EC9D3 Advanced Microeconomics, Part I: Lecture 1

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Course Outline

- **Lecture 1:** Individual Preferences, Utility Representation.
- **Lecture 2:** Utility Maximization, Expenditure Minimization, Demand.
- **Lecture 3:** Revealed Preferences, Choice under Uncertainty.
- **Lecture 4:** Intertemporal Choice, Production, Profit Maximization.
- **Lecture 5:** Cost Minimization, General Equilibrium Introduction.

Course Outline (2)

- **Lecture 6:** Exchange Economies, Existence, Welfare Theorems.
- **Lecture 7:** Production Economies, Externalities, Incomplete Markets.
- **Lecture 8:** Social Choice, May Theorem, Arrow Theorem.
- **Lecture 9:** Interpersonal Comparisons, Manipulabity, Liberty.

Textbooks

- Andreu Mas-Colell, Michael Whinston and Jerry Green (1995): *Microeconomic Theory*, Oxford: Oxford University Press.
- Peter C. Ordeshook (2008): *Game Theory and Political Economy: An Introduction*, Cambridge University Press.
- Geoffrey A. Jehle and Philip J. Reny (2010): *Advanced Microeconomic Theory*, FT/Prentice-Hall.

Microeconomic Theory

It is the analysis of the behaviour of individual economic agents and the aggregation of their actions in an institutional framework.

- *individual agents:* typically a consumer or a firm (producer);
- *behaviour:* traditionally utility maximization or profit maximization;
- *the institutional framework:* traditionally, the price mechanism in an impersonal market place or a game theoretic setting,
- *the mode of analysis:* equilibrium analysis.

What do we intend to get out?

- In a positive sense: a better understanding of individual agent's behaviour in certain situations.
- In normative sense: the ability to intervene or not, both at the government level and at the institutional level.
- The models we analyze are highly simplified hence, although they have some general predictive power, they are *not directly empirically testable* (lab environment).
- However, these models represent the building blocks of more complex and realistic testable models.

Consumer Theory

- *The agent:* individual (consumer);

- *The activity:* consume a whole set of commodities (goods and services). We focus on L commodities $l = 1, \ldots, L$;

- *The framework: consumption feasible set*

$$X \subset \mathbb{R}^L$$

  where $x \in X$ is a *consumption bundle* which specifies the amounts of the different commodities;

- Time and location are included in the definition of a commodity.

Consumption Feasible Set

Let X be the set of commodity bundles that the individual can conceivably consume given the physical constraints imposed by the environment.

Example of physical constraints: Impossibility to have negative amounts of bread, water,..., indivisibility.

Constraints may be physical but also institutional (legal requirements).

Example: non-negative orthant.

$$X = \left\{ x \in \mathbb{R}^L \mid x_l \geq 0, \forall l = 1, \ldots, L \right\} = \mathbb{R}^L_+$$

Properties of the Consumption Feasible Set

1. Non-negativity: $X \subset \mathbb{R}^L_+$
2. Closed set: it includes its own boundary;
3. Convexity: if $x \in X$ and $y \in X$ than for every $\alpha \in [0, 1]$:

$$x'' = \alpha x + (1 - \alpha) y \in X$$

Preference Relation

- Each consumer is endowed with *a preference relation* $\succeq$ *defined on the consumption feasible set* X.
- These preferences represent the *primitive* of our analysis.
- The expression:

$$x \succeq y$$

means that *"x is at least as good as y".*

- From this weak preference relation two relevant binary relations may be derived:

Strong Preference and Indifference Relations

- *The strong preference relation* $\succ$ defined as follows.

$$x \succ y \quad \text{iff} \quad x \succeq y \quad \text{and not} \quad y \succeq x;$$

- *The indifference relation* $\sim$ defined as follows.

$$x \sim y \quad \text{iff} \quad x \succeq y \quad \text{and} \quad y \succeq x.$$

Axioms of Choice

1. **Completeness:** for every $x, y \in X$ either $x \succeq y$ or $y \succeq x$, or both.
2. **Transitivity:** for every $x, y, z \in X$ if $x \succeq y$ and $y \succeq z$ then
$$x \succeq z.$$
3. **Reflexivity:** for every $x \in X$
$$x \succeq x.$$

A preference relation satisfying completeness, transitivity and reflexivity is termed *rational*.

