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Random orderings of the integers and card shuffling

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Abstract

In this paper we study random orderings of the integers with a certain invariance property. We describe all such orders in a simple way. We define and represent random shuffles of a countable set of labels and then give an interpretation of these orders in terms of a class of generalized riffle shuffles.
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1. Introduction

In this paper we define random shuffles on $\mathbb{N}$ and represent their laws in terms of the laws of pairs of random variables with uniform marginals (Theorem 4.2). A natural subclass of random shuffles are those whose restrictions to $\{1, \dots, n\}$ induce, for every n, a random walk on the permutation group $\mathcal{S}_n$. Partly in order to study such shuffles, and partly because they are of substantial interest in their own right, we introduce and study the class of $\mathcal{I}$-invariant orderings: random orderings of $\mathbb{Z}$ whose laws are invariant under increasing relabellings. Section 3 is devoted to defining and representing $\mathcal{I}$-invariant orderings in terms of quasi-uniform measures (Theorem 3.4).

2. Preliminaries

We denote by $\mathcal{O}$ the class of all strict total orderings of $\mathbb{Z}$. This inherits a natural measurable structure as a subset of $2^{\mathbb{Z}\times\mathbb{Z}}$. We will denote a generic element of $\mathcal{O}$ by $\lhd$, and write $m \lhd n$ if

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m is less than n under $\lhd$. By a strict ordering we mean $m \lhd n$ and $n \lhd m$ implies $m = n$ for every pair of integers.

Given any strictly increasing map $f : \mathbb{Z} \mapsto \mathbb{Z}$ there is a naturally induced map $\hat{f} : \mathcal{O} \mapsto \mathcal{O}$ defined by

$$m \overset{\hat{f}}{\lhd} n \quad \text{if and only if} \quad f(m) \lhd f(n),$$

where we are denoting the image of the ordering $\lhd$ under $\hat{f}$ by $\overset{\hat{f}}{\lhd}$.

Definition 2.1. A probability measure $\mathbb{P}$ on $\mathcal{O}$ is said to be $\mathcal{I}$-invariant if $\mathbb{P} \circ \hat{f}^{-1} = \mathbb{P}$, for all strictly increasing f.

Our purpose is to give an explicit description of all such invariant random orderings. This is closely related to work of Gnedin, [5], and Hirth and Ressel [7], who considered random *weak* total orderings of the integers with laws that are *exchangeable* rather than $\mathcal{I}$-invariant. The laws of such orderings are mixtures of extremal laws, which may be parameterized by open subsets of (0, 1), and constructed by a generalization of Kingman's paintbox construction for exchangeable partitions, [8]. A similar situation holds for us. We parameterize extremal $\mathcal{I}$-invariant laws by means of what we call quasi-uniform measures, which are in one-to-one correspondence with *pairs* of disjoint open subsets of (0, 1).

It is worth noting that exchangeable strict orderings are a subclass of exchangeable weak orderings and so their description is covered by existing results. In fact the law of such an ordering is unique. On the other hand, $\mathcal{I}$-invariant weak orderings have not been described, though one may anticipate that some generalized paintbox representation exists for these also.

Our original motivation for studying random orderings came from thinking about a celebrated method of shuffling a pack of n cards, known as the Gilbert–Shannon–Reeds shuffle. This has the following simple description. Cut the pack into two portions and then interleave the two together in such a way that all possible outcomes are equally likely. This shuffle and various generalizations have been well studied; see, for example, Bayer and Diaconis [2], Diaconis [3], and Lalley [9].

The GSR shuffle has an interesting "geometric" representation. Let our cards be labelled by the integers $1, 2, \ldots, n$. Take n independent random variables $U_1, \ldots, U_n$ each uniformly distributed on [0, 1]. We suppose that the cards are initially ordered so that the card carrying label k is above the card labelled l whenever $U_k > U_l$. Let $f : [0, 1] \to [0, 1]$ be given by $f(x) = 2x \bmod 1$. We now permute the cards so that the card labelled k lies above that labelled l if $f(U_k) > f(U_l)$. The random permutation so generated is a GSR shuffle. This representation makes its clear that the GSR shuffle is naturally thought of a consistent family of shuffles. If we perform it on a pack of n cards initially lying in a uniformly random order and consider the shuffle that has been applied to cards carrying labels $1, 2, \ldots, (n-1)$ then this too is a GSR shuffle. More generally if we apply a sequence of independent GSR shuffles to a pack of size n initially in a uniformly random order, then the ordering of cards carrying labels $1, 2, \ldots, (n-1)$ evolves as if a sequence of GSR shuffles had been applied to it. Moreover after any number of shuffles have been applied, the location of the missing card carrying label n is uniform within the pack even knowing the entire history of the ordering of the first $(n-1)$ cards. In fact, as we will describe in the second half of this paper, there is a one-to-one correspondence between generalized GSR shuffles, meaning a consistent family of shuffles with this conditional independence property, and the laws of $\mathcal{I}$-invariant orderings.

