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When are plurality rule voting games dominance-solvable?

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Abstract

This paper studies the dominance-solvability (by iterated deletion of weakly dominated strategies) of plurality rule voting games. For any number of alternatives and at least four voters, we find sufficient conditions for the game to be dominance-solvable (DS) and *not* to be DS. These conditions can be stated in terms of only one aspect of the game, the largest proportion of voters who agree on which alternative is worst in a sequence of subsets of the original set of alternatives. When the number of voters is large, "almost all" games can be classified as either DS or not DS. When the electorate is sufficiently replicated, then if the game is DS, a Condorcet winner always exists, and the outcome is the Condorcet winner.

JEL classification: C72; D72; D71

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1. Introduction

Plurality voting is the dominant electoral rule in many democracies. Nevertheless, its properties are still not well-understood. One major problem is that with plurality voting, there are often incentives for voters to vote strategically[1] (i.e., not for their

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[1] In practice, strategic (non-sincere) voting seems to be quite common where plurality rule voting is used. For example, in parliamentary elections in the UK and Germany evidence suggests that candidates who were perceived to be running third were deserted by their supporters (Cox, 1997, Chapter 4). Moreover there is some experimental evidence that voters do vote strategically in three candidate elections (Forsythe et al., 1996).

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most preferred alternative). But then, with strategic voting, multiple voting equilibria are pervasive. For example, consider the "canonical" plurality voting game where voters vote simultaneously, preferences are common knowledge, and ties are broken fairly.[2] It is obvious that with at least three voters, *any* candidate may win in a Nash equilibrium: if all other voters vote for this candidate, then it is a (weak) best response for any voter to also vote for that candidate, as she cannot affect the outcome, however she votes. The multiple equilibrium problem also arises[3] when agents have incomplete information about some aspect of the structure of the game (Myatt, 1999; Myerson and Weber, 1993; Myerson, 2002).

The reason this problem arises is that Nash equilibrium allows *any* possible beliefs on the part of voters, as long as they are consistent. For example, suppose that it is common knowledge that a candidate, z is worst for all voters. Nevertheless, there is a Nash equilibrium where every voter votes for z because he believes that all other voters will vote for z. The obvious response to this problem is to look for equilibrium refinements, such as ruling out weakly dominated voting strategies (Besley and Coate, 1997). However, it turns out that standard refinements have little bite in this canonical plurality rule game. For example, De Sinopoli (2000) shows that with more than four voters, if an alternative is not a strict Condorcet loser, there is a perfect Nash equilibrium where that alternative is an outcome with probability at least 0.5. Moreover, there is by definition only one strict Condorcet loser in any set of alternatives. It follows from this result that imposing the weaker refinement of weakly undominated Nash equilibrium (as Besley and Coate do) can rule out at most one alternative as a Nash outcome.

We take a different approach to this problem of multiplicity of Nash equilibria in this paper. First, we argue below that eliminating weakly dominated strategies is very reasonable in the plurality rule game; it simply amounts to no-one voting for her worst-ranked alternative.[4] But, there is nothing to stop voters going a step further and recalculating which strategies are weakly dominated for them given that *other* voters will not use weakly dominated strategies. In other words, if we *iteratively* eliminate weakly dominated strategies, it is possible that we could substantially narrow down the set of possible outcomes in the plurality voting game. Indeed, it is possible that so many strategies could be eliminated via iterated deletion that the remaining strategies can generate only one outcome: that is, the plurality voting game could be *dominance-solvable*.

Our paper investigates the conditions under which the plurality rule voting game is dominance solvable. The main contribution is to derive conditions that are sufficient for the game to be dominance-solvable and *not* to be dominance-solvable. Moreover, as the

[2] If there are K alternatives with the most number of votes, then each of these alternatives is selected as winner with probability $1/K$ (Myerson, 2002).

[3] For example, Myerson (2002) studies "scoring" voting rules (of which plurality voting is a special case) in an environment where there are three alternatives, each voter is equally likely to have three possible preference orderings over these alternatives (in the base case), and the number of voters is a Poisson random variable. The equilibrium is defined for the limiting case as the expected number of voters becomes large, and allows voters to make small mistakes. Even in this setting, plurality rule generates multiple equilibria; there is an equilibrium where any one of the three candidates can win with probability one.

[4] Lemma 1 below shows that with more than three voters, the only voting strategy that is weakly dominated is the one where the voter votes for her worst alternative.

number of voters, n, becomes large, these conditions are asymptotically necessary and sufficient for dominance-solvability. The conditions are most easily stated in the case of three alternatives,[5] when they involve just one summary statistic of the game, namely the largest fraction of players that agree on which alternative is worst, q. When this fraction is greater than $2/3$, the game is *always* dominance-solvable; when this fraction is less than or equal to $2/3 - \pi_n$, for some $\pi_n > 0$, the game is *never* dominance-solvable. Moreover, π_n goes to zero asymptotically with n, i.e., the number of voters.

The intuition for the sufficiency condition is straightforward. First, voting for one's worst alternative is weakly dominated, so if a sufficient fraction of the voters agree on which is worst, all voters can deduce that this alternative cannot win if voters do not vote for weakly dominated alternatives. But if this alternative cannot win, a vote for it is "wasted," i.e., weakly dominated *wherever* it appears in a voter's preference ordering, so the game is reduced to one of just two alternatives by iterated deletion, and two-alternative voting games are always dominance-solvable.

The intuition behind the sufficient condition for the game *not* to be dominance-solvable is more subtle. When sufficient disagreement on the worst alternative is allowed, the space of weakly undominated strategy profiles is rich enough to ensure that for any voter i, voting for her middle-ranked (or best) alternative is a unique best response (i.e., not weakly dominated) to *some* weakly undominated profile of voting strategies of the other players. This means that iterated deletion cannot proceed beyond deleting the strategy of voting for one's worst alternative.

Moreover, if we increase the number of voters without changing the distribution of preferences across alternatives (replicating the electorate), for a large enough electorate, we can find necessary *and* sufficient conditions for the game to be dominance-solvable. Finally, when the sufficient conditions for dominance-solvability hold, we show that the *only* strategies that survive iterated deletion involve *every* voter voting for one of two alternatives (a strong form of Duverger's Law, Cox (1997)), and moreover, every voter votes "sincerely" over this pair, i.e., for her more-preferred alternative of the two.

A key question is the nature of the winning alternative(s) when the game is dominance-solvable. Here, we show the following. When the number of voters is at least four, dominance-solvability implies that a Condorcet Winner (CW) exists, and if the sufficient conditions for dominance-solvability of the game also hold, then the solution is a CW. Indeed, when the electorate is sufficiently replicated, a sharper result is possible: the outcome is *always* a CW whenever the game is dominance-solvable. However, with three voters, even if the game is dominance-solvable, and a CW exists, the outcome may not be the CW!

[5] In the general case with n voters and K alternatives, let q_K be the largest fraction of players who agree on which alternative (say z_K) is worst. When z_K is deleted from the feasible set, let q_{K-1} be the largest fraction of players who agree on which remaining alternative (say z_{K-1}) is worst, and so on. This procedure generates a sequence $q_K, q_{K-1}, \ldots, q_3$. Our sufficient condition for dominance-solvability is that each element in the sequence be sufficiently large: $q_k > (k-1)/k$. Our sufficient condition for non-dominance-solvability is that there is an $3 \leqslant l \leqslant K$, such that for all $k > l$, q_k is sufficiently large, but q_l fails to be sufficiently large (i.e., $q_l \leqslant ((l-1)/l) - \pi_n^l$, for some $\pi_n^l > 0$). Moreover, as $n \to \infty$, $\pi_n^l \to 0$.