Axioms of Choice (2)

4. **Continuity:** the preference relation $\succeq$ in X is continuous if it is preserved under the limit operation.

   In other words, for every converging sequence of pairs of commodity bundles $\{(x^n, y^n)\}_{n=0}^{\infty}$ such that

   $$x^n \succeq y^n \qquad \forall n$$

   where

   $$x = \lim_{n\to\infty} x^n \qquad y = \lim_{n\to\infty} y^n$$

   then

   $$x \succeq y.$$

Alternative Formulations of Continuity

There exist two alternative formulations of such axiom.

4. **Continuity II:** Given a bundle z both the upper contour set $\{y \in X \mid y \succeq z\}$ and the lower contour set $\{y \in X \mid z \succeq y\}$ are *closed sets.*

4. **Continuity III:** Both the strict upper contour set $\{y \in X \mid y \succ z\}$ and the strict lower contour set $\{y \in X \mid z \succ y\}$ are *open sets.*

Utility Function

Definition

A utility function is a mapping

$$u : X \to \mathbb{R}.$$

This mapping summarizes and represents the preference of a consumer in an *ordinal fashion*.

One of the key results of consumer theory is: the Representation Theorem.

Representation Theorem

Theorem (Representation Theorem)

If preferences are

- *rational (complete, reflexive and transitive) and*
- *continuous;*

then there exists a continuous utility function that represents such preferences.

A utility function represents a preference relation $\succeq$ if the following holds:

$$x \succeq y \qquad \textit{iff} \qquad u(x) \geq u(y)$$

Proof of Representation Theorem

- The proof of such theorem is rather lengthy.

- We prove an easier theorem that makes the following extra assumption on the preference relation $\succeq$.

5. **Strong monotonicity:** for every $x, y \in X$ if $x \geq y$ (meaning $x_l \geq y_l$ for every $l = 1, \ldots, L$) but $x \neq y$ (meaning that there exists an l such that $x_l > y_l$) then

$$x \succ y.$$

Easier Representation Theorem

Theorem (Easier Representation Theorem)

If preferences are:

- *rational (complete, reflexive and transitive),*
- *continuous and*
- *strongly monotonic then*

there exists a continuous utility function that represents them.

Proof of Representation Theorem

Proof:

- Let

$$e = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

- For given $x \in X$ let

$$B(x) = \{t \in \mathbb{R} \mid (t\ e) \succeq x\}$$

be a restricted upper contour set, where

$$(t\ e) = \begin{pmatrix} t \\ \vdots \\ t \end{pmatrix}$$

Proof of Representation Theorem (2)

- Let
$$W(x) = \{t \in \mathbb{R} \mid x \succeq (t\ e)\}$$
be the restricted lower contour set.

- By *strong monotonicity:*
  - $B(x)$ is *non-empty;*
  - $W(x)$ is *non-empty* since $0 \in W(x)$;

Proof of Representation Theorem (3)

- By *continuity:*
  - $B(x)$ and $W(x)$ are *both closed.*
- By *completeness:*
  - the set $B(x) \cup W(x) = \mathbb{R}$
- By *connectedness* of $\mathbb{R}$ (divisibility theorem):
  - there exists a $t_x \in \mathbb{R}$ such that $(t_x\ e) \sim x$

Proof of Representation Theorem (4)

Definition (Utility Function)

$$u(x) = t_x.$$

Claim

The utility function $u(\cdot)$ represents the preference relation $\succeq$. In other words, given $x \in X$ and $y \in X$:

$$u(y) \geq u(x) \qquad \text{iff} \qquad y \succeq x$$

Proof of Representation Theorem (5)

Proof (Sufficiency): Assume $u(y) \geq u(x)$

- by definition of $u(\cdot)$ it implies

$$t_y \geq t_x;$$

- by strong monotonicity

$$(t_y \ e) \succeq (t_x \ e);$$

- by definition of $u(\cdot)$

$$y \sim (t_y \ e) \qquad (t_x \ e) \sim x;$$

- by transitivity:

$$y \succeq x.$$

Proof of Representation Theorem (6)

