EC9D3 Advanced Microeconomics, Part I: Lecture 3

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Demands for Goods with an Endowment

- *Typical example: Labour supply*

- Define the endowment of the consumer:

$$M = m + p\,\omega$$

- The Marshallian demand is then:

$$x^*(p, m) = x(p, m + p\,\omega)$$

Demands for Goods with an Endowment (2)

- Differentiation gives:

$$\frac{\partial x_l^*(p,m)}{\partial p_l} = \frac{\partial x_l(p, m + p\omega)}{\partial p_l} + \frac{\partial x_l(p, m + p\omega)}{\partial M}\omega_l$$

- Standard Slutsky decomposition gives:

$$\frac{\partial x_l(p, m + p\omega)}{\partial p_l} = \frac{\partial h_l(p, U)}{\partial p_l} - \frac{\partial x_l(p, m + p\omega)}{\partial M}x_l$$

- Substituting one in the other, and using $\partial x_l^* / \partial m = \partial x_l / \partial M$, we get:

$$\frac{\partial x_l^*(p,m)}{\partial p_l} = \frac{\partial h_l(p, U)}{\partial p_l} + \frac{\partial x_l^*(p,m)}{\partial m}[\omega_l - x_l]$$

Demands for Goods with an Endowment (3)

- If, as in the labour supply case,

$$[\omega_l - x_l] \geq 0$$

- Then we get:

$$\frac{\partial x_l}{\partial w} = \frac{\partial h_l}{\partial w} + \frac{\partial x_l}{\partial m} [\omega_l - x_l]$$

- This equation may yield *backward bending labour supply* (backward bending leisure demand) when leisure a *normal good*.

Cross-price Effects

- Commodity i and j are *net substitutes* iff

$$\frac{\partial h_j}{\partial p_i} = \frac{\partial h_i}{\partial p_j} > 0$$

- Commodity i and j are *net complements* (price effect) iff

$$\frac{\partial h_j}{\partial p_i} = \frac{\partial h_i}{\partial p_j} < 0$$

Cross-price Effects (2)

- Commodity *i is a gross substitute of j* iff

$$\frac{\partial x_i}{\partial p_j} > 0$$

- Commodity *i is a gross complement of j* iff

$$\frac{\partial x_i}{\partial p_j} < 0$$

- Notice the wording, these effects are not symmetric.

Properties of the Marshallian Demand

- The Slutsky decomposition can be used to identify a new property of the Marshallian demand.

- The substitution matrix can be written in terms of Marshallian demand:

$$S = \begin{pmatrix} \frac{\partial x_1}{\partial p_1} + x_1 \frac{\partial x_1}{\partial m} & \cdots & \frac{\partial x_1}{\partial p_L} + x_L \frac{\partial x_1}{\partial m} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_L}{\partial p_1} + x_1 \frac{\partial x_L}{\partial m} & \cdots & \frac{\partial x_L}{\partial p_L} + x_L \frac{\partial x_L}{\partial m} \end{pmatrix}$$

- Such a matrix is *symmetric* and *negative semi-definite*.

Integrability Problem

Question: Given a set of observed (Marshallian) demands $x(p, m)$ under which conditions are we sure that there exists a consumer's utility function from which these demands are derived?

Answer: The answer is that $x(p, m)$ satisfy:

1. adding up;
2. homogeneity of degree 0 in (p, m);
3. the Slutsky (substitution) matrix is symmetric and negative semi-definite.

Revealed Preferences

The integrability problem is stated in terms of observed *demand functions*, however what we actually observe is a finite set of consumer's choices.

Question: Given a finite set of *demand data:*

$$(p^1, m^1), \dots, (p^n, m^n)$$

are the consumer choices we observe

$$x^1, \dots, x^n$$

consistent with the standard model of the consumer maximizing a (quasi-concave) utility function subject to a budget constraint?