This paper builds on an established literature. It has long been recognized that iterated deletion of weakly dominated strategies may be a powerful tool for predicting outcomes in voting games. In a seminal contribution, Farquharson (1969) called this procedure "sophisticated voting," and he called a voting game "determinate" if sophisticated voting led to a unique outcome. However, this procedure has recently received perhaps less attention than it merits. This may be for two reasons. First, generally, the *order* of deletion of weakly dominated strategies matters. We deal with this by assuming that voters have strict preferences over alternatives; this is sufficient to ensure that order of deletion does not affect the outcomes (Marx and Swinkels, 1997). Second, until recently, game theory has lacked a "common knowledge" justification[6] as to why players would not play iteratively weakly dominated strategies: the recent work of Rajan (1998) fills that gap.

More recent related literature[7] is as follows. The only work of which we are aware on refinements of Nash equilibrium with plurality voting is De Sinopoli (2000), as described above. De Sinopoli and Turrini (2002) showed that iterated deletion of weakly dominated strategies may be applied to eliminate some of the Nash equilibria in the citizen-candidate model of Besley and Coate (1997). They show that in a four candidate example, that iterated weak dominance eliminates all the Nash equilibria except for one. Dhillon and Lockwood (2002), building on the results of their paper, show that this possibility is restricted to the case of four (or more) candidate equilibria: for any political equilibrium with up to three candidates, one can find another equilibrium with an identical outcome where strategies at the voting stage are iteratively weakly undominated.

The layout of the paper is as follows. The model is outlined in Section 2. Our analysis of the three alternative case is in Section 3, and the more general case in Section 4. Section 5 discusses some extensions and concludes.

2. The model

2.1. Preliminaries

There is a set $N = \{1, \ldots, n\}$ of voters with $n \geqslant 4$ and a set $X = \{x_1, \ldots, x_K\}$ of alternatives. The voting game is as follows. Each voter has one vote, which she can cast for any one of the K alternatives (i.e., no abstentions are allowed). The alternative with the largest number of votes wins (plurality rule). If two or more alternatives have the greatest number of votes, the tie-breaking rule is that every alternative in this set is selected with equal probability. All voters vote simultaneously.

[6] It is well known that if the structure of the game and rationality of the players are common knowledge, then players must play only those strategies that survive iterated deletion of *strictly* dominated strategies.

[7] A more weakly related literature is as follows. Borgers (1992), Borgers and Janssen (1995) have results on the dominance-solvability of Bertrand and Cournot games. For example, Borgers (1992) shows that in a model of Bertrand price competition, under some conditions, the set of prices that survive iterated deletion is close to the Walrasian price, and Borgers and Janssen (1995) have similar results for the Cournot case. More recently, Mariotti (2000) has provided a class of games (called maximum games) which are dominance solvable.

Let $\mathcal{L}$ denote the set of lotteries (i.e., probability distributions) over X. By the tie-breaking rule just stated, the set of possible outcomes with plurality voting is the subset of *homogeneous* lotteries $\mathcal{L}_H \subset \mathcal{L}$. Here, $L \in \mathcal{L}_H$ iff for some $Y \subseteq X$, every alternative in Y has probability $1/\#Y$, and every alternative not in Y has probability zero. Voter $i \in N$ has a preference ordering over $\mathcal{L}$, denoted $\succeq_i$, which is assumed to satisfy the von Neumann–Morgenstern axioms, implying a utility representation with utility function $u_i : X \to \Re$.

This game can be written more formally in strategic form as follows. Let $V_i = X$ be the strategy set of i, with generic element v_i. If $v_i = x_k$, voter i votes for alternative x_k. Let v be the strategy profile $v = (v_1, \ldots, v_n)$. Let $\omega_k(v)$ be the number of votes for alternative x_k if the strategy profile is v. Also, let the *winset* $W(v) \subset X$ be defined as

$$W(v) = \left\{x_k \in X \;\middle|\; \omega_k(v) \geqslant \omega_l(v),\ x_l \in X\right\}.$$

This is the set of alternatives that receive the most number of votes. Every alternative in $W(v)$ wins with equal probability.

So, given the assumptions on preferences, we can write the expected utility of i as a function of the strategy profile v as

$$Eu_i(v) = \frac{1}{\#W(v)} \sum_{x_k \in W(v)} u_i(x_k).$$

This completes the description of the plurality rule game in strategic form. We denote the game formally by $\Gamma = (u_i, V_i)_{i \in N}$ where of course $V_i = X$, so sometimes we write $\Gamma = (u_i, X)_{i \in N}$. Finally, we will assume:[8]

(A1) Every voter has strict preferences over $\mathcal{L}_H$, i.e., for all $L, L' \in \mathcal{L}_H$, either $L \succ_i L'$ or $L' \succ_i L$.

An immediate implication of (A1) is that no player is indifferent between any two different winsets, i.e., for all strategy profiles v, v' if $W(v) \neq W(v')$, then $Eu_i(v) \neq Eu_i(v')$, $i \in N$. In Section 2.2 below, we show that this fact implies that the order of deletion of weakly dominated strategies does not matter. Note that (A1) holds generically as it only rules out a finite set of equalities.

The following notation will be useful. Let $\omega(v_{-i})$ be a vector recording the total votes for each alternative in X given a strategy profile v_{-i}, i.e., when individual i is not included. Also, let $\Omega_{-i} = \{\omega(v_{-i}) \mid v_{-i} \in V_{-i}\}$. We suppress the dependence of ω on v_{-i} except when needed by writing $\omega(v_{-i}) = \omega_{-i}$, and refer to ω_{-i} as a *vote distribution*. Clearly i's best response to v_{-i} depends only on the information in ω_{-i}.

Finally, define an alternative $x \in X$ to be a *Condorcet winner* (CW) if $\#\{i \in N \mid x \succ_i y\} \geqslant \#\{i \in N \mid y \succ_i x\}$, all $y \neq x$, and say that x is a *strict* CW if all the inequalities hold strictly, and *weak* otherwise. As we have assumed strict preferences, if the number of voters, n, is odd, the CW is strict, i.e., unique, but if n is even, this is not necessarily the case (Moulin, 1983, p. 29). In the former case, denote the unique CW by x^{cw}, and in the latter case, denote the set of CWs by X^{cw}.

[8] This form of (A1) is due to the neutral tie-breaking rule, which generates lotteries over X. If a deterministic tie-breaking rule were used, then it would be sufficient to assume that preferences over X were strict.

Two comments are in order at this point. First, we do not allow voters to abstain; this is without loss of generality because abstention is always a weakly dominated strategy for any voter (Brams, 1994), and so will be deleted at the first round of the iterated deletion process. Second, we have assumed at least four voters: the case of three voters is somewhat special, and is covered in detail in Dhillon and Lockwood (1999).

2.2. Iterated deletion of weakly dominated strategies

By our assumption (A1), the transference of decision-maker indifference (TDI) condition of Marx and Swinkels (1997) is satisfied in the game $\Gamma = (u_i, X)_{i\in N}$. TDI says that if a player i is indifferent between two strategy profiles v, v' differing in his own strategy only, then all other players $j \neq i$ are also indifferent between v, v'. This is satisfied in our game because by (A1), i can be indifferent between v, v' only if $W(v) = W(v')$, in which case other players are indifferent also. Then, let $V^\infty, \widehat{V}^\infty$ be two sets of strategy profiles obtained by iterated deletion of weakly dominated strategies (with different orders of deletion) in the plurality voting game. By Corollary 1 of Marx and Swinkels (1997), $V^\infty, \widehat{V}^\infty$ only differ by the addition or removal of strategies that are (for any player i) payoff equivalent to some other strategy (of player i) in $V^\infty, \widehat{V}^\infty$, respectively (Marx and Swinkels, 1997, Definition 5). By (A1), payoff-equivalent strategies must give the same outcome. So, the set of winsets generated by $V^\infty, \widehat{V}^\infty$ is the same, i.e., if $W(S) = \{W(v) \mid v \in S\}$, then $W(V^\infty) = W(\widehat{V}^\infty)$. In this sense, the order of deletion of weakly dominated strategies does not matter.