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Another natural application of $\mathcal{I}$-invariant orderings has been given by Gnedin and Olshanski, [6]. They define the infinite graph of zigzag diagrams $\mathcal{Z}$, a relative of Young's lattice, and identify the boundary of $\mathcal{Z}$ by using the representation of $\mathcal{I}$-invariant orderings described here. They also relate this boundary problem to the characterization of certain positive characters on the algebra of quasisymmetric functions.

3. Describing $\mathcal{I}$-invariant orderings

Definition 3.1. A probability measure μ on [0, 1] is called quasi-uniform if it satisfies

$$\mu\{x \in [0,1] : \mu[0,x) \leq x \leq \mu[0,x]\} = 1.$$

Remark 3.2. It is not hard to show that the set of all quasi-uniform measures is closed with respect to the topology of weak convergence of probability measures on [0, 1].

Such a measure is really quite a simple object, and it may be described as follows.

Lemma 3.3. *Suppose that F is a closed subset of* [0, 1]*, and λ_F is the measure with density 1_F with respect to Lebesgue measure. Corresponding to each open component G_i of its complement F^c is a point mass $m_i\delta_{x_i}$, of size m_i equal to the length of the interval G_i, situated at position x_i, which is either the left or right hand end of G_i. Then the measure μ given by*

$$\mu = \lambda_F + \sum_i m_i \delta_{x_i}, \tag{3.1}$$

is quasi-uniform. Moreover every quasi-uniform μ can be decomposed in this fashion, and the decomposition is unique (up to the labelling of the intervals).

We omit the proof which is elementary.

Notice that it is possible for two distinct masses in the above decomposition to be placed at the same point.

Suppose μ is quasi-uniform with representation (3.1); then it is naturally paired with another quasi-uniform distribution μ', having representation

$$\mu' = \lambda_F + \sum_i m_i \delta_{x_i'}, \tag{3.2}$$

where, for each i, x_i' is the right hand endpoint of the interval G_i if x_i is the left hand endpoint, and vice versa. Thus each mass m_i in the decomposition of μ is switched to the opposite end of the interval to which it corresponds. If X and Y are random variables on the same probability space with the law of X being μ and the law of Y being μ' and so that X and Y are equal or take values at either end of a component of F^c, then let us say that such X and Y form a conjugate pair. Notice that their joint law is specified completely by the above description. Such a pair may contain a little more information than either variable separately: whenever they are not equal they together determine an interval of F^c.

Now for any quasi-uniform μ we construct an $\mathcal{I}$-invariant ordering whose law we denote by $\mathbb{P}^\mu$. Consider an infinite sequence of independent pairs of random variables $(X_n, Y_n)_{n\in\mathbb{Z}}$. For each n the variables X_n and Y_n form a conjugate pair with X_n distributed according to μ. Now,

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supposing that $m < n$ (with respect to the natural ordering of the integers), take $m \lhd n$ if and only if one of the following happens:

$$\begin{cases} X_m < X_n, \\ Y_m < Y_n, \\ X_m = X_n > Y_m = Y_n. \end{cases} \tag{3.3}$$

It is easy to check that this defines an $\mathcal{I}$-invariant ordering. Moreover the strong law of large numbers implies that

$$\begin{cases} X_n = \lim_{N\to\infty} \frac{1}{N} \sum_{-N}^{n} 1_{(k \lhd n)} \\ Y_n = \lim_{N\to\infty} \frac{1}{N} \sum_{n}^{N} 1_{(k \lhd n)}, \end{cases} \tag{3.4}$$

exist almost surely. Notice that, since μ can be recovered from $\lhd$ as the empirical distribution of the sequence X_n, $\mathbb{P}^{\mu_1} \neq \mathbb{P}^{\mu_2}$ if $\mu_1 \neq \mu_2$. The following theorem says that by taking mixtures of orderings of this form we obtain all possible $\mathcal{I}$-invariant orderings.

Theorem 3.4. *Suppose that $\lhd$ is an $\mathcal{I}$-invariant ordering. Then almost surely, the random variables defined by (3.4) exist, and for any m and n the relation $m \lhd n$ holds if and only if (3.3) does. Moreover the sequence of random variables $(X_n)_{n\in\mathbb{Z}}$ is exchangeable and with probability one it admits an empirical distribution $\mu(X)$ which is quasi-uniform. Conditional on $\mu(X) = \mu$ the law of $\lhd$ is $\mathbb{P}^{\mu}$.*

In general any ordering $\lhd$ belonging to $\mathcal{O}$ projects to an equivalence relation, $\sim$, on $\mathbb{Z}$ defined by

$$n \sim m \Leftrightarrow \text{ there are only finitely many } k \text{ between (with respect to } \lhd) \; n \text{ and } m.$$

If the ordering is $\mathcal{I}$-invariant then it follows from the above theorem that this partition is exchangeable as studied by Kingman [8].

