \nu < n$, then

$$C_h = S_0^1 \mathcal{P}_k\left(\frac{u_1(\rho - d_1)}{\rho(u_1 - d_1)}\right) - K_1 \rho^{-n} \mathcal{P}_k\left(\frac{\rho - d_1}{u_1 - d_1}\right)$$

$$+ S_0^2 \mathcal{P}_{n-k+1}\left(\frac{u_2(\rho - d_2)}{\rho(u_2 - d_2)}\right) - K_2 \rho^{-n} \mathcal{P}_{n-k+1}\left(\frac{\rho - d_2}{u_2 - d_2}\right),$$

where k is the minimal integer such that

$$d_1^{n-k} u_1^k S_0^1 - K_1 > d_2^k u_2^{n-k} S_0^2 - K_2.$$

This formula for for C_h is similar to (8), but even if each M_i consists of only two points, this hedge price is not correct. As in the previous section, one can represent the maximal income of the investor who uses his hedge strategy by the formula

$$\rho^n(X_0 - (\mathcal{B}^n_{min} f)(S^1_0, S^2_0))$$

with

$$(\mathcal{B}_{min} f)(z_1, z_2) = \frac{1}{\rho} \min_{\xi^1 \in M_1} \min_{\xi^2 \in M_2} [f(\xi^1 z_1, \xi^2 z_2) - \gamma^1 z_1(\xi^1 - \rho) - \gamma^2 z_2(\xi^2 - \rho)]|_{\gamma^1 = \gamma^{h1}, \gamma^2 = \gamma^{h2}}. \tag{21}$$

The corresponding minimal price of the option is

$$C_{min} = ((\mathcal{B}_{min})^n f)(S^1_0, S^2_0). \tag{22}$$

Supposing as in the previous section that one can estimate the probability p of the numbers ξ^i_k to be near the boundaries of the corresponding sets M_i (the case when this probability is different for each type of common stocks can be evidently covered in the same way) one gets for the mean price of the option (corresponding to the mean income of the investor playing with his hedge strategy) is

$$C_{mean} = ((\mathcal{B}_{mean})^n f)(S^1_0, S^2_0), \tag{23}$$

where (when supposing $\kappa = 0$ as above) $(\mathcal{B}_{mean} f)(z_1, z_2)$ is equal to

$$\frac{1}{\rho}\left[p\frac{\rho - d_1}{u_1 - d_1} f(u_1 z_1, d_2 z_2) + (1-p) f(\rho z_1, \rho z_2) + p\frac{\rho - d_2}{u_2 - d_2} f(d_1 z_1, u_2 z_2)\right]. \tag{24}$$

The explicit formula for (23) is similar to (12).

3. Continuous-time limit.

As was shown in [CRR], the binomial CRR formula for option prices (8) tends to the Black-Sholes formula under an appropriate limit procedure. We find similar limits for formulas of the previous section. Following our methodology we make it in a simplest way ruling out all probability theory. The only "trace" of the geometric Brownian motion model of Black-Sholes will be the assumption (which is clearly more rough than the usual assumptions of the standard Black-Sholes model) that the logarithm of the relative growth of the stock prices is proportional to $\sqrt{\tau}$ for small intervals of time τ. More exactly, if τ is the time between the successive evaluations of common stock prices, then the bounds d_i, u_i of M_i are given by the formulas $\log u_i = \sigma_i \sqrt{\tau} + \mu_i \tau$ and $\log d_i = -\sigma_i \sqrt{\tau} + \mu_i \tau$, where the coefficients $\mu_i > 0$ stand for the systematic growth and the coefficients σ_i (so called volatilities) stand for "random oscillations". Moreover, as usual, $\log \rho$ is proportional to τ, i.e. $\log \rho = r\tau$ for some constant $r \geq 1$. Let $\mathcal{B}(\tau)$ denote the corresponding operator (16).