However, for expositional convenience, for the most part, we will assume an order of deletion as in Moulin (1983). Let $NWD_i(S_i, S_{-i}) \subseteq V_i$ be the set of strategies for i which are not weakly dominated by any $v_i' \in S_i$, given $S_{-i} \subseteq V_{-i}$. That is, $v_i \in NWD_i(S_i, S_{-i})$ has the property that there is not any $v_i' \in S_i$ with

$$u_i\bigl(v_i', v_{-i}\bigr) \geqslant u_i(v_i, v_{-i}) \quad \forall v_{-i} \in S_{-i}, \tag{1}$$

where the inequality in (1) is strict for some $v_{-i} \in S_{-i}$. Let $V_i^0 = V_i$, and define recursively

$$V_i^m = NWD_i\bigl(V_i^{m-1}, V_{-i}^{m-1}\bigr), \quad i \in N,\ m = 1, 2, \ldots. \tag{2}$$

Also, say that a v_i is *weakly dominated relative to* V^{m-1} if it is not in V_i^m. As X is finite, this algorithm converges after a finite number of steps to V^∞, the set of iteratively weakly undominated strategy profiles. The set of *iteratively weakly undominated winsets* is $W(V^\infty) = \{W(v) \mid v \in V^\infty\}$. The game is said to be *dominance-solvable* (*DS*) if $W(V^\infty)$ contains a single element, W^∞ which we refer to as the *solution winset*. If alternative $x \in W^\infty$, it is a *solution outcome*. We distinquish between a solution winset and a solution outcome as the former, in general, may contain several alternatives.

2.3. Characterizing undominated strategies

The following useful preliminary result characterizes weakly dominated voting strategies in the plurality voting game.

Lemma 1. *In the plurality voting game $\Gamma = (u_i, X)_{i \in N}$, voting for one's worst alternative is the only weakly dominated strategy.*

This generalizes existing results, which show that in the plurality rule game, the strategy of voting for one's worst alternative is always weakly dominated, and the strategy of voting for one's best alternative is never weakly dominated (Brams, 1994). This result, along with all others, is proved in the Appendix. The intuition for the result is simply[9] that the set of preference profiles is rich enough so that when i votes for *any* alternative in X except his worst-ranked, we can find a $v_{-i} \in V_{-i}$ such that this strategy for i is a unique best response to v_{-i}.

3. Results for three alternatives

The case of three alternatives is of course special, but in this case, our results can presented in a simple and intuitive way, which helps prepare for discussion of the general many-alternative case in the next section. Moreover, comparative studies of voting systems tend to work with the three-alternative case as it is simplest case that serves to differentiate alternative systems (e.g., majority voting, plurality voting, approval voting)—see, for example, Myerson and Weber (1993), Myerson (2002)—and it is also the simplest case where strategic voting may occur. In practice, some important political contests typically have three candidates or less, e.g., presidential elections in the US (Levin and Nalebuff, 1995).

3.1. Sufficient conditions for dominance-solvability and non-dominance solvability

Let the set of alternatives be $X = \{x, y, z\}$. Let N_x, N_y, N_z be the sets of voters that rank x, y or z respectively as worst, and let n_x, n_y, n_z be the numbers of voters in each set. Also, define $q = \max_{a \in X} n_a/n$; this is the largest fraction of voters who agree on which alternative is worst, and let $b = \arg\max_{a \in X} n_a$ be the alternative that most rank worst. So, b is easily remembered as denoting a "bottom-ranked" alternative.

Now define a critical value of q as:

$$q_n = \begin{cases} 1 - \dfrac{1}{n} - \dfrac{1}{n}\left[\dfrac{n+1}{3}\right], & n \text{ odd}, \\ 1 - \dfrac{1}{n}\left[\dfrac{n+2}{3}\right], & n \text{ even}, \end{cases} \tag{3}$$

where $[x]$ denotes the smallest integer greater than or equal to x. Note that $q_n < 2/3$, and $\lim_{n\to\infty} q_n = 2/3$. Finally, say that in game $\Gamma = (u_i, X)_{i \in N}$, preferences are *polarized over alternative* $x \in X$ if there is an $M \subset N$ such that all $i \in M$ rank x highest, and $i \in N/M$ rank x lowest. Preferences over alternative x are *non-polarized* otherwise.

[9] This requires the assumption of at least four players: the case of three voters and three alternatives, whether a voter's middle-ranked alternative is weakly dominated or not depends on cardinal preferences (Dhillon and Lockwood, 1999).

We then have the following result, which follows directly from Theorems 1 and 2 below, setting $K = 3$.

Proposition 1. *Assume that* $K = 3$. *If* (i) $q > 2/3$, *or* (ii) $q = 2/3$, *and preferences are not polarized over* b, *the game is dominance-solvable. If* $q \leqslant q_n$, *then the game is not dominance-solvable.*

The intuition for this result is as described in the introduction. However, it is probably worth saying more about the somewhat less intuitive condition for non-dominance solvability, $q \leqslant q_n$. A sufficient condition for non-dominance solvability is that for every $i \in N$, we can find a vote distribution $\widetilde{\omega}_{-i}$ such that:

(i) i's unique best response to ω_{-i} is to vote for her second-ranked alternative;
(ii) $\widetilde{\omega}_{-i}$ does not have any $j \neq i$ voting for her worst alternative.

Condition (i) ensures that no voter's second-ranked alternative is weakly dominated, implying that iterated deletion stops after the first round; condition (ii) ensures that the construction of the $\widetilde{\omega}_{-i}$ are internally consistent, i.e., do not involve any voter voting for a weakly dominated alternative. Conditions (i) and (ii) place a number of linear restrictions on the $\widetilde{\omega}_{-i}$; a sufficient condition for them all to be satisfied is $q \leqslant q_n$. The proof of Theorem 2 gives the details.

The first question that one might ask at this stage is whether the sufficient conditions for dominance-solvability in Proposition 1 are also necessary. The example below answers this question negatively, by presenting a game which is not classified as either dominance-solvable or not by Proposition 1, and showing that it is dominance-solvable.

Example 1. The ordinal preferences of five voters are as follows:

$$\begin{aligned} 1, 2, 3&: \quad x \succ y \succ z, \\ 4&: \quad x \succ z \succ y, \\ 5&: \quad z \succ x \succ y. \end{aligned}$$

Note that $q_n = 2/5$, $q = 3/5$, so, $q = q_n + 1/n < 2/3$. Also, preferences are not polarized over $b = y$ so the game is not classified by Proposition 1. We will show that the game is dominance-solvable.

First, note that after the first round of deletion, by Lemma 1, $V_i^1 = \{x, y\}$, $i = 1, \ldots, 3$, $V_i^1 = \{x, z\}$, $i = 4, 5$. We show (in the reduced game) that for voter 4, $v_4 = z$ is weakly dominated by $v_4 = x$. Suppose to the contrary that there exists a $\omega_{-4} \in \Omega_{-4}^1$ such that z is a unique best response to ω_{-4}. This requires that 4 must be able to affect the outcome by voting z, given some $\omega_{-4} \in \Omega_{-4}^1$. The only such vote distributions are $\omega_{-4} = (2, 1, 1)$, $(1, 2, 1)$, and $(2, 2, 0)$. But it is clear that voter 4 does better voting for x in response to each of these, a contradiction. (For example, if he votes for x rather than z against $(2, 1, 1)$, the outcome is x rather than a homogeneous lottery over $\{x, z\}$, which voter 4 obviously prefers.)

So, at the end of the second round of deletion, z can get at most one vote i.e., from voter 5, and so cannot win, in which case $v_5 = z$ is weakly dominated for voter 5. The game is then reduced to one where each player can vote for (at most) one of two alternatives x, y and is thus dominance-solvable, with x being the solution outcome.

The next example clarifies the role of the non-polarization condition by showing that it is needed for dominance-solvability when $q = 2/3$.