Revealed Preference Relationship

To be able to answer we need to define a new binary relationship: *revealed preference relationship.*

1. If x is chosen and $p\,x' \leq m$ then x is revealed preferred to x'

In the case in which the preferences we are trying to recover satisfy a *local non-satiation assumption* then

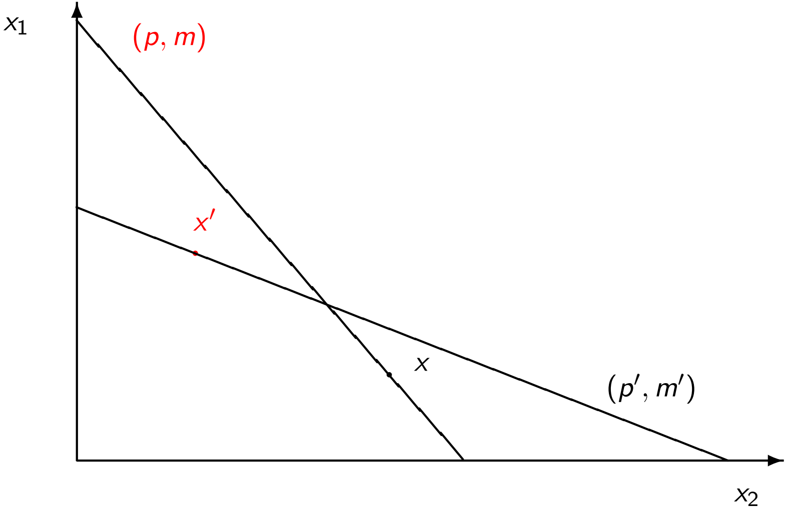
1. If x is chosen and $p\,x' < m$ then x is strictly revealed preferred to x'

Revealed Preference Argument

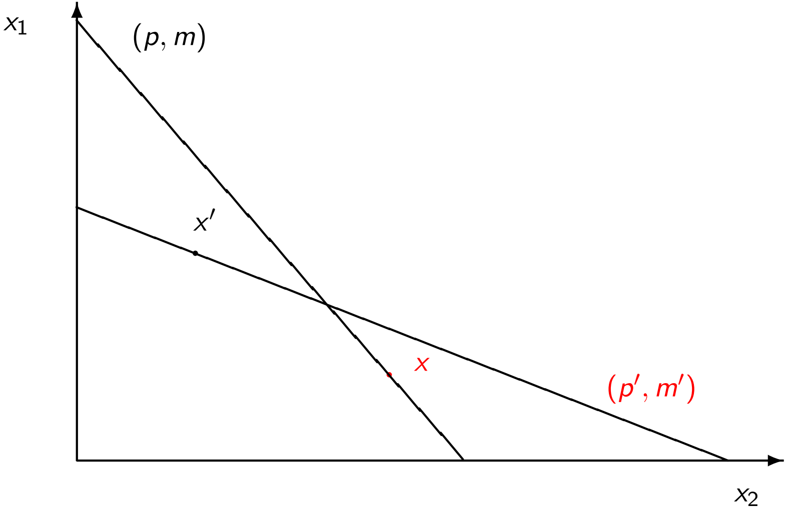
Consider two data points (x, x') such that:

- for (p, m): x chosen and $p\,x' < m$
  or x strictly revealed preferred to x';
- for (p', m'): x' chosen and $p'\,x \leq m'$
  or x' revealed preferred to x;
- We therefore have to conclude that *the data (x, x') observed are not consistent with the consumer maximizing his/her preferences* (satisf. local non-satiation) subject to budget constraint.

Revealed Preference Argument (2)



Revealed Preference Argument (2)



Revealed Preference Argument (3)

In the event, however, that:

- The two data points are such that the following relationship holds: $px' > m$ and $p'x > m'$
- Then *the information available is compatible with the consumer maximizing his preferences* (satisf. local non-satiation) subject to budget constraint.
- Of course, this is not a proof that the consumer is indeed maximizing his/her preferences.

Weak Axioms of Revealed Preferences

We now have the elements to introduce the following *weak axiom of revealed preferences.*

Axiom (Weak Axiom of Revealed Preferences)

The Marshallian demand function $x(p, m)$ satisfies the Weak Axiom of Revealed Preferences (WA) if, for any pair of price-income situations (p, m) and (p', m'), the following property holds:

$$\textit{if } p\, x(p', m') \leq m \textit{ and } x(p', m') \neq x(p, m) \textit{ then } p'\, x(p, m) > m'.$$

Therefore, *if $x(p, m)$ is weakly revealed preferred to $x(p', m')$* and they are different consumption bundles then *$x(p', m')$ cannot be weakly revealed preferred to $x(p, m)$*.