The proof of Theorem 3.4 hinges on the elementary observation of the next lemma, which begins to explain the role of quasi-uniform measures. We state it in terms of Y_n, given in (3.4); there is an obvious analogue for X_n.

Lemma 3.5. *Suppose that $\lhd$ is some fixed ordering. Suppose that, for each $n \in \mathbb{Z}$, the limit*

$$Y_n = \lim_{N\to\infty} \frac{1}{N} \sum_{n}^{N} 1_{(k \lhd n)}$$

exists. Suppose also that the sequence of empirical measures $\nu^{(N)}$ defined by

$$\nu^N[0, x] = \frac{1}{N} \sum_{1}^{N} 1_{(Y_k \le x)} \quad x \in [0, 1],$$

converges weakly to a probability measure ν as N tends to infinity. Then

$$\nu[0, Y_0) \le Y_0 \le \nu[0, Y_0].$$

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Proof. It is an easy consequence of the transitivity of $\lhd$ that:

$$Y_k < Y_0 \Rightarrow k \lhd 0 \Rightarrow Y_k \le Y_0,$$

for any $k \in \mathbb{Z} \setminus \{0\}$. Thus

$$\frac{1}{N}\sum_1^N 1_{(Y_k<Y_0)} \le \frac{1}{N}\sum_1^N 1_{(k\lhd 0)} \le \frac{1}{N}\sum_1^N 1_{(Y_k\le Y_0)}.$$

But the left hand side is $\nu^{(N)}[0, Y_0)$ while the right hand side is $\nu^{(N)}[0, Y_0]$ and by virtue of weak convergence:

$$\nu[0, Y_0) \le \liminf \nu^{(N)}[0, Y_0)$$
$$\nu[0, Y_0] \ge \limsup \nu^{(N)}[0, Y_0]. \quad \square$$

The second key ingredient in proving Theorem 3.4 is that $\mathcal{I}$-invariance of an infinite sequence of random variables, also known as contractibility, defined below in Lemma 3.7, gives rise to the apparently stronger property of exchangeability. As a first example of this we have the following lemma for which we give an elementary proof.

Lemma 3.6. *Suppose that $\lhd$ is an $\mathcal{I}$-invariant ordering. Then each of the families of random variables*

$$(1_{(k\lhd 0)}; k > 0) \quad \textit{and} \quad (1_{(k\lhd 0)}; k < 0)$$

is exchangeable.

Proof. To verify the first family is exchangeable it suffices to check that for finite collections of positive integers $j_1 \cdots j_m$ and $k_1 \cdots k_n$ the value of

$$\mathbb{E}\left[1_{(0\lhd j_1)} \cdots 1_{(0\lhd j_m)} 1_{(k_1\lhd 0)} \cdots 1_{(k_n\lhd 0)}\right]$$

depends only on n and m. Now replace $1_{(k_i\lhd 0)}$ by $1-1_{(0\lhd k_i)}$, multiply out and apply $\mathcal{I}$-invariance to obtain an expression involving terms: $\mathbb{E}[1_{(0\lhd 1)}1_{(0\lhd 2)} \cdots 1_{(0\lhd k)}]$ for $m \le k \le m+n$. A similar argument holds for the second family. $\square$

This lemma is actually a special case of the next result, originally observed by Ryll-Nardzewski. For a proof we refer the reader to the recent book by Kallenberg [4], where several different ones are given. See also Aldous [1].

Lemma 3.7. *Suppose that a sequence of random variables $(X_k; k \in \mathbb{Z})$ is $\mathcal{I}$-invariant, in the sense that for any increasing function $f : \mathbb{Z} \mapsto \mathbb{Z}$*

$$(X_k; k \in \mathbb{Z}) \overset{\text{law}}{=} (X_{f(k)}; k \in \mathbb{Z}).$$

Then in fact $(X_k; k \in \mathbb{Z})$ are exchangeable — the sequence admits with probability one an empirical distribution and conditional on it the random variables are independent and identically distributed.

Proof of Theorem 3.4. We begin by observing that the variables (X_k, Y_k) exist by virtue of the exchangeability property of Lemma 3.6 and De Finetti's Theorem. The $\mathcal{I}$-invariance of the ordering implies that the law of the sequence of pairs (X_k, Y_k) is $\mathcal{I}$-invariant, and consequently

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we may deduce from Lemma 3.7 that it is, in fact, an exchangeable sequence. It follows from Lemma 3.5 that the empirical distributions for both X_k and Y_k must be quasi-uniform.