Proof (Necessity): Assume $y \succeq x$;

- by definition of t_y and t_x:
$$(t_y\ e) \sim y \qquad x \sim (t_x\ e);$$
- by transitivity:
$$(t_y\ e) \succeq (t_x\ e);$$
- by strong monotonicity:
$$t_y \geq t_x;$$
- by definition of $u(\cdot)$:
$$u(y) \geq u(x).$$

□

Proof of Representation Theorem (7)

- The final step is to prove the *continuity of the utility function $u(\cdot)$*.
- *Continuity of $u(\cdot)$* means that for any sequence $\{x^n\}_{n=0}^{\infty}$ with $x = \lim_{n\to\infty} x^n$ we have

$$\lim_{n\to\infty} u(x^n) = u(x).$$

- Notice first that continuity of the utility function $u(\cdot)$ is a more restrictive property of continuity of preferences.
- Consider for example

$$v(x) = \begin{cases} u(x) & x_h \leq 3 \\ u(x) + 4 & x_h > 3. \end{cases}$$

Proof of Representation Theorem (7)

- Therefore we do need to prove continuity of the specific utility function we constructed $u(x) = t_x$.
- Consider a sequence $\{x^n\}_{n=0}^{\infty}$ with $x = \lim_{n \to \infty} x^n$.
- We prove first that the sequence $\{u(x^n)\}_{n=0}^{\infty}$ has a converging subsequence.
- Monotonicity implies that for all $\varepsilon > 0$ the utility value $u(x')$ lies in a compact set $[\underline{t}, \overline{t}]$ for every x' such that $\| x' - x \| \leq \varepsilon$ where $\| x' - x \|$ denotes the Euclidean distance between x' and x.

Proof of Representation Theorem (8)

- Since $x = \lim_{n \to \infty} x^n$ then there exists $\overline{n}$ such that $u(x^n) \in [\underline{t}, \overline{t}]$ for every $n > \overline{n}$.
- An infinite sequence that lies in a compact set has a converging subsequence.
- We prove next that all converging subsequences of $\{x^m\}_{m=0}^{\infty}$ are such that $\lim_{m \to \infty} u(x^m) = u(x)$.
- Assume by way of contradiction that there exists a subsequence $\{x^m\}_{m=0}^{\infty}$ such that $\lim_{m \to \infty} u(x^m) = q \neq u(x)$.

Proof of Representation Theorem (9)

Consider first the case $q > u(x)$.

- Monotonicity implies that $q\,e \succ u(x)\,e$.
- Consider now $p = [q + u(x)]/2$ then by monotonicity $p\,e \succ u(x)\,e$.
- Then there exists $\hat{m}$ such that for every $m > \hat{m}$ it is the case that $u(x^m) > p$ and $x^m \sim u(x^m)\,e \succ p\,e$.
- Continuity of preferences imply then $x \succeq p\,e$ and from $x \sim u(x)\,e$ also $u(x)\,e \succeq p\,e$ a contradiction of $p\,e \succ u(x)\,e$.
- The proof in the case $q < u(x)$ is symmetric. □

Preferences without Utility Representation

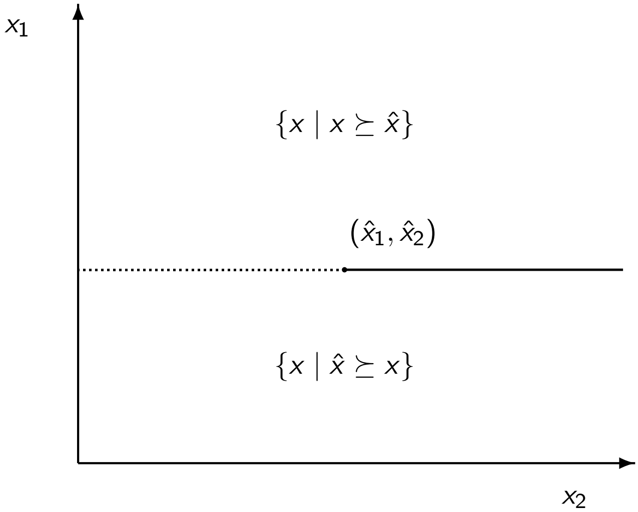
- Notice that there exists preferences that have *no utility representation*.
- Consider for example the following *lexicographic preferences:*

$$(x_1, x_2) \succ (y_1, y_2)$$

if and only if either $x_1 > y_1$ or if $x_1 = y_1$ then $x_2 > y_2$.