Under these assumptions, the calculation of the coefficient κ from (17) and the strategies γ^h from the Lemma for small τ yields

$$\kappa = \frac{1}{2}\left(\frac{\sigma_1+\sigma_2}{2} + \frac{\mu_1 - r}{\sigma_1} + \frac{\mu_2 - r}{\sigma_2}\right)\sqrt{\tau} + O(\tau^{3/2}),$$

$$\gamma^{hj} = \frac{\partial f}{\partial z_j}(z_1, z_2)(1 + O(\tau), \quad j = 1, 2.$$

Suppose that the coefficient at $\sqrt{\tau}$ in this formula is not negative and one can use formula (18) for $\mathcal{B}$. (In fact, the opposite assumption would lead to the same resulting differential equation.) Calculating the coefficients of (18) for small times one obtains for $(\mathcal{B}_\tau f)(z_1, z_2)$ the expression

$$\frac{1-r\tau}{2}\left(\left(\frac{\sigma_1+\sigma_2}{2} + \frac{\mu_1 - r}{\sigma_1} + \frac{\mu_2 - r}{\sigma_2}\right)\sqrt{\tau} + O(\tau^{3/2})\right) f\left(e^{-\sigma_1\sqrt{\tau}+\mu_1\tau}z_1,\, e^{-\sigma_2\sqrt{\tau}+\mu_2\tau}z_2\right)$$

$$+\frac{1-r\tau}{2}\left(1 - \left(\frac{\sigma_1}{2} + \frac{\mu_1 - r}{\sigma_1}\right)\sqrt{\tau} + O(\tau^{3/2})\right) f\left(e^{\sigma_1\sqrt{\tau}+\mu_1\tau}z_1,\, e^{-\sigma_2\sqrt{\tau}+\mu_2\tau}z_2\right)$$

$$+\frac{1-r\tau}{2}\left(1 - \left(\frac{\sigma_2}{2} + \frac{\mu_2 - r}{\sigma_2}\right)\sqrt{\tau} + O(\tau^{3/2})\right) f\left(e^{-\sigma_1\sqrt{\tau}+\mu_1\tau}z_1,\, e^{\sigma_2\sqrt{\tau}+\mu_2\tau}z_2\right).$$

Due to the Taylor formula, one has for any σ_1, σ_2:

$$f\left(e^{\sigma_1\sqrt{\tau}+\mu_1\tau}z_1,\, e^{\sigma_2\sqrt{\tau}+\mu_2\tau}z_2\right) = \frac{1}{2}\frac{\partial^2 f}{\partial z_1^2}(z_1, z_2)z_1^2\sigma_1^2\tau + \frac{1}{2}\frac{\partial^2 f}{\partial z_2^2}(z_1, z_2)z_2^2\sigma_2^2\tau$$

$$+f(z_1, z_2) + \frac{\partial f}{\partial z_1}(z_1, z_2)z_1(\sigma_1\sqrt{\tau} + \mu_1\tau + \sigma_1^2\tau/2) + \frac{\partial f}{\partial z_2}(z_1, z_2)z_2(\sigma_2\sqrt{\tau} + \mu_2\tau + \sigma_2^2\tau/2).$$

Substituting these expansions in the previous formula one sees that all terms proportional to $\sqrt{\tau}$ vanish and therefore one can write down a differential equation for the function

$$F_h(t, z_1, z_2) = \lim_{n\to\infty}(\mathcal{B}^n(t/n)f)(z_1, z_2),$$

which actually has the form

$$\frac{\partial F}{\partial t} = \frac{1}{2}\sigma_1^2 z_1^2\frac{\partial^2 F}{\partial z_1^2} + \frac{1}{2}\sigma_2^2 z_2^2\frac{\partial^2 F}{\partial z_2^2} + rz_1\frac{\partial F}{\partial z_1} + rz_2\frac{\partial F}{\partial z_2} - rF \tag{25}$$

with initial condition $F(0, z_1, z_2) = f(z_1, z_2)$. Rewriting equation (25) in terms of the function R defined by the formula

$$F(t, z_1, z_2) = e^{-rt}R(t, rt + \log z_1, rt + \log z_2) \tag{26}$$

yields a linear diffusion equation with constant coefficients

$$\frac{\partial R}{\partial t} = \frac{1}{2}\sigma_1^2\left(\frac{\partial^2 R}{\partial p_1^2} - \frac{\partial R}{\partial p_1}\right) + \frac{1}{2}\sigma_2^2\left(\frac{\partial^2 R}{\partial p_2^2} - \frac{\partial R}{\partial p_2}\right).$$