Weak Axioms of Revealed Preferences (2)

- The best use of the WA requires a special kind of price changes (eliminate income effects).
- Consider a change in price from p to p' accompanied by an associated change in income (from m to m') that makes the consumer's initial consumption bundle just affordable at the new prices p'.
- That is, the income level m' is such that $m' = p'x(p, m)$ or the corresponding *income m' changes so that*

$$\Delta m = \Delta p \, x(p, m)$$

where $\Delta p = (p' - p)$ and $\Delta m = (m' - m)$.

Weak Axioms of Revealed Preferences (3)

The change Δm is known as *Slutsky income compensation* and Δp *Slutsky income compensated price changes*.

Result

Suppose that the demand function $x(p, m)$ satisfies:

- *homogeneity of degree zero,*
- *the underlying preferences are monotonic (locally non-satiated),*

then $x(p, m)$ satisfies the weak axiom of revealed preferences if and only if for any compensated price change from (p, m) to $(p', p'x(p, m))$ we have:

$$(p' - p)[x(p', m') - x(p, m)] \leq 0$$

with strict inequality whenever $x(p, m) \neq x(p', m')$.

Weak Axioms of Revealed Preferences (4)

Proof: Assume WA holds. Consider the strict inequality result.

We can rewrite the condition as:

$$(p' - p)[x(p', m') - x(p, m)] =$$

$$p'[x(p', m') - x(p, m)] - p[x(p', m') - x(p, m)]$$

Consider the first term, we know $p'x(p, m) = m'$ and by monotonicity we get $p'x(p', m') = m'$ therefore

$$p'[x(p', m') - x(p, m)] = 0.$$

Weak Axioms of Revealed Preferences (5)

Consider the second term. By construction $x(p,m)$ is affordable under p', the WA therefore implies:

$$p\,x(p',m') > m$$

since

$$p\,x(p,m) = m$$

we conclude:

$$p[x(p',m') - x(p,m)] > 0$$

which implies:

$$(p'-p)[x(p',m') - x(p,m)] < 0.$$

Weak Axioms of Revealed Preferences (6)

The opposite implication follows from the observation that the WA holds if it holds for every compensated price change.

Assume that this is not the case.

There exists a compensated price change from (p', m') to (p, m), such that $p\,x(p', m') = m$, and:

$$x(p, m) \neq x(p', m') \quad \text{and} \quad p'x(p, m) \leq m'.$$

By monotonicity:

$$p[x(p', m') - x(p, m)] = 0 \quad \text{and} \quad p'[x(p', m') - x(p, m)] \geq 0.$$

Hence:

$$(p' - p)[x(p', m') - x(p, m)] \geq 0$$

which is a contradiction. $\square$

Compensated Law of Demand

The inequality we have obtained can be written as:

$$\Delta p \, \Delta x \leq 0$$

This is known as the *compensated law of demand*.

When $x(p, m)$ is differentiable the compensated law of demand becomes:

$$dp \, dx \leq 0$$

where $dm = dp \, x(p, m)$.

We then obtain:

$$dx = D_p x(p, m) dp^T + D_m x(p, m) dm$$

Compensated Law of Demand (2)

or

$$dx = D_p x(p,m) dp^T + D_m x(p,m) dp\, x(p,m)$$

or

$$dx = \left[D_p x(p,m) + D_m x(p,m) x(p,m)^T \right] dp^T$$

which from $dp\, dx \leq 0$ gives us:

$$dp \left[D_p x(p,m) + D_m x(p,m) x(p,m)^T \right] dp^T \leq 0$$

$$dp\, S(p,m)\, dp^T \leq 0$$

for every dp.

Compensated Law of Demand (3)

This statement is equivalent to:

- *the WA holds if and only if the substitution matrix is negative semi-definite.*

Recall however that for the consumer's utility function consumer, from which the observed choices are derived, to exists we need:

- the substitution matrix to be negative semi-definite and *symmetric*.

Strong Axiom of Revealed Preferences

Question: what is a set of necessary and sufficient conditions that rationalize demand behavior as derived from a consumer max utility subject to budget constraint?

Answer: Strong Axiom of Revealed Preferences.