- Discontinuity follows from the fact that the upper contour set and the lower contour set are both neither closed nor open.

Lexicographic Preferences



Local Non-Satiation

Consider a weaker assumption than strong monotonicity, but enough for a Representation Theorem:

6. **Local non-satiation:** A preference relation $\succeq$ is *locally non-satiated* if for every $x \in X$ and every $\varepsilon > 0$, there exists $y \in X$ such that:

$$\| y - x \| \leq \varepsilon \quad \text{and} \qquad y \succ x$$

where $\| y - x \|$ denotes the Euclidean distance between points x and y in an L-dimensional vector space:

$$\| y - x \| = \left[\sum_{l=1}^{L} (x_l - y_l)^2 \right]^{\frac{1}{2}}.$$

Continuous Utility Function

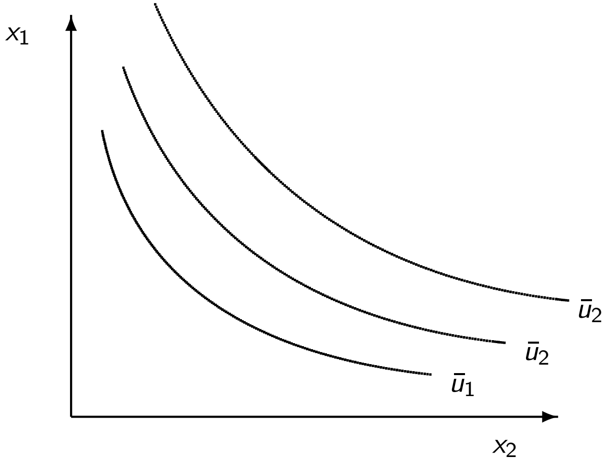
From now on we shall assume that:

- the consumer's preference relation is continuous
- the consumer's preferences satisfy strong monotonicity (local non-satiation),

Hence preferences are representable by a *continuous utility function.*

Indifference Curves

A relevant feature of a utility function is its *map of indifference curves:*



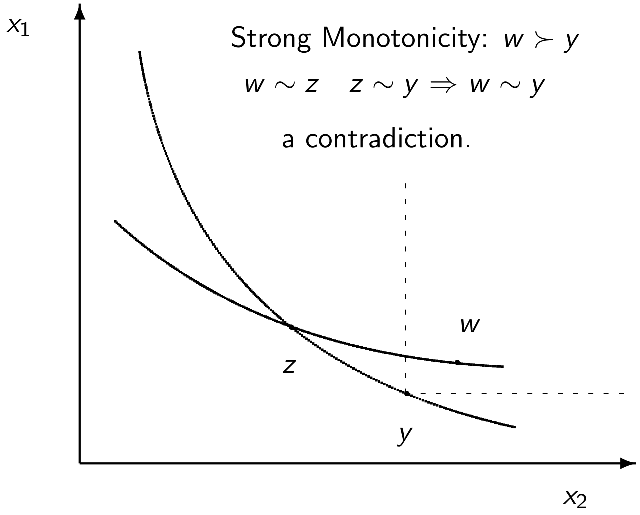
Properties of Indifference Curves

1. Downward sloping (implied by strict monotonicity).

2. Each consumption bundle is part of an indifference curve (implied by the completeness of preferences).

3. Two indifference curves *cannot* cross (it violates transitivity):

Indifference Curves cannot Cross



Convexity of Indifference Curves

4. Convexity (to the origin), implied by the *convexity* of the preference relation $\succeq$.

Definition (Convex Preferences)

The preference relation $\succeq$ is convex if for every $x \in X$ the upper contour set $\{y \in X \mid y \succeq x\}$ is convex.