The next step is to show that the variables (X_k, Y_k) determine the ordering $\lhd$. Divide $\mathbb{Z} \setminus \{0\}$ into three classes.

$$U_0 = \{k : \text{ either } X_k > X_0 \text{ or } Y_k > Y_0\}$$
$$E_0 = \{k : X_k = X_0 \text{ and } Y_k = Y_0\}$$
$$B_0 = \{k : \text{ either } X_k < X_0 \text{ or } Y_k < Y_0\}.$$

Notice that, since $j \lhd k$ implies $X_k \geq X_j$ and $Y_k \geq Y_j$, we must have any element of U_0 ordered above any element of E_0 which in turn must be ordered above any element of B_0. Because of the exchangeability of X and Y, the three classes have limiting sizes:

$$|U_0| = \lim \frac{1}{N} \sum_{1}^{N} 1_{(k \in U_0)} = \lim \frac{1}{N} \sum_{-N}^{-1} 1_{(k \in U_0)}$$

and similarly for $|E_0|$ and $|B_0|$. The exchangeability of X and Y also implies that if the size $|E_0|$ of E_0 is zero then it is, in fact, empty. Otherwise the empirical distributions of X and Y have atoms at the values of X_0 and Y_0. We claim the restriction of $\lhd$ to E_0 either preserves or reverses the natural order: it then follows that in the former case: $Y_0 = X_0 - |E_0|$, while in the latter case: $Y_0 = X_0 + |E_0|$. This then establishes that the ordering $\lhd$ is determined by the sequence (X_k, Y_k) according to (3.3).

To prove the claim of the previous paragraph suppose that $0 < j < k$, and let p be the probability $\mathbb{P}\,(0 \lhd k \lhd j, \text{ and } j \in E_0)$. Now

$$\frac{1}{N-j} \sum_{r=j+1}^{N} 1_{(0 \lhd r \lhd j, \text{ and } j \in E_0)} = \frac{1_{(0 \lhd j \text{ and } j \in E_0)}}{N-j} \sum_{r=j+1}^{N} (1_{(r \lhd j)} - 1_{(r \lhd 0)}),$$

must converge to $1_{(0 \lhd j \text{ and } j \in E_0)}(Y_j - Y_0) = 0$ in L^1 (by bounded convergence), yet its expectation is, for all N, equal to p. This and similar versions show that if $r \in E_0$ then the only s between 0 and r with respect to $\lhd$ are also between 0 and r in the natural ordering. But now we may replace 0 by t in this statement; then by noting that if $t \in E_0$ then $E_t = E_0$ we deduce that whenever r and t both belong to E_0 then s being between them with respect to $\lhd$ implies s is between them with respect to the natural order.

To complete the proof of the theorem impose the condition on the joint empirical measure of (X, Y) of reducing to the iid case. The arguments in the previous step show that each X_k and Y_k must form a conjugate pair, and that the conditional distribution of $\lhd$ is $\mathbb{P}^\mu$ where μ is the empirical measure of X. □

4. Shuffling an infinite set of cards

The first half of this section is an account of what it might mean to shuffle an infinite set of cards. The state of an infinite pack of cards will be represented by an ordering of the natural numbers. The second half of the section considers classes of Markov processes (indexed by discrete time) taking values in the space of such orderings and shows how one such class is naturally associated with the class of $\mathcal{I}$-invariant orderings that we have studied in the previous section.

Recall the standard model for shuffling cards: n cards carrying *labels* 1 through to n each have a distinct *position* 1 through to n in the pack. We associate the state of the pack with a permutation ρ belonging to the permutation group on n objects $\mathcal{S}_n$. If $\rho(k) = m$ then we say that the card carrying label k is in position m in the pack. A completely randomized pack simply means choosing ρ according to the uniform measure on $\mathcal{S}_n$. A *shuffle* S is a possibly random permutation (belonging to $\mathcal{S}_n$!) of the positions in the pack. Thus $S(m) = m'$ means that the card that was in position m is moved to position m'. Consequently the state of the pack is changed from ρ to $S\rho$. In this way S induces a map $\hat{S} : \mathcal{S}_n \mapsto \mathcal{S}_n$ defined by $\hat{S}(\rho) = S\rho$. Such an $\hat{S}$ ignores the labelling of the pack. If r (*also* belonging to $\mathcal{S}_n$) is used to change the labels so that the card that now carries the label k is the card that previously carried the label $r(k)$ and we denote by $\hat{r}$ the induced map $\hat{r}(\rho) = \rho r$ then we obtain the commutation relation

$$\hat{S} \circ \hat{r} = \hat{r} \circ \hat{S}, \tag{4.1}$$

for all $r \in \mathcal{S}_n$. Moreover any map $\hat{S}$ that commutes with all relabellings is induced by some $S \in \mathcal{S}_n$.