It allows to write the solution of the Cauchy problem for equation (25) explicitly, which yields the two-dimensional version of the Black-Sholes formula for hedging option price in continuous time

$$F_h = e^{-rt}(2\pi)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} du_1 du_2$$

$$\times f(S_0^1\exp\{u_1\sigma_1\sqrt{t}+(r-\sigma_1^2/2)t\}, S_0^2\exp\{u_2\sigma_2\sqrt{t}+(r-\sigma_2^2/2)t\})\exp\{-(u_1^2+u_2^2)/2\}. \quad (27)$$

The same procedure for the one-dimensional model from section 2 gives for the continuous version of (8) the standard Black-Sholes formula

$$C_h = e^{-rt}(2\pi)^{-1/2}\int_{-\infty}^{\infty} f(S_0\exp\{u\sigma\sqrt{t} + (r-\sigma^2/2)t\})\exp\{-u^2/2\}\, du,$$

which for the function f of the form (1.5) reduces (after simple manipulations) to a more explicit form

$$C_h = S_0\Phi(u_1) - Ke^{-rt}\Phi(u_2),$$

where

$$u_{1,2} = \frac{\log(S_0/K)}{\sigma\sqrt{t}} + \sqrt{t}\left(\frac{r}{\sigma} \pm \frac{\sigma}{2}\right).$$

In its turn, formula (27) can also be rewritten in a more explicit way for the function f of form (15) (with $I = 2$). Namely, in that case,

$$C_h = \frac{1}{2\pi}\int\int_{A_1(t)}\left(S_0^1 e^{-(u_1-\sigma_1\sqrt{t})^2/2} + K_1 e^{-rt}e^{-u_1^2/2}\right)e^{-u_2^2/2}\, du_1 du_2$$

$$+\frac{1}{2\pi}\int\int_{A_2(t)}\left(S_0^2 e^{-(u_2-\sigma_2\sqrt{t})^2/2} + K_2 e^{-rt}e^{-u_2^2/2}\right)e^{-u_1^2/2}\, du_1 du_2, \quad (28)$$

where the sets $A_1(t), A_2(t)$ are defined by the following formulae

$$A_i(t) = \{(u_1, u_2) : S_0^i e^{\sigma_i y_1\sqrt{t}+(r-\sigma_i^2/2)t} - K_i \geq \max(0, S_0^j e^{\sigma_j y_1\sqrt{t}+(r-\sigma_j^2/2)t} - K_j)\}$$

with j being equal to 2 for $i = 1$ and conversely.

The continuous limit of the estimates (12),(14) or (23) of the option prices for corresponding discrete models can be found in the same way as above. For instance, in two dimensional case, for the function

$$F_{mean}(t, z_1, z_2) = (\mathcal{B}_{mean}^t f)(z_1, z_2) = \lim_{n\to\infty}(\mathcal{B}_{mean}^n(t/n)f)(z)$$

one obtains the same equation (25) but with volatilities $\sqrt{p}\sigma_1$, $\sqrt{p}\sigma_2$ instead of σ_1 and σ_2 respectively. For the continuous limit of the minimal price

$$F_{min}(t,z_1,z_2) = (\mathcal{B}^t_{min} f)(z_1,z_2) = \lim_{n\to\infty} (\mathcal{B}^n_{min}(t/n) f)(z)$$

(which is therefore equal to the difference between the hedge price F_h and the maximal unpredictable surplus of an investor) one obtains by the same procedure a more difficult, essentially nonlinear, equation

$$\frac{\partial F}{\partial t} = \frac{1}{2} \max_{s_1\in[0,\sigma_1]} s_1^2 z_1^2 \frac{\partial^2 F}{\partial z_1^2} + \frac{1}{2} \max_{s_2\in[0,\sigma_2]} s_2^2 z_2^2 \frac{\partial^2 F}{\partial z_2^2} + rz_1 \frac{\partial F}{\partial z_1} + rz_2 \frac{\partial F}{\partial z_2} - rF. \tag{29}$$

Under transformation (26) this reduces to

$$\frac{\partial R}{\partial t} = \frac{1}{2} \max_{s_1\in[0,\sigma_1]} \left(\frac{\partial^2 R}{\partial p_1^2} - \frac{\partial R}{\partial p_1} \right) + \frac{1}{2} \max_{s_2\in[0,\sigma_2]} \left(\frac{\partial^2 R}{\partial p_2^2} - \frac{\partial R}{\partial p_2} \right), \tag{30}$$

which is a two-dimensional version of the equation obtained in [L] by means of stochastic analysis and under certain probabilistic assumptions on the evolution of the underlying common stocks.