The convexity property of the indifference curves can be restated in the following manner.

Marginal Rate of Substitution

Definition (Marginal Rate of Substitution)

The *marginal rate of substitution* is the slope of an indifference curve:

$$\text{MRS} = \left|\frac{dx_2}{dx_1}\right| = \frac{\partial u/\partial x_1}{\partial u/\partial x_2} = \frac{u_1}{u_2}$$

- The convexity to the origin of indifference curves may be interpreted as *diminishing MRS*.
- Alternatively, the indifference curves are convex to the origin if and only if the utility function $u(\cdot)$ is *quasi-concave*.

Quasi-Concave Utility Function

Definition (Quasi-Concavity)

The function $u(\cdot)$ is *quasi-concave* if and only if the set:

$$\{y \in X \mid u(y) \geq k\}$$

is convex for every $k \in \mathbb{R}$.

Notice that if you choose x so that $k = u(x)$:

- the set above is the upper-contour set of x,
- the definition of quasi-concavity of the utility function coincides with the definition of convexity of preferences.

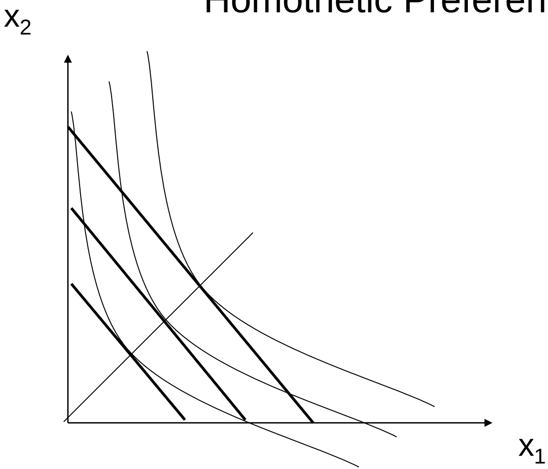
Diminishing MRS

- *Notice* that diminishing MRS is sometimes interpreted as diminishing marginal utility. This is *meaningless*.
- Indeed, given that utility function are characterized in an ordinal fashion, they are *defined up to a monotonic transformation:* the MRS is independent of monotonic transformation (proof by differentiation).
- Notice that for the same reason concavity of the utility function $u(\cdot)$ is meaningless (subsequent convex transformations of the $u(\cdot)$).

Homotheticity

- Preferences are homothetic if indifference is invariant to scaling up consumption bundles: $q^0 \sim q^1$ implies $\lambda q^0 \sim \lambda q^1$ for any $\lambda > 0$.
- This imposes no restriction on the shape of any one indifference curve, but all indifference curves have the same shape: those further out from the origin are magnified versions of those further in.
- Marginal rates of substitution are constant along rays through origin.
- Homotheticity holds if the utility function is homogeneous of degree one: $u(\lambda q) = \lambda u(q)$ for $\lambda > 0$.
- Up to increasing transformation, this is the only class of utility functions with homothetic preferences.
- Preferences are homothetic if and only if $u(q) = \phi(v(q))$ where $v(\lambda q) = \lambda v(q)$ for $\lambda > 0$.

Homothetic Preferences



Income expansion paths are rays through the origin.

Quasilinearity

- Quasilinearity implies that indifference curves all to have the same shape in the sense of being translated versions of each other.

- Indifference is invariant to adding quantities to a particular good.

- Preferences are quasilinear with respect to the i-th good if $\mathrm{q}^0 \sim \mathrm{q}^1$ implies $\mathrm{q}^0 + \lambda \mathrm{e}_i \sim \mathrm{q}^1 + \lambda \mathrm{e}_i$ for any $\lambda > 0$ and e_i is the n-vector with zeroes in all places except the i-th.

- In terms of the utility function, preferences are quasilinear if and only if $u(\mathrm{q}) = \phi(v(\mathrm{q}))$ where $v(\mathrm{q} + \lambda \mathrm{e}_i) = v(\mathrm{q}) + \lambda$ for $\lambda > 0$.

Quasi-linear Indifference Curves