Example 2. The ordinal preferences of the six voters are as follows:

$$1, 2, 3, 4: \quad x \succ y \succ z,$$
$$5, 6: \quad z \succ x \succ y.$$

Also, voters 1–4 prefer their second-ranked alternative, y, to a homogeneous lottery over $\{x, y, z\}$. Note that $q = 4/6 = 2/3$, and preferences over z are polarized, so this game is not classified by Proposition 1. In fact, the game is not dominance-solvable. To show this, we prove that for any voter, it is a weakly undominated strategy relative to V^1 to vote for her second-ranked alternative. A similar argument (left to the reader) then shows that it is a weakly undominated strategy relative to V^1 to vote for her first-ranked alternative. These two statements together then imply that iterated deletion stops at the first round.

By Lemma 1, $V_i^1 = \{x, y\}$, $i = 1, \dots, 4$, $V_i^1 = \{x, z\}$, $i = 5, 6$. Define $\Omega_{-i}^1 = \{\omega(v_{-i}) \mid v_{-i} \in V_{-i}^1\}$. We show that for every voter i, there exists $\tilde{\omega}_{-i} \in \Omega_{-i}^1$ such that her second-ranked alternative is a unique best response to $\tilde{\omega}_{-i}$. Specifically, for $i = 1, \dots, 4$, y is a unique best response to $\tilde{\omega}_{-i} = (1, 2, 2)$. To see this, note that if i responds to $(1, 2, 2)$ with y, the outcome is y, but if i responds with x, the outcome is the homogeneous lottery over $\{x, y, z\}$. Also, for $i = 5, 6$, x is a unique best response to $\tilde{\omega}_{-i} = (2, 3, 0)$. Finally, it is easily checked that $\tilde{\omega}_{-i} \in \Omega_{-i}^1$, all $i = 1, \dots, 6$.

Another question is how "close" Proposition 1 comes to classifying all games as dominance-solvable or not for a fixed profile of voter preferences. One way of looking at this is to note that Proposition 1 can classify the game as dominance-solvable or not except when

(i) when $q = 2/3$ and preferences over b are polarized;
(ii) $q_n + 1/n = q < 2/3$;
(iii) $q_n + 1/n < q < 2/3$.

It is possible to show that (iii) never occurs,[10] so we see that there are *at most* two values (out of n possible values) of q such that the game cannot be classified as dominance-solvable or not. If we hold the distribution of preferences across voters constant as n increases, we can prove a sharper result, namely we can provide *necessary and sufficient conditions* for games with more than a critical number of voters to be DS: the general statement of this result is given in Theorem 3 below.

[10] See Corollary 1 of Dhillon and Lockwood (1999).

3.2. Dominance-solvability and Condorcet winners

We now turn to a characterization of the solution outcome in the event that the game is dominance-solvable, and in particular how this outcome relates to the CW, whenever the latter exists. First, there is the question of whether dominance-solvability implies existence of a Condorcet winner, or vice-versa. Here, from Proposition 1, as the game is dominance-solvable, $q > q_n$, i.e., $q \geqslant q_n + 1/n$. It is easy to check[11] that $q_n + 1/n > 0.5$. So, if the game is dominance-solvable, there is a Condorcet loser. Consequently, as $K = 3$, by a well-known result there is a Condorcet winner.[12] On the other hand, it is clear that the reverse implication is not true. For example, the game in Example 2 is not dominance-solvable, but there is clearly a Condorcet winner, namely x. We can summarize our discussion as follows:

Proposition 2. *If the game is dominance-solvable, a Condorcet winner exists.*

The next, and key, question is whether the solution outcome is a Condorcet winner in the event that the game is dominance-solvable. As stated in Theorem 1 below, if our *sufficient* conditions for dominance-solvability are satisfied, then this is the case. That is, in the case of three alternatives, we have the following result.

Proposition 3. *If* (i) $q > 2/3$, *or* (ii) $q = 2/3$, *and preferences are not polarized over* b, *any solution outcome is a CW, i.e.,* $W^{\infty} \subset X^{cw}$.

However, what if the game is dominance-solvable, but the sufficient conditions for dominance-solvability do *not* hold? This is a possibility, as Example 2 shows. In that example, the solution outcome, x, is again the CW. We conjecture, but have not been able to prove, that this is true generally: that is, *every* solution outcome must be a Condorcet winner. We certainly have an asymptotic result of this kind, i.e., Theorem 3 below. This says that if the electorate is replicated sufficiently often, and if the game is dominance-solvable, the outcome is a Condorcet winner.

However, it should be emphasized that the above results and conjecture only apply to the case of four or more voters. With only three voters, it is possible to have a voting game which is dominance-solvable, but where the solution is *not* the CW, as the following example shows. As the example indicates, this arises because Lemma 1 does not hold with three voters, and in particular, more strategies can be ruled out as weakly dominated at the first round of iteration.

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[11] In the odd case, $q_n + 1/n > 0.5$ if $0.5n > [(n + 1)/3]$. As $[(n + 1)/3] < (n + 1)/3 + 1$, it is sufficient that $0.5n > (n + 1)/3 + 1$. This holds for $n \geqslant 9$. Finally, in the cases $n = 5, 7$, $q_n + 1/n = 3/5, 4/7$, respectively. The proof in the even case is similar.

[12] If $n = 3$, it is not even the case that dominance-solvability requires the existence of a Condorcet winner, as Example 4 of Dhillon and Lockwood (1999) shows.

Example 3. Ordinal preferences over the three alternatives are as follows.

$$1: \quad x \succ z \succ y,$$
$$2: \quad z \succ x \succ y,$$
$$3: \quad y \succ z \succ x.$$

The unique CW is z. Say that a voter has *dominated middle alternative* (DMA) preferences if he prefers an equal-probability lottery over the three alternatives to his second-ranked alternative. It is easy to show[13] that the strategy of voting for one's second-ranked alternative is weakly dominated if and only if that voter has DMA preferences (so, Lemma 1 above does not apply to the case of three voters). Assume now that only voters 1 and 3 have DMA preferences. Then, by the result just stated, $V_1^1 = \{x\}$, $V_3^1 = \{y\}$. Moreover, as voter 2 has non-DMA preferences, his unique best response to $v_{-2} = (x, y)$ is $v_2 = x$. So the game is DS, and $W^\infty = \{x\} \neq \{z\}$.

4. General results

We now consider the case of an arbitrary number $K \geqslant 3$ of alternatives. For any $Y \subset X$, define $\Gamma = (u_i, Y)_{i \in N}$ to be the plurality game defined in Section 1 above, with a fixed set of n players, but a set $Y \subset X$ of alternatives. The preferences of players are the restriction of the preferences over the set X, to the subset Y. For any such game, let $Q(Y)$ be the largest set of voters who agree on a worst alternative in Y, and define

$$q(Y) = \frac{\#Q(Y)}{n}. \tag{4}$$

This fraction plays a crucial role in what follows. Denote the worst alternative in Y for voters in $Q(Y)$ by $b(Y)$. Without loss of generality, we will restrict our attention to games where $b(Y)$ is a Condorcet loser, i.e., $q(Y) > 0.5$, so $Q(Y)$ is unique.

Let $X \equiv X_K$, and define the following sets recursively:

$$X_{l-1} = X_l / \{b(X_l)\}, \quad l = K, \ldots, 2. \tag{5}$$

Each set is obtained from the previous one by deleting the alternative in the previous set that is worst-ranked by the most players, and the initial set is just X. These sets are uniquely defined for any sequence of games where for each game, at least a simple majority agree on the worst alternative (there exists a Condorcet loser). Note that $\#X_l = l$. We now have our general sufficient conditions for dominance-solvability.

[13] Consider for example voter 1. The only possible profiles where 1 is pivotal (i.e., changes the outcome with his vote) are where voters 2 and 3 vote for different alternatives, i.e., $\omega_{-1} = (1, 1, 0), (1, 0, 1)$ or $(0, 1, 1)$. In all other (non-pivotal) profiles, all strategies of voter 1 give him the same payoff. Thus x is as good as z in all the non-pivotal profiles, but on the pivotal profiles we have: if 1 votes for x against these profiles, he gets outcome x, x, H, respectively, where H is the equal-probability lottery over all alternatives. If he votes for z against these profiles, he gets outcome H, z, z, respectively. If 1 has DMA preferences, he strictly prefers the outcome arising from voting for x in every case: hence z is dominated by x. If his preference is the opposite, then the strategy z is a unique best response to profile $(0, 1, 1)$, hence undominated.