5. Conclusion.

Let $C(X)$ be the space of continuous functions on some metric space X, and let D be a subspace of $C(X)$. A mapping $\mathcal{B} : D \mapsto C(X)$ is said to be nonexpansive and homogeneous, if $\sup_x |\mathcal{B}f(x) - \mathcal{B}g(x)| \leq \sup_x |f(x) - g(x)|$ whenever $\mathcal{B}f$ and $\mathcal{B}g$ are defined, and $\mathcal{B}(a+f) = a + \mathcal{B}f$ for any constant a. The theory of such mappings in the case of finite set X, ie when $C(X) = \mathcal{R}^n$, has natural applications in the study of games, discrete event systems and timed event graphs (see e.g. [G], [BCOQ], [KM]), since it was shown that any such mapping in $\mathcal{R}^n$ can be presented as the Bellman operator of some (stochastic) game with a value. The main problem in the study of nonexpansive maps is the study of the iterations $\mathcal{B}^k$ and its asymptotic behaviour as $k \to \infty$. In the case of the mappings in $\mathcal{R}^n$ a big progress in these studies was achieved by means of the investigation of the corresponding "generalised eigenvalue problem" $\mathcal{B}f = a + f, f \in \mathcal{R}^n, a \in \mathcal{R}$. One sees that all three types of prices, C_h, C_{min}, C_{mean}, are expressed in terms of the iterations of some nonexpansive maps, which act not in a finite dimensional space but in the space of continuous functions on the real line or on the plane. Other reasonable generalisations lead to the same result. For example, it was supposed above (which is a commonly used assumption) that the number of stock units γ, which an investor chooses in every moment of time, is arbitrary (no restrictions are posed, this number can even be negative). However, in reality, the boundaries on possible values of γ seem to exist either due to the general boundary on the existing common stock units (one should suppose then that $\gamma \leq \gamma_0$ for some fixed γ_0), or due to the bounds on the possibilities of an investor to make (friction-free) borrowing (one should suppose then the restrictions of the type $\gamma_k \leq X_k/S_k$, say, when no borrowing is allowed). On the other hand, one can omit the assumption of the friction-free exchange of the market securities. In all cases, one proves the existence of hedge strategies and the formula of type (19), (22) for the hedging or minimal price by the same arguments, and in all cases, the Bellman operator $\mathcal{B}$ is a nonexpansive homogeneous mapping on the space of continuous functions on some metric space. However, the formula for this $\mathcal{B}$ would be more complicated. Therefore, in order to be able to find the asymptotic formulas for hedging or minimal prices in various situations one needs to expand the theory of nonexpansive maps iterations to the infinite dimensional case.

Concluding remarks. This paper is an improved version of the author's preprint [K]. As the author learned from a referee report, a deterministic approach to the evaluation of stock prices was discussed recently also in [Mc].

References.

[BCOQ] F.Baccelli, G.Cohen, G. Olsder, and J-P. Quadrat. Synchronisation and Linearity: an Algebra for Discrete Event Systems. John Wiley Publ., N.Y., 1992.

[CRR] J.C. Cox, S.A. Ross, M. Rubinstein. *Option Pricing: A Simplified Approach.* J. Financial Economics **7** (1979), 229-263.

[G] J.Gunawardena (Ed.). Proc. Intern. Workshop "Idempotency", held in Bristol, Oct. 1994. Cambridge Univ. Press., 1998

[K] V. Kolokoltsov. A formula for option prices on a market with unknown volatility. Research Report **9/96**, 1996, Dep. Math. Stat. and O.R., Nottingham Trent University.

[KM] V.N. Kolokoltsov, V.P. Maslov. Idempotent Analysis and its Applications. Kluwer Publisher, 1997.

[L] T. Lions. *Uncertain volatility and the risk-free synthesis of derivatives.* Applied Mathematical Finance **2** (1995), 117-133.

[Mc] McEneaney. Math. Oper. Research **22**, 1997.