Strong Axiom of Revealed Preferences (2)

Axiom (Strong Axiom of Revealed Preferences)

The demand $x(p, m)$ satisfies the SA if and only if for any list

$$(p^1, m^1), \ldots, (p^N, m^N)$$

with $x(p^{n+1}, m^{n+1}) \neq x(p^n, m^n)$ for all $n \leq N - 1$ we have:

$$p^N x(p^1, m^1) > m^N$$

whenever for any $n \leq N - 1$:

$$p^n x(p^{n+1}, m^{n+1}) \leq m^n$$

If $x(p^1, m^1)$ is directly or indirectly revealed preferred to $x(p^N, m^N)$ then $x(p^N, m^N)$ cannot be directly or indirectly revealed preferred to $x(p^1, m^1)$.

Strong Axiom of Revealed Preferences (3)

Essentially SA implies that given *any* finite set of demand data, it is not possible to construct a cycle of the type:

$$x^{n_1} \text{ r.p. } x^{n_2} \text{ r.p. } \ldots \text{ r.p. } x^{n_1}$$

where r.p. is strict in at least one case.

Theorem

SA is both a necessary and sufficient condition for the existence of an underlying utility function (justifying observed choices) that has a symmetric and negative semi-definite substitution matrix.

Strong Axiom of Revealed Preferences (4)

Using the set of conditions specified above we can conclude that:

Result

For an homogeneous of degree zero Marshallian demand that satisfies adding up conditions the SA — also known as the Generalized Axiom of Revealed Preferences (GARP) — is equivalent to the symmetry and negative semi-definiteness of the substitution matrix.

The hard part of the proof is sufficiency.

Instead of proving this result we consider the following example of how to use GARP.

How to use GARP

Consider the following data:

	$x^1 = \begin{pmatrix} 10 \\ 10 \\ 10 \end{pmatrix}$	$x^2 = \begin{pmatrix} 9 \\ 25 \\ 7.5 \end{pmatrix}$	$x^3 = \begin{pmatrix} 15 \\ 5 \\ 9 \end{pmatrix}$
$p^1 = (10, 10, 10)$	300	415	290
$p^2 = (10, 1, 2)$	130	130	173
$p^3 = (1, 1, 10)$	120	109	110

where $m^1 = 300$, $m^2 = 130$ and $m^3 = 110$.

How to use GARP (2)

- Notice that in period $t = 1$ the price was p_1, x_1 was chosen but x_3 was affordable:
$$x^1 \text{ s.r.p. } x^3.$$
- In period $t = 2$ the price was p_2, x_2 was chosen but x_1 was affordable:
$$x^2 \text{ w.r.p. } x^1.$$
- In period $t = 3$ the price was p_3, x_3 was chosen but x_2 was affordable:
$$x^3 \text{ s.r.p. } x^2.$$
- Hence
$$x^2 \text{ w.r.p. } x^1 \text{ s.r.p. } x^3 \text{ s.r.p. } x^2$$
- which *violates GARP* (but satisfies WA).

Compensated Variations

- Assume that p changes from p^0 to p^1 (price decrease $p^0 \geq p^1$).
- We cannot measure the consumer's gain in terms of utility (utility is not cardinal) however we can ask either of these alternative questions:

1. At the new price level p^1 what change in income would restore the original level of utility for the consumer?

   This change in income is known as *compensating variation CV.*

   CV is implicitly defined by:

   $$V(p^0, m) = V(p^1, m - \text{ CV })$$

   Notice that is $p^0 \geq p^1$ then CV > 0.

Equivalent Variations

2. At the old price level p^0 what change in income would induce the new level of utility for the consumer?

This change is known as *equivalent variation EV.*

EV is implicitly defined by:

$$V(p^1, m) = V(p^0, m + \text{ EV })$$

Notice that is $p^0 \geq p^1$ then EV > 0.

Compensated and Equivalent Variations

- CV and EV can be defined through the expenditure function.

$$CV = e(p^0, u^0) - e(p^1, u^0)$$

where u^0 is the level of utility achieved when $p = p^0$

$$EV = e(p^0, u^1) - e(p^1, u^1)$$

where u^1 is the level of utility when $p = p^1$.

These two measures refer to rather different situations:

- CV suitable to compensate individuals once a project has gone ahead;
- EV useful to compare in advance the effect of different projects.