Theorem 1. *Assume that for all* $l = 3, \ldots, K$, *either* (i) $q(X_l) > (l-1)/l$, *or* (ii) $q(X_l) = (l-1)/l$, *and preferences are not polarized over* $b(X_l)$ *in game* $\Gamma = (u_i, X_l)_{i \in N}$. *Then the game* $\Gamma = (u_i, X)_{i \in N}$ *is dominance solvable. Moreover, the solution winset* W^∞ *is that alternative in* X_2 *which is preferred to the other by a strict majority of voters, or* X_2 *if equal numbers of voters prefer each alternative in* X_2. *Also, whenever* (i) *or* (ii) *hold, then*

(a) *if* n *is odd, a unique CW* x^{cw} *exists, and* $W^\infty = \{x^{cw}\}$;
(b) *if* n *is even, at least one Condorcet winner exists* (*i.e.,* $X^{cw} \neq \emptyset$) *and* $W^\infty \subset X^{cw}$.

The conditions require that for the sequence of sets of alternatives $(X_K, X_{K-1}, \ldots, X_3)$, there is sufficient agreement amongst the voters about which alternative is worst. Moreover, the solution outcome is generated by sincere voting over the set of the two-element alternatives, X_2, that remains when the alternatives ranked worst by the most voters have been sequentially deleted.

Three further remarks are appropriate at this point. First, if the game is DS, then the only iteratively undominated strategies involve voting for one of two alternatives in X_2. This is consistent with Duverger's Law, which asserts that "plurality rule tends to produce a two-party system" (Cox, 1997). Second, our sufficient conditions for DS are quite strong in that they imply the existence of a Condorcet winner, but they have the attractive feature that any alternative in the solution outcome W^∞ is always a CW. Third, the sufficiency conditions are quite strong. For example, if $K = 4$, we need that at least $3/4$ of the voters agree on which alternative is worst, and once that alternative has been deleted from the set, $2/3$ of voters must agree which of the remaining three alternatives are worst.

We now present sufficient conditions for the game $\Gamma(u_i, X)_{i \in N}$ *not* to be DS, and a characterization of $W(V^\infty)$, the set of iteratively weakly undominated outcomes in this case. Consider the sequence of sets (5) above, and the associated sequence of fractions $\{q(X_l)\}_{l=3}^{K}$. Also, for any game with l alternatives, define the critical fractions:

$$q_n^l = \begin{cases} q_n \text{ in Eq. (3)}, & l = 3, \\ 1 - \dfrac{1}{n} - \dfrac{1}{n}\left[\dfrac{n+3l-8}{l}\right], & l > 3,\ n \text{ odd}, \\ 1 - \dfrac{1}{n}\left[\dfrac{n+3l-7}{l}\right], & l > 3,\ n \text{ even}, \end{cases} \tag{6}$$

where $[x]$ denotes the smallest integer larger than x. Note that $q_n^l < (l-1)/l$, and $\lim_{n\to\infty} q_n^l = (l-1)/l$. Obviously, q_n^3 in (6) is equal to q_n in (3). Then we have:

Theorem 2. *If there exists an* $l \in \{3, 4, \ldots, K\}$ *such that* (i) $q(X_k) > (k-1)/k$, *all* $k > l$; (ii) $q(X_l) < q_n^l$, *or* $q(X_l) = q_n^l + 1/n$, *and preferences over* $b(X_l)$ *are polarized, then the game* $\Gamma = (u_i, X)_{i \in N}$ *is not DS. In this case, the set of iteratively undominated winsets is* $W(V^\infty) = \{W(v) \mid v_i \in X_l/b_i\}$, *where* b_i *is voter i's bottom-ranked alternative in* X_l.

Note from Theorem 2 that we are also able to characterize the set of iteratively weakly undominated winsets even if the game is not dominance-solvable.

Theorems 1 and 2 together provide conditions under which a game is classifiable as dominance-solvable or not.[14] If we hold the distribution of preferences across voters constant as n increases, we can prove a sharper result, namely we can provide *necessary and sufficient conditions* for games with more than a critical number of voters to be DS. This can be formalized as follows. Let $\Gamma_n = (u_i, X)_{i \in N}$ be the plurality voting game with a fixed number $n \geqslant 3$ players. Note that in any such game, there are $K!$ possible strict preference orderings over the K alternatives. Let ϕ_l^n, $l = 1, \ldots, K!$, be the fractions of players in Γ_n who have the lth possible preference ordering. So a distribution of preferences on X across players is characterized by $\phi^n = \{\phi_l^n\}_{l=1}^{K!}$. Define the *$m$-replica game* $\Gamma_{n,m} = (u_i, X)_{i=1}^{nm}$, $m = 1, 2, \ldots,$ to be a game with nm voters but with $\phi^{nm} = \phi^n$, all m, i.e., where the different "types" of voters in Γ_n are replicated by the factor m. The key feature of the m-replica game is that the distribution of preference profiles in the population of players does not change as n changes.

For any preference distribution ϕ, and set of alternatives $Y \subset X$, define $q(\phi, Y)$ as in (4) to be the largest fraction of voters who agree on the worst alternative in Y. Also, recall the definition of the sequence of subsets of alternatives $X_K, X_{K-1}, \ldots, X_3$ defined in (5) above. We make the following assumption about Γ_n which rules out some "non-generic" cases.

(A2) $q(\phi^n, X_l) \neq (l-1)/l$, $l = 3, \ldots, K$.

Then, we have the following result.

Theorem 3. *Consider any game Γ_n for which* (A2) *holds. Then there is an m_0 such that for all $m > m_0$, $\Gamma_{n,m}$ is dominance-solvable iff $q(\phi^n, X_l) > (l-1)/l$, $l = 3, \ldots, K$.*

In other words, if the replicated electorate is large enough, and condition (A2) holds, then the game can *always* be classified as DS or not DS. An obvious corollary of Theorems 1 and 3 is the following:

Corollary 2. *Consider any game Γ_n for which* (A2) *holds. Then there is an m_0 such that if $m > m_0$, and $\Gamma_{n,m}$ is dominance-solvable, at least one Condorcet winner exists* ($X^{cw} \neq \emptyset$) *and any solution outcome is a Condorcet winner, i.e., $W^\infty \subset X^{cw}$.*

This is the most general statement of the relationship between dominance-solvability and Condorcet winners.

[14] These conditions leave few games unclassified. Indeed, it can be shown that for any $k \in \{3, 4, \ldots, K\}$, there is at most *one* possible value of $q(X_k)$, namely $q_n^k + 1/n$, for which $q_n^k < q(X_k) < (k-1)/k$. Consequently, from inspection of Theorems 1 and 2, there are at most *two* possible values of each $q(X_l)$ for which the game cannot always be classified as DS or not, $q_n^k + 1/n$ and $(k-1)/k$. That is to say, if $q(X_k) \neq q_n^k + 1/n$, $(k-1)/k$, $k \in \{3, 4, \ldots, K\}$, then the game can *always* be classified.

5. Extensions and conclusions

5.1. Some extensions

First, we have ruled out indifference over elements of X, and also certain lotteries over X, by (A1). When voters are indifferent over outcomes, in general, the order of deletion of dominated strategies matters. There are two alternatives here. One is to make assumptions sufficient to ensure that the Marx and Swinkels (1997) Transference of Decision Maker Indifference (TDI) condition is satisfied (as discussed in Section 2.2). An assumption[15] which implies TDI in our model is that if some $i \in N$ is indifferent between winsets $W(Y)$, $W(Z)$, $Y, Z \subset X$, then so are all $j \in N$. With this assumption, all voters are indifferent between the same subset of alternatives.