We have rather laboured the point in the previous paragraph so as to motivate our model for shuffling an infinite pack of cards. We have seen that for a finite pack the permutation group plays three distinct roles — it describes the state of the pack, it gives rise to shuffles, and it can be used to relabel the pack. We proceed to the description of three different objects that play these roles in the infinite framework. First note that for a finite pack we may also specify the state of the pack by giving an ordering $\lhd^{(n)}$ of $\{1, \ldots, n\}$ related to our previous description by means of a permutation ρ via

$$k \lhd^{(n)} k' \quad \text{iff} \quad \rho(k) < \rho(k'). \tag{4.2}$$

With this approach we note that we may restrict the ordering $\lhd^{(n)}$ to the first $n-1$ cards to obtain an ordering $\lhd^{(n-1)}$. Moreover, if $\lhd^{(n)}$ is chosen uniformly then $\lhd^{(n-1)}$ is uniformly distributed also. Because of this consistency there is a unique measure λ on the space of total orderings of $\mathbb{N}$ so that the restriction $\lhd^{(n)}$ of the ordering to $\{1, \ldots, n\}$ is uniform. It is well known how to construct a random ordering distributed according to λ. Let $U_1, U_2, \ldots$ be an infinite sequence of independent random variables uniformly distributed on [0, 1]. Then put:

$$k \lhd k' \quad \text{iff} \quad U_k < U_{k'}. \tag{4.3}$$

It is immediate that the restriction $\lhd^{(n)}$ is uniform whence, by the uniqueness property of the projective limit, $\lhd$ has λ as its distribution. Notice that, for each k,

$$U_k = \lim_{n\to\infty} \frac{1}{n} \sum_{i\le n, i\ne k} 1_{(i \lhd k)} \quad \text{a.s.}, \tag{4.4}$$

and thus we may regard U_k as being the (relative) position of the card carrying label k. A slightly different way of thinking about this: Eqs. (4.3) and (4.4) set up a measure isomorphism between the space of total orderings endowed with λ and the space $[0, 1]^\infty$ endowed with the infinite product of uniform measure on [0, 1]. It is often much easier to think about things in the $[0, 1]^\infty$ world, as we shall see.

Suppose that r is an arbitrary bijection of $\mathbb{N}$ onto itself and define the induced relabelling $\hat{r}$ via

$$k \overset{\hat{r}}{\lhd} k' \quad \text{iff} \quad r(k) \lhd r(k'). \tag{4.5}$$

It is easy to see that $\hat{r}$ preserves the uniform measure λ and in fact this invariance property characterizes λ. It is also true that if we use (4.4) to define a random variable U_k on the space of orderings then

$$U_k \circ \hat{r} = U_{r(k)} \tag{4.6}$$

almost surely under λ. This means that the action of a relabelling on the space $[0, 1]^\infty$ is just to permute the coordinates. To see Eq. (4.6), just note that given r, for λ almost all $\lhd$, we may define a new order $\lhd'$ by $k \lhd' k'$ iff $U_{r(k)} < U_{r(k')}$ which agrees with $\overset{\hat{r}}{\lhd}$.

Suppose that $\hat{S}$ is a map from $\tilde{\mathcal{O}}$, the space of orderings of $\mathbb{N}$, into itself. We will call $\hat{S}$ a shuffle when the commutation property (4.1) holds λ almost surely for each relabelling of the infinite pack as defined in the previous paragraph. In this case, by an application of the Hewitt–Savage zero–one law, there exists a function, $S : [0, 1] \mapsto [0, 1]$, preserving Lebesgue measure, such that, for each k,

$$U_k \circ \hat{S} = S \circ U_k \tag{4.7}$$

λ almost surely. Moreover, each such S corresponds to some $\hat{S}$.

We define a random shuffle as a suitable generalization of such functions:

Definition 4.1. A random shuffle is described by a transition kernel κ on the space of orderings of $\mathbb{N}$, satisfying the following generalization of the commutation relation for any r: whenever A is a measurable subset of the space of total orderings, and $\hat{r}$ a relabelling,

$$\kappa(\hat{r}(\lhd), \hat{r}(A)) = \kappa(\lhd, A) \quad \text{for } \lambda \text{ almost all } \lhd\ . \tag{4.8}$$

Here we have written $\hat{r}(\lhd)$ for the ordering $\overset{\hat{r}}{\lhd}$. We remark that in this definition a relabelling is an arbitrary bijection of $\mathbb{N}$, but the following theorem will show that in fact we could have restricted ourselves to relabellings that permute only finitely many labels.

As with deterministic shuffles, we can express κ using the card positions.

Theorem 4.2. *Suppose that ν is a probability measure on $[0, 1]^2$ having both marginals uniform on $[0, 1]$. Take a sequence of independent pairs of random variables $((U_1, V_1), \ldots, (U_k, V_k), \ldots)$, each pair distributed according to ν. This then determines, by virtue of (4.3), the joint law of a pair of orderings $(\lhd, \lhd')$. Take $\hat{\nu}(\lhd, \cdot)$ to be a regular conditional probability for $\lhd'$ given $\lhd$. Then $\kappa = \hat{\nu}$ satisfies the commutation relation (4.8). Moreover, any κ satisfying the relation (4.8) is a mixture of kernels constructed in this manner.*