Compensated and Equivalent Variations (2)

- By Shephard's lemma, we can write

$$
\begin{aligned}
CV &= \sum_{l=1}^{L} \int_{p_l^1}^{p_l^0} \frac{\partial e(p, u^0)}{\partial p_l} dp_l = \sum_{l=1}^{L} \int_{p_l^1}^{p_l^0} h_l(p, u^0) dp_l \\
EV &= \sum_{l=1}^{L} \int_{p_l^1}^{p_l^0} h_l(p, u^1) dp_l
\end{aligned}
$$

- When only one price p_l changes, from p_l^0 to $p_l^1 \leq p_l^0$:

$$
CV = \int_{p_l^1}^{p_l^0} h_l(p, u^0) dp_l, \qquad EV = \int_{p_l^1}^{p_l^0} h_l(p, u^1) dp_l
$$

Consumer Surplus

- Define the *consumer surplus* for commodity l when the price is p_l:

$$\text{CS} \;=\int_{p_l}^{\bar{p}_l} x_l(p,m)dp_l$$

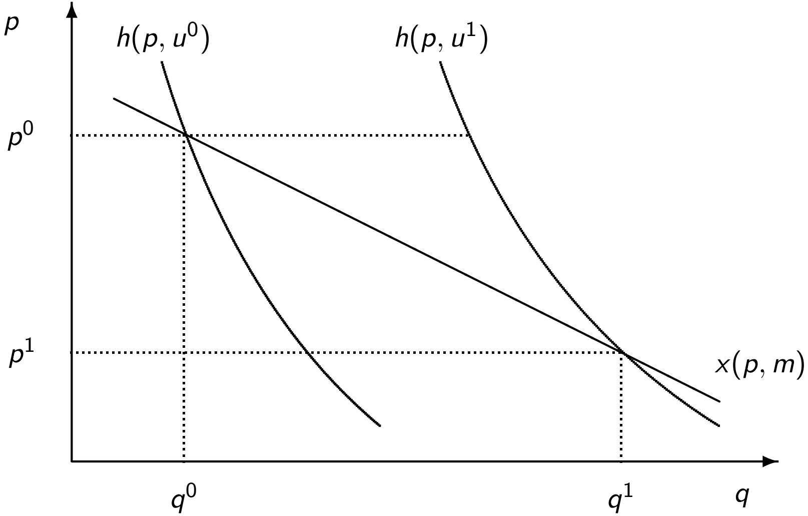
- If commodity l is normal the Marshallian demand is more steep than the Hicksian demand (by Slutsky)

$$\frac{\partial x_l}{\partial p_l}-\frac{\partial h_l}{\partial p_l}=-\frac{\partial x_l}{\partial m}\,x_l<0$$

- For a *normal* good:

$$\text{CV} \;<\Delta\text{CS} \;<\; \text{EV}$$

CV, EV and ΔCS



CV, EV and ΔCS (2)

- For an *inferior* good:

$$\text{CV} \; > \Delta\text{CS} \; > \; \text{EV}$$

- When *the income effect is zero:*

$$\text{CV} \; = \Delta\text{CS} \; = \; \text{EV}$$

this is the case for quasi-linear utility functions

$$u(x_1, x_2) = u(x_1) + x_2$$

where

$$\frac{\partial x_1}{\partial m} = 0$$

Uncertainty

- Choice under uncertainty is modeled by treating commodities consumed in different possible states of the world, $s = 0, 1, ..., S$, as different goods, q_s.
- Interpret q_s more generally as a description of any feature of s, not only quantities of commodities consumed.
- Let $q = (q_0, q_1, ..., q_S)$ be the vector of outcomes.
- The budget constraint links what can be consumed in different states.
- It is determined by means for transferring wealth between states of the world such as insurance, gambling, risky investment.
- The relative prices on consumption in different states of the world are then set by the premia in insurance contracts, betting odds and so on.

Expected utility

- The probabilities π_s of states $s = 0, 1, ..., S$, enter not the budget constraint but preferences.
- Define preferences over combinations of vectors of outcomes and probabilities known as lotteries, gambles or prospects.
- A simple lottery $L = (\mathrm{q}, \pi)$ is a list of outcomes q and associated probabilities π.
- This is a context in which separability assumptions are often regarded as extremely persuasive.