The second is to accept that the order matters, and focus on the outcome with some "plausible" order of deletion. The order of iteration we used to prove Theorems 2 and 3 is of some interest. Iterated deletion is applied to the game $\Gamma = (u_i, X)_{i \in N}$ until the alternative ranked worst by the highest number of voters (say b) and only that alternative, is deleted from all strategy sets, so the game is reduced to $\Gamma = (u_i, X/b)_{i \in N}$, and so on. This procedure is known as the *Coombs social choice function* (Moulin, 1983, p. 24). If we want to apply this order of deletion with indifference, the problem is that b may not be uniquely defined. But, given some tie-breaking rule, we may be able to proceed as before.

A second extension would be to consider different scoring rules, other than plurality voting, to see whether well-known scoring rules can be "ranked" in terms of the strength of the conditions required to make them dominance-solvable. This is the subject of our current research. One other simple extension of plurality voting that can be studied using the methods of this paper is plurality voting with a *runoff*: with this rule, if no alternative gets more than 50% of the vote, then there is a second round when voters vote only for the two alternatives with most votes.[16] With only two alternatives at the second stage, there will be no strategic voting, so every voter rationally anticipates the same winner at the second stage. So, one can write down expected payoffs just as functions of first-round votes, and analyse the resulting game using the methods of this paper.

5.2. Conclusions

This paper has presented conditions sufficient for a plurality voting game to be dominance-solvable, and sufficient for it not to be dominance-solvable. These conditions can be stated in terms of only one sequence of statistics of the game, the largest proportion of voters who agree on which alternative is worst in a sequence of subsets of the original

[15] This assumption is satisfied, for example, in a citizen-candidate voting game where voters are of $K \leqslant n$ types, and every voter has strict preferences over different types, satisfying (A1), and is indifferent between two candidates of a given type. Then any voter is only indifferent between $L(Y)$, $L(Z)$ if Z can be obtained from Y by deleting candidates of some type and replacing them by others of the same type, in which case all voters are indifferent.

[16] To break ties, we need to assume that if the winset $W(v)$ has more than two members, two are selected randomly.

set of alternatives, where each subset is derived from the previous one by deleting the alternative that most voters rank as worst in the previous subset. When the number of voters is large, "almost all" games can be classified as either dominance-solvable or not dominance-solvable. If the game is dominance-solvable, the outcome is usually but not always the Condorcet winner, whenever it exists.

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Appendix

Proof of Lemma 1. Suppose w.l.o.g. that voter i's preferences are: $x_1 \succ_i x_2 \succ_i \cdots \succ_i x_j \succ_i x_{j+1} \succ_i \cdots \succ_i x_K$. So, it is sufficient to show that for any $j < K$, there exists some $\omega_{-i}^j \in \Omega_{-i}$ such that $v_i = x_j$ is a unique best response to ω_{-i}^j; for then, $v_i = x_j$ cannot be weakly dominated.

Let $\omega_{-i}^j = (\omega_1^j, \omega_2^j, \dots, \omega_K^j)$ where ω_l^j is the number of votes (excluding i's) for alternative x_l. If n is odd, construct ω_{-i}^j so that $\omega_j^j = \omega_{j+1}^j = (n-1)/2$, $\omega_l^j = 0$, $\forall l \neq j, j+1$. As $n > 3$, note that $\omega_j^j, \omega_{j+1}^j > \omega_l^j + 1$, $\forall l \neq j, j+1$. So, if i plays x_j against ω_{-i}^j, the outcome is x_j, if i plays x_{j+1} against ω_{-i}^j, the outcome is x_{j+1}, and finally if i plays x_l, $l \neq j, j+1$ against ω_{-i}^j, the outcome is x_j or x_{j+1} with equal probability. As i strictly prefers the first outcome to the second or third, x_j is a unique best response to ω_{-i}^j, as claimed.

If n is even, construct ω_{-i}^j so that $\omega_j^j = n/2 - 1$, $\omega_{j+1}^j = n/2$, $\omega_l^j = 0$, $\forall l \neq j, j+1$. As $n > 3$, note that $\omega_j^j, \omega_{j+1}^j > \omega_l^j$, $\forall l \neq j, j+1$. So, if i plays x_j against ω_{-i}^j, the outcome is x_j or x_{j+1} with equal probability. If i plays x_{j+1} or x_l, $l \neq j, j+1$, against ω_{-i}^j, the outcome is x_{j+1}. As i strictly prefers the first outcome to the second, x_j is a unique best response to ω_{-i}^j, as claimed. □

Proof of Theorem 1. Let $Y \subset X$, and define

$$T_i(Y) = \begin{cases} Y/b(Y), & i \in Q(Y), \\ Y, & i \notin Q(Y), \end{cases}$$

and also $T(Y) = \bigtimes_{i \in N} T_i(Y)$. Let $y = \#Y$. We can now state and prove three additional lemmas.

Lemma A.1. *In the game $\Gamma = (u_i, Y)_{i \in N}$, if $q(Y) > (y-1)/y$, then $v_i = b(Y)$ is weakly dominated for all $i \in N/Q(Y)$ relative to $T(Y)$.*

Proof. (i) First, we show that $b(Y) \notin W(v)$ for all $v \in T(Y)$. Suppose to the contrary that $b(Y) \in W(v)$ for some $v \in T(Y)$. Then $b(Y)$ must get at least as many votes as all other alternatives in Y. But $b(Y)$ can get at most $n(1-q(Y))$ votes, as $nq(Y)$ players have deleted $b(Y)$ from their strategy sets. So, as no alternative gets more votes than $b(Y)$, and there are y alternatives, the total number of votes cast is at most $T = yn(1-q(Y))$. Now, if $q(Y) > (y-1)/y$, $yn(1-q(Y)) < n$, so $T < n$, a contradiction since we do not allow abstentions.

(ii) Fix a profile of votes $v_{-i} \in \bigtimes_{j \neq i} T_j(Y)$. Given this profile, any $i \in N/Q(Y)$ can always do weakly better by voting for her most preferred alternative in $Y/b(Y)$—say s_i—rather than for $b(Y)$. This is because a vote for $b(Y)$ can never affect the winset by part (i), so a switch from $b(Y)$ to s_i by i will either:

(i) not affect the outcome, or
(ii) add s_i to the winset, or
(iii) eliminate all alternatives other than s_i from the winset.

In cases (ii) and (iii), i strictly gains. □

Lemma A.2. *In the game $\Gamma = (u_i, Y)_{i \in N}$, if $q(Y) = (y-1)/y$, and preferences are non-polarized, then $v_i = b(Y)$ is weakly dominated for some $i \in N/Q(Y)$ relative to $T(Y)$.*

Proof. (i) First we prove that if $b(Y) \in W(v)$ for some $v \in T(Y)$, then, *all* alternatives must be in $W(v)$. For suppose not; let $x \neq b(Y)$ and $x \notin W(v)$. So, there are at most $y-1$ alternatives in $W(v)$, and as $b(Y) \in W(v)$, all of these alternatives can get no more votes than $b(Y)$. Moreover, $b(Y)$ can get at most $n(1-q(Y))$ votes, by definition of $q(Y)$ and the fact that $v \in T(Y)$. Thus, the total number of votes cast is at most $T = (y-1)n(1-q(Y)) + n(1-q(Y)) - 1$ where $n(1-q(Y)) - 1$ is the most votes that x can get and not be in $W(v)$. Now, as $q(Y) = (y-1)/y$, $T = n-1$, a contradiction since we do not allow abstentions.