Proof. Suppose that (U_k, V_k) for $k \geq 1$ form a sequence of independent pairs of random variables, each pair having the distribution ν on $[0, 1]^2$. Then the sequence of independent uniform variables $(U_k; k \geq 1)$ gives rise to, with probability one, an ordering $\lhd$ distributed according to λ, and similarly $(V_k; k \geq 1)$ gives rise to an ordering $\lhd'$. Fix a relabelling r. Since the ordering $\hat{r}(\lhd)$ corresponds to the sequence of random variables $\tilde{U}_k = U_{r(k)}$ and similarly the ordering $\hat{r}(\lhd')$ corresponds to the sequence of random variables $\tilde{V}_k = V_{r(k)}$ we see that:

$$(\hat{r}(\lhd), \hat{r}(\lhd')) \overset{\text{law}}{=} (\lhd, \lhd').$$

From this it follows that $\hat{\nu}$, defined as a regular conditional probability for $\lhd'$ given $\lhd$, satisfies (4.8).

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To see the last claim of the theorem, suppose that κ satisfies (4.8), and consider a pair of orderings $(\lhd, \lhd')$ determined as follows. Let $\lhd$ be distributed according to λ, and let the conditional distribution of $\lhd'$ given $\lhd$ be $\kappa(\lhd, \cdot)$. It follows from the invariance of λ under relabellings and (4.8) that, for any relabelling r,

$$(\hat{r}(\lhd), \hat{r}(\lhd')) \stackrel{\text{law}}{=} (\lhd, \lhd').$$

Now, as we remarked above, λ is characterized by its invariance under relabellings and so the law of $\lhd'$ must also be λ. Let the card positions corresponding to $\lhd$ be $(U_1, \ldots, U_k, \ldots)$, and those corresponding to $\lhd'$ be $(V_1, \ldots, V_k, \ldots)$. Then the sequence of pairs $((U_1, V_1), \ldots, (U_k, V_k), \ldots)$ is exchangeable. So the sequence admits a random empirical measure Ξ on $[0, 1]^2$. Let the law of Ξ be

$$\int \alpha(\mathrm{d}\nu)\nu,$$

the integral being with respect to a probability measure α on the space of probability measures on $[0, 1]^2$. Applying the strong law of large numbers to each of the sequences U_k and V_k, we deduce that the marginals of Ξ are, with probability one, uniform. Thus α must be supported on the set of measures ν having uniform marginals. Finally observe that, since, conditional on $\Xi = \nu$, the pairs (U_k, V_k) are independent and distributed as ν,

$$\kappa(\lhd, \cdot) = \int \alpha(\mathrm{d}\nu)\hat{\nu}(\lhd, \cdot),$$

for λ almost all $\lhd$. By choosing an appropriate version for $\hat{\nu}$ we can obtain equality for all $\lhd$. □

Notice how this relates to the representation, (4.7), of deterministic shuffles. Corresponding to a measure-preserving $S : [0, 1] \mapsto [0, 1]$ is the ν with

$$\nu(A \times B) = \int_A 1_B(S(x))\mathrm{d}x, \quad \text{for any Borel subsets } A \text{ and } B \text{ of } [0, 1]. \tag{4.9}$$

We turn now to the topic of shuffles on $\mathbb{N}$ that are generalizations of the GSR riffle shuffle. Shuffling an infinite set of cards was really a two-step procedure. First we built the pack as the limit of a consistent family of finite packs, then we discussed appropriate transformations of the limiting object as shuffles. But we could do this differently. In what follows we consider transformations on the finite packs first, look for some consistency of the resulting processes, and then we pass to the limit.

Suppose that $(\rho_h^{(n)}; h \geq 0)$ is a random walk on $\mathcal{S}_n$, starting from a uniformly chosen $\rho_0^{(n)}$. Think of this as describing the state of a pack of n cards at times $h = 0, 1, 2, \ldots$. Now let $m < n$ and imagine that only the cards carrying the labels $1, 2 \ldots, m$ are observed. Recall that, via (4.2), $\rho_h^{(n)}$ determines an ordering $\lhd_h^{(n)}$ and let $\lhd_h^{(m)}$ be the restriction of this ordering to $1, 2 \ldots, m$. Then, using (4.2) again, we associate with $\lhd_h^{(m)}$ a permutation $\rho_h^{(m)}$ belonging to $\mathcal{S}_m$. Clearly, for each h we have that $\rho_h^{(m)}$ is uniformly distributed but it is easy to construct examples so that the process $(\rho_h^{(m)}; h \geq 0)$ is not a random walk. What are the weakest conditions that must be placed on the jump distribution of $\rho^{(n)}$ to ensure that it *is* a random walk? We do not know. But here are two special cases for which it works.

Case 1: $\lhd_{h+1}^{(m)}$ is conditionally independent of $\lhd_h^{(n)}$ given $\lhd_h^{(m)}$.

Case 2: $\lhd_{h-1}^{(m)}$ is conditionally independent of $\lhd_h^{(n)}$ given $\lhd_h^{(m)}$.

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It is immediate that if case 1 holds then $(\rho_h^{(m)}; h \geq 0)$ is a random walk, and, of course, case 2 is just case 1 run backwards!

Now what we really want to do is construct an infinite family of *random walks* $((\rho_h^{(n)}; h \geq 0); n \geq 1)$ so that the associated orderings $\lhd_h^{(n)}$ are consistent, that is, $\lhd_h^{(m)}$ is the restriction of $\lhd_h^{(n)}$ whenever $m < n$. Such a consistent family of processes determines a limiting process $(\lhd_h; h \geq 0)$ taking values in the space of orderings of $\mathbb{N}$. Such a process is Markovian with a transition kernel κ which satisfies (4.8). When we express κ as a mixture:

$$\kappa(\lhd, \cdot) = \int \alpha(\mathrm{d}\nu)\hat{\nu}(\lhd, \cdot), \tag{4.10}$$

the measure α is supported on ν having a special form. We investigate this special form next with the help of $\mathcal{I}$-invariant orderings.

Definition 4.3. Let us say that $(\lhd_h; h \geq 0)$ is of type 1 if case 1 holds for each pair $m < n$, and let us say it is of type 2 if case 2 holds for each pair $m < n$. We always assume that $\lhd_h$ is distributed according to λ. We call the corresponding kernels type 1 and type 2 shuffles.

Remark 4.4. It is obvious from the preceding discussion that if κ is a type 1 shuffle then $\kappa(\lhd, \{\tilde{\lhd} : \tilde{\lhd}^{(n)} = \sigma\})$ is λ almost surely constant over $\{\lhd: \lhd^{(n)} = \rho\}$. We denote the common value by $\kappa_n(\rho, \sigma)$. It is clear from the preceding comments that κ is determined by the $(\kappa_n)_{n\geq 1}$.

Suppose that μ is any quasi-uniform measure. Let X and Y be a conjugate pair of random variables with the law of X being μ. Let U be an independent random variable uniformly distributed on [0, 1], and let ν_μ be the law of the pair $(U, UX + (1-U)Y)$. It is easy to check that the measure ν_μ has uniform marginals and so, by Theorem 4.2, there is a corresponding kernel, satisfying (4.8), which we denote by $\hat{\nu}_\mu$.

Proposition 4.5. *There is a one-to-one correspondence between the laws of $\mathcal{I}$-invariant orderings and the type 1 shuffles. Under this bijection, the law of the ordering*

$$\int \theta(\mathrm{d}\mu)\mathbb{P}^\mu,$$

corresponds to the kernel

$$\int \theta(\mathrm{d}\mu)\hat{\nu}_\mu.$$

Proof. Suppose that a type 1 shuffle is given by a kernel κ. Then we obtain the law of an $\mathcal{I}$-invariant ordering $\lhd$ of $\mathbb{Z}$, which we denote $P(\kappa)$, as follows. Given integers $k_1 < k_2 < \cdots < k_n$ and a permutation $\rho \in \mathcal{S}_n$ then the probability that $\lhd$ orders k_i so that

$$k_i \lhd k_j \quad \text{iff} \quad \rho(i) < \rho(j) \tag{4.11}$$

is the probability that $\rho_{h+1}^{(n)}(\rho_h^{(n)})^{-1} = \rho$. Notice that $\lhd$ is defined in such a way as to be automatically $\mathcal{I}$-invariant. In checking that this definition of $\lhd$ is meaningful we need the conditional independence asserted by case 1.

In the converse direction, suppose we are given the law $\mathbb{P}$ of an $\mathcal{I}$-invariant order $\lhd$. By Theorem 3.4 this is a mixture:

$$\int \theta(\mathrm{d}\mu)\mathbb{P}^\mu, \tag{4.12}$$

for some probability measure θ on the space of probability measures on [0, 1]. In fact, θ is the law of the random empirical measure, $\mu(X)$, of Theorem 3.4, and is supported on the set of quasi-uniform measures. We define a kernel $\mathcal{K}(\mathbb{P})$, by

$$\int \theta(\mathrm{d}\mu)\hat{\nu}_\mu.$$

We must check that $\mathcal{K}(\mathbb{P})$ is a type 1 shuffle. Recall that $\hat{\nu}_\mu$ is the regular conditional probability law for $\lhd'$ given $\lhd$, where $\lhd$ and $\lhd'$ are the orders of $(U_n)_{n\geq 1}$ and $(V_n)_{n\geq 1}$ respectively and the $(U_n, V_n)_{n\geq 1}$ are iid with common law ν_μ. The result now follows from the fact (which we leave to the reader to check) that the order of $V_1, \ldots, V_n$ is independent of $(U_k)_{k\geq 1}$, conditional on the order of $U_1, \ldots, U_n$.