(ii) As the game is non-polarized, $b(Y)$ is not ranked best in Y by some $i \notin Q(Y)$. From part (i), given any vote profile of the other voters $v_{-i} \in \bigtimes_{j \neq i} T_j(Y)$, there are two possibilities for this voter i: *either* when $v_i = b(Y)$, $b(Y) \notin W(v)$, in which case he does weakly better by voting for his most preferred alternative, by the argument in the proof of Lemma A.1, *or* when $v_i = b(Y)$, $b(Y) \in W(v)$, in which case all alternatives have equal numbers of votes, in which case, he could do strictly better by voting for his most preferred alternative. So, choice of $v_i = b(Y)$ is weakly dominated as claimed. □

Lemma A.3. *In the game $\Gamma = (u_i, Y)_{i \in N}$ if* (i) $q(Y) > (y-1)/y$, *or* (ii) $q(Y) = (y-1)/y$ *and preferences are non-polarized, then this game can be reduced to the game $\Gamma = (u_i, Y/b(Y))_{i \in N}$ by iteratively deleting weakly dominated strategies.*

Proof. (i) Assume first that condition (i) of the lemma holds. Then, from Lemma 1, $b(Y)$ is weakly dominated relative to $V = Y^n$ for all players in $Q(Y)$, and so can be deleted from their strategy sets to get $V^1 = T(Y)$. Then, from Lemma A.1, $b(Y)$ is weakly dominated relative to $T(Y)$ for all players in $N/Q(Y)$. So, we can delete $b(Y)$ from the strategy sets of all $i \in N/Q(Y)$ to get $V^2 = (Y/b(Y))^n$ after two rounds of deletion, as required.

(ii) Now assume that condition (ii) of the lemma holds. Again, from Lemma 1, $b(Y)$ is weakly dominated relative to $V = Y^n$ for all players in $Q(Y)$, and so can be deleted to get $V^1 = T(Y)$. Now from Lemma A.2, $b(Y)$ is weakly dominated relative to $T(Y)$ for some $i \in N/Q(Y)$. So, we can delete $b(Y)$ from the strategy set of this $i \in N/Q(Y)$ to give a set of strategy profiles $V^2 = (V_i^2, V_{-i}^2) = (Y/b(Y), \bigtimes_{j \neq i} T_j(Y))$. But then by the argument of Lemma A.1, $b(Y) \notin W(v)$, $v \in V^2$, as $b(Y)$ can now get at most $n(1 - q(Y)) - 1$ votes. So, $b(Y)$ is weakly dominated for *all* players relative to V^2 and thus can be deleted from all the remaining players' strategy sets to get $V^3 = (Y/b(Y))^n$ after three rounds of deletion, as required. □

We can now return to the proof of Theorem 1. Under condition (i) or (ii) in the theorem, by Lemma A.3, $\Gamma = (u_i, X_l)_{i \in N}$ can be reduced to $\Gamma = (u_i, X_{l-1})_{i \in N}$ by iterated deletion of weakly dominated strategies. So, iterating, $\Gamma = (u_i, X_l)_{i \in N}$ can be reduced to $\Gamma = (u_i, X_2)_{i \in N}$ where each player has only two strategies. In the game $\Gamma = (u_i, X_2)_{i \in N}$, the only undominated strategy is to vote sincerely, and so the game is dominance-solvable, with an outcome as described in the theorem.

To prove the last part, let $X_2 = \{x, y\}$. First suppose that n is odd. Then w.l.o.g., suppose x beats y in a majority vote, so $W^\infty = \{x\}$. Then, x must be a CW. For suppose not. Then, x must be ranked worse than some $w \notin X_2$ by a majority of voters, and thus must be in X/X_2, contrary to assumption.

Now suppose that n is even. Then, either the above argument applies (i.e., $W^\infty = \{x\}$, where x is a CW), or equal numbers of voters prefer x to y and vice versa, in which case $W^\infty = \{x, y\}$. Again, x, y must both be CWs. For suppose not. Then, one or both of x, y must be ranked worse than some $w \notin X_2$ by a majority of voters, and thus must be in X/X_2, contrary to assumption. □

Proof of Theorem 2. Let l be the first $k \in \{3, 4, \ldots, K\}$ for which $q(X_k) \leqslant q_n^k$. Then, by the proof of Theorem 1, the game $\Gamma = (u_i, X)_{i \in N}$ can be reduced to $\Gamma = (u_i, X_l)_{i \in N}$ by iterated deletion of weakly dominated strategies. Let b_i be i's bottom-ranked alternative in X_l. Then, by Lemma 1, $v_i = b_i$ is weakly dominated in $\Gamma = (u_i, X_l)_{i \in N}$, so we can delete b_i from player i's strategy set to get $T_i = X_l/b_i$, $i \in N$. We now show that no $v_i \in T_i$ is weakly dominated relative to $T = \bigtimes_i T_i$: this implies that T is a *full reduction of V by weak dominance* (Marx and Swinkels, 1997, Definition 3) and hence by Corollary 1 of Marx and Swinkels, the set of iteratively undominated winsets is $W = \{W(v) \mid v_i \in T_i, i \in N\}$, which proves the theorem, given that $T_i = X_l/b_i$.

W.l.o.g., let $X_l = \{x_1, \ldots, x_l\}$, and let $N(x_m) = \{i \in N \mid b_i = x_m\}$, $1 \leqslant m \leqslant l$. So,

$$T_i = X_l/x_m, \quad i \in N(x_m). \tag{A.1}$$

It now suffices to show that for every $i \in N$, there exists $v_{-i} \in T_{-i}$ such that it is a unique best response for i to vote for any alternative $x \in T_i$. For then, no alternative in T_i can be weakly dominated for i.

W.l.o.g., let $i \in N(x_l)$, so $T_i = \{x_1, x_2, \ldots, x_{l-1}\}$, and assume that $x_1 \succ_i x_2 \succ_i \cdots \succ_i x_j \succ_i x_{j+1} \succ_i \cdots \succ_i x_{l-1}$. Also, let $\omega_{-i} = (\omega_1, \omega_2, \ldots, \omega_l)$ be a vote distribution over X_l, where ω_j is the number of votes for alternative x_j, and $\Omega_{-i}(T) = \{\omega(v_{-i}) \mid v_{-i} \in T_{-i}\}$. So, we need to show that there exists $\omega_{-i} \in \Omega_{-i}(T)$ such that it is a unique best response for i to vote for any alternative $x_1, \ldots, x_{l-1}$. Note two properties of a vote distribution ω_{-i} that must hold if it is to belong to $\Omega_{-i}(T)$. First, the total number of votes must add up to $n-1$:

$$\sum_{k=1}^{l} \omega_k = n - 1. \tag{A.2}$$

Second, ω_k must be no greater than the number of voters (excluding i) who do *not* rank x_k worst in X_l, i.e., for whom $x_k \in T_i$. Inspection of (A.1) implies that this requires

$$\omega_l \leqslant \sum_{k \neq l} n_k, \qquad \omega_j \leqslant \sum_{k \neq j} n_k - 1, \quad j \neq l, \tag{A.3}$$

where $n_k = \#N(x_k)$.

Case 1 (*n odd*). We know from the proof of Lemma 1 that $v_i = x_j$, $j < l$, is a unique best response in T_i to $\omega_{-i} = (\omega_1, \omega_2, \ldots, \omega_l)$ if

$$\omega_j = \omega_{j+1} > \omega_k + 1, \quad k \neq j, j+1. \tag{A.4}$$

So, it suffices to show that we can find $\widetilde{\omega}_{-i} \in \Omega_{-i}(T)$ where (A.2)–(A.4) are satisfied. We construct $\widetilde{\omega}_{-i}$ as follows. First, set $\omega_j = \omega_{j+1} = \omega$, where:

$$\omega = \begin{cases} \left[\dfrac{n+1}{3}\right] & \text{if } l = 3, \\ \left[\dfrac{n+3l-8}{l}\right] & \text{if } l > 3, \end{cases} \tag{A.5}$$

where $[x]$ is the smallest integer greater than or equal to x. Let $t = (n-1) - 2\omega$ be the number of remaining votes.[17] Note that we can always write $t = s(l-2) + r, \geqslant 0$, $0 \leqslant r < l-2$, where s, r are integers. Now, distribute the remaining t votes over the remaining $l-2$ alternatives as evenly as possible. That is, if $r = 0$, $s \geqslant 0$, give every remaining alternative s votes; if $r > 0$ and give every remaining alternative s votes and an additional vote to r of the $l-2$ remaining alternatives. Clearly, $\widetilde{\omega}_{-i}$ satisfies (A.2) by construction.