All that remains is to establish that the maps P and $\mathcal{K}$ satisfy

$$P \circ \mathcal{K} = id \tag{4.13}$$

and

$$\mathcal{K} \circ P = id. \tag{4.14}$$

To establish (4.14), given a type 1 shuffle, κ, set $\hat{\kappa} = \mathcal{K} \circ P(\kappa)$. Recall from Remark 4.4 that κ is characterized by $(\kappa_n(id, \rho); \rho \in S_n, n \geq 1)$ and observe that

$$\kappa_n(id, \rho) = \mathbb{P}(\lhd^{(n)} = \rho) = \int \theta(\mathrm{d}\mu)\mathbb{P}^\mu(\lhd^{(n)} = \rho) = \int \theta(\mathrm{d}\mu)\hat{\nu}_\mu(id, \rho) = \hat{\kappa}_n(id, \rho),$$

where $\mathbb{P} = P(\kappa)$. The proof of (4.13) is similar. □

Remark 4.6. The proof of Proposition 4.5 now makes clear the role of quasi-uniform pairs μ, μ' in constructing type 1 shuffles. An extremal type 1 shuffle is realized by taking the appropriate quasi-uniform μ, constructing a corresponding sequence of conjugate iid pairs (X_n, Y_n) and then setting $V_n = U_n X_n + (1 - U_n)Y_n$, where U_n and V_n are, respectively, the initial and final positions of card n. This definition still makes sense even if there are ties in final card positions. For suppose that $V_n = V_m$; notice that this can only happen if either the corresponding initial positions are the same and the corresponding conjugate pairs (X_n, Y_n) and (X_m, Y_m) are equal or if the initial positions take values in {0, 1} and the conjugate pairs lie on adjacent components of G. In the latter case we resolve the tie by ordering m above n iff (X_m, Y_m) belongs to the higher/rightmost component of G. In the former case, we preserve the initial ordering between m and n if $Y_n = Y_m < X_m = X_n$ and otherwise reverse it (just as in (3.3)). The corresponding kernel, ν_μ, is defined on all $\lhd \in \tilde{\mathcal{O}}$, and $\nu_\mu(\lhd, \cdot)$ is a probability measure on $\tilde{\mathcal{O}}$ for every $\lhd$. Under $\mathbb{P}^\mu$, the law of the restriction of $\lhd$ to $\mathbb{N}$ is equal to $\nu_\mu(id, \cdot)$.

For type 2 shuffles the story is similar. An $\mathcal{I}$-invariant ordering is determined as follows. For integers $k_1 < k_2 < \cdots < k_n$ and a permutation $\rho \in \mathcal{S}_n$ then the probability that $\lhd$ orders k_i so that

$$k_i \lhd k_j \quad \text{iff} \quad \rho(i) < \rho(j) \tag{4.15}$$

is the probability that $\rho_h^{(n)}(\rho_{h+1}^{(n)})^{-1} = \rho$. The law of this ordering determines the transition kernel as in the proof of Proposition 4.5. If μ is a quasi-uniform measure, let ν^μ be the measure on $[0, 1]^2$ defined by

$$\nu^\mu(\mathrm{d}x, \mathrm{d}y) = \nu_\mu(\mathrm{d}y, \mathrm{d}x), \tag{4.16}$$

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and let $\hat{\nu}^\mu$ be the associated kernel.

Proposition 4.7. *There is a one-to-one correspondence between the laws of $\mathcal{I}$-invariant orderings and type 2 shuffles. Under this bijection, the law of the ordering*

$$\int \theta(\mathrm{d}\mu)\mathbb{P}^\mu,$$

corresponds to the kernel

$$\int \theta(\mathrm{d}\mu)\hat{\nu}^\mu.$$

Proof. The result follows immediately from Proposition 4.5 by time reversal. □

It is this result which generalizes the GSR shuffle. In particular, if the Markov chain corresponds to a measure θ which puts all of its mass on a single quasi-uniform measure μ, and μ is purely atomic, then there exists a function $S : [0,1] \mapsto [0,1]$ such that (4.9) holds for ν^μ, and the chain is in fact deterministic and is obtained by iterating the shuffle $\hat{S}$ associated by Eq. (4.7) with S. If G corresponding to μ is decomposed as

$$G = \bigcup_n (l_n, L_n) \cup \bigcup_m (r_m, R_m),$$

where the atoms of μ are situated on $\{l_n;\ n \geq 1\} \cup \{R_m;\ m \geq 1\}$ then, at least for $x \in G$,

$$S(x) = \sum_n \frac{L_n - x}{L_n - l_n} 1_{(l_n, L_n)}(x) + \sum_m \frac{x - r_m}{R_m - r_m} 1_{(r_m, R_m)}(x).$$

Thus, for example, the GSR shuffle corresponds to the quasi-uniform measure with atoms of size $\frac{1}{2}$ at $\frac{1}{2}$ and 1.

We end with the following question. Do there exist Markov processes, other than those constructed above, on the space of orderings of $\mathbb{N}$ such that, for each n, the restriction of the ordering to $\{1, \dots, n\}$ evolves as if induced by a random walk on $\mathcal{S}_n$?

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