[17] It is easy to show that $t \geqslant 0$. For this, we require $n - 1 \geqslant 2[(n+3l-8)/l] = 6 + 2[(n-8)/l]$. Now the right-hand side of this inequality is largest when $l = 3$, so we only need $n \geqslant 6 + 2[(n-8)/3]$. It can easily be checked that this holds for $n \geqslant 6$, the case of $n = 5$ can be checked separately. Similarly, when $l = 3$, we can show that $n - 1 \geqslant 2[(n+1)/3]$ for $n \geqslant 9$, the cases $n = 5, 7$, can be checked separately.

Also, $\widetilde{\omega}_{-i}$ satisfies (A.4). To see this, note first that the maximum number of votes for any of the remaining alternatives ω_k, $k \neq j,\ j+1$ is s if $r = 0$, and $s+1$ if $r > 0$. Also, $s = (t-r)/(l-2)$. But then, noting that if $l = 2$, $r \equiv 0$, (A.4) requires simply that

$$\left[\frac{n+1}{3}\right] > \frac{t}{l-2} + 1 \quad \text{if } l = 3, \tag{A.6}$$

$$\left[\frac{n+3l-8}{l}\right] > \frac{t-r}{l-2} + 2 \quad \text{if } l > 3,\ l-2 > r \geqslant 1. \tag{A.7}$$

Using the definition of t, the inequality (A.6) requires

$$\left[\frac{n+1}{3}\right] > \frac{n-1-2[(n+1)/3]}{l-2} + 1$$

which, using $[x] \geqslant x$, certainly holds. Also, the inequality (A.7) requires

$$\left[\frac{n+3l-8}{l}\right] > \frac{n-2-2[(n+3l-8)/l]}{l-2} + 2$$

which, again, using $[x] \geqslant x$, certainly holds for $l > 2$.

It remains to check that $\widetilde{\omega}_{-i}$ satisfies (A.3). From (A.4), a sufficient condition for (A.3) to be satisfied is that

$$\omega \leqslant \sum_{k \neq j} n_k - 1. \tag{A.8}$$

Now let $n_k \leqslant \theta n$, $\forall k$. Note that as $\sum_{k=1}^{l} n_k = n$, $n_k \leqslant \theta n$ implies $\sum_{k \neq j} n_k \geqslant (1-\theta)n$, for any j. Then (A.8) is certainly satisfied if the following holds:

$$\omega \leqslant (1-\theta)n - 1. \tag{A.9}$$

Now let q_n^l be the largest value of θ such that (A.9) holds. So, q_n^l satisfies (A.9) with equality, i.e., $q_n^l = 1 - 1/n - \omega/n$. Substituting out ω from (A.5), we get the expression (3) for $l = 3$, and the expression (6) for $l > 3$, for the case of n odd.

Case 2 (n *even*). Here, the argument is the same, except we now choose $\widetilde{\omega}_{-i} \in T_{-i}$ such that (A.10) below, rather than (A.4) is satisfied;

$$\omega_j = \omega_{j+1} - 1 > \omega_k, \quad k \neq j, j+1. \tag{A.10}$$

For then, if (A.10) holds, by Lemma 1, $v_i = x_j$, $j < l$, is a unique best response by i to $\widetilde{\omega}_{-i} \in T_{-i}$. The required vote distribution $\widetilde{\omega}_{-i}$ is constructed as follows. First, we set $\omega_{j+1} = \omega$, $\omega_j = \omega - 1$, where

$$\omega = \begin{cases} \left[\dfrac{n+2}{3}\right] & \text{if } l = 3, \\[2ex] \left[\dfrac{n+3l-7}{l}\right] & \text{if } l > 3. \end{cases} \tag{A.11}$$

Also, distribute the remaining $t = n - 2\omega$ votes over the remaining $l - 2$ alternatives as evenly as possible,[18] as before.

Clearly, $\widetilde{\omega}_{-i}$ satisfies (A.2) by construction. Also, $\widetilde{\omega}_{-i}$ satisfies (A.4). To see this, note by the argument in the odd case, that the maximum number of votes for any of the remaining alternatives ω_k, $k \neq j, j+1$, is s if $r = 0$, and $s + 1$ if $r > 0$, where $s = (t - r)/(l - 2)$. But then, noting that if $l = 2$, $r \equiv 0$, (A.4) requires simply that

$$\left[\frac{n+2}{3}\right] - 1 > \frac{t}{l-2} \quad \text{if } l = 3, \tag{A.12}$$

$$\left[\frac{n+3l-7}{l}\right] - 1 > \frac{t-r}{l-2} + 1 \quad \text{if } l > 3,\ l-2 > r \geqslant 1, \tag{A.13}$$

where $t = s(l - 2) + r$, as before. Using the definition of t, (A.12) requires

$$\left[\frac{n+2}{3}\right] - 1 > n - 2\left[\frac{n+2}{3}\right] + 1,$$

which, using $[x] \geqslant x$, certainly holds. Noting that the RHS of (A.13) is maximized when $r = 1$, the inequality (A.13) requires

$$\left[\frac{n+3l-7}{l}\right] > \frac{n - 2[(n+3l-7)/l]}{l-2} + 2,$$

which, again using $[x] \geqslant x$, certainly holds for $l > 3$. Finally, (A.3) is certainly satisfied by $\widetilde{\omega}_{-i}$ if

$$\omega_{j+1} \leqslant \sum_{k \neq j} n_k,$$

which, from (A.11), reduces to

$$\omega \leqslant (1 - \theta)n, \tag{A.14}$$

where $n_k \leqslant \theta n$, $\forall k$. Now let q_n^l be the largest value of θ such that (A.14) holds. So, q_n^l satisfies (A.14) with equality. Solving this expression for q_n^l,and using (A.11), we get the expression in (6) for the case of n even, and the expression in (3) for n even and $l = 3$. □

Proof of Theorem 3. (Sufficiency) If $q(\phi^n, X_l) > (l - 1)/l$, $l = 3, \ldots, K$, obviously $q(\phi^{nm}, X_l) > (l-1)/l$, $l = 3, \ldots, K$, $m \geqslant 1$, so $\Gamma_{n,m}$ is DS for all $m \geqslant 1$ from Theorem 1.

(Necessity) Assume that n is odd. The proof for the even case is similar. If it is *not* the case that $q(\phi^n, X_l) > (l - 1)/l$, $l = 3, \ldots, K$, then, by Lemma A.3, then there is some $l \in \{3, \ldots, K\}$ such that

$$q(\phi^n, X_l) < \frac{l-1}{l}, \qquad q(\phi^n, X_k) > \frac{k-1}{k}, \quad k > l. \tag{A.15}$$

[18] It can be checked, as for the odd case that $t \geqslant 0$, since this is true iff $n - 2 \geqslant 2\omega$, i.e., $n - 2 \geqslant 4 + 2[(n-7)/3]$. The RHS is maximized when $l = 3$, thus it is sufficient to show that $n - 2 \geqslant 4 + 2[(n - 7)/3]$. The latter holds for $n \geqslant 6$, and the case $n = 4$ can be checked separately.

But then as $q_n^l < (l-1)/l$, and $\lim_{n\to\infty} q_n^l = (l-1)/l$, there exists a m_0 such that

$$q(\phi^{nm}, X_l) = q(\phi^n, X_l) \leqslant q_{nm}^l, \quad m \geqslant m_0. \tag{A.16}$$

So, we conclude from (A.15), (A.16) that

$$q(\phi^{nm}, X_l) \leqslant q_{nm}^l, \qquad q(\phi^{nm}, X_k) \geqslant \frac{k-1}{k}, \quad k > l,$$

for all $m \geqslant m_0$. So, by Theorem 2, $\Gamma_{n,m}$ is not DS for all $m \geqslant m_0$. □

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