Expected utility (2)

- Consider the choice between two lotteries:

$$L^0 = \left(q_0^0, q_1, ..., q_S, \pi\right), \quad L^1 = \left(q_0^1, q_1, ..., q_S, \pi\right).$$

- The outcome is the same in states other than $s = 0$.
- These outcomes should not matter to the choice.
- If $L^0 \succeq L^1$, then this should be so for any $q_1, ..., q_S$.
- This is the sure thing principle (because it says choice should ignore outcomes in states of the world where the outcome is a "sure thing").
- The sure thing principle is a strong separability assumption.

Expected utility (3)

- The sure thing principle implies that preferences have an additive utility representation

$$u(L) = \sum_{s} v_s(q_s, \pi).$$

- The sure thing principle implies the MRS between outcomes in two different states be independent of outcomes in any third state.
- The sure thing principle rules out considerations such as potential for regret.
- The assumption is incompatible with behaviour in examples such as the Allais paradox.

Expected utility (4)

- Assume that the preference relation is continuous in the probabilities.
- Let's consider combinations of lotteries known as compound lotteries.
- Let $\pi \circ L^0 + (1-\pi) \circ L^1$ denote the lottery which gives a chance π of entering lottery L^0 and a chance $(1-\pi)$ of entering lottery L^1.
- The betweenness axiom says that

  if $L^0 \succeq L^1$ then $L^0 \succeq \pi \circ L^0 + (1-\pi) \circ L^1 \succeq L^1$.

  Any compound lottery mixing the two lies between them in the consumer's preference ordering.
- This implies linearity of indifference curves in probability space since

  if $L^0 \sim L^1$ then $u\left(L^0\right) = u\left(L^1\right) = \pi u\left(L^0\right) + (1-\pi)\, u\left(L^1\right) = u\left(\pi \circ L^0 + (1-\pi) \circ L^1\right)$.

Expected utility (5)

- If we combine the betweenness axiom with the sure thing principle then we get the strong independence axiom:

  $L^0 \succeq L^1$ if and only if $\pi \circ L^0 + (1-\pi) \circ L^2 \succeq \pi \circ L^1 + (1-\pi) \circ L^2$ for any third lottery L^2.

- Given strong independence, preferences are both additive across states and linear in probabilities, and take the expected utility form:

$$u(L) = \sum_s \pi_s v_s(\mathrm{q}_s).$$

- If we add the assumption that the description of the state of the world s is irrelevant to the utility gained from the bundle q_s, then

$$u(L) = \sum_s \pi_s v(\mathrm{q}_s).$$

Risk Aversion

- Suppose the outcomes are monetary amounts, y_s, so that

$$u(L) = \sum_s \pi_s v(y_s).$$

- Attitudes towards risk are captured through the Bernoulli utility function $v(\cdot)$.
- One is risk averse if she prefers to take the expected monetary value of a gamble with certainty to participating in the gamble.
- Thus a person is risk averse if

$$Ev(y_s) = \sum_s \pi_s v(y_s) \leq v(Ey_s) = v\left(\sum_s \pi_s y_s\right)$$

for any monetary lottery.

Risk Aversion (2)

- Taking a case with two outcomes, for $0 \leq \pi \leq 1$,

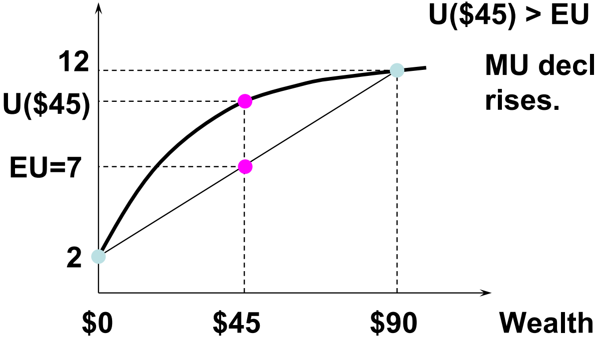
$$\pi v(y_0) + (1-\pi) v(y_1) \leq v(\pi y_0 + (1-\pi) y_1).$$

- Risk aversion is represented by concavity of the Bernoulli utility function $v(\cdot)$.

- Concavity is not a property preserved under arbitrary increasing transformations of $v(\cdot)$.

- If two alternative expected utility functions represent the same preferences then it must be that the Bernoulli utility functions are affine, i.e. linear, transformations of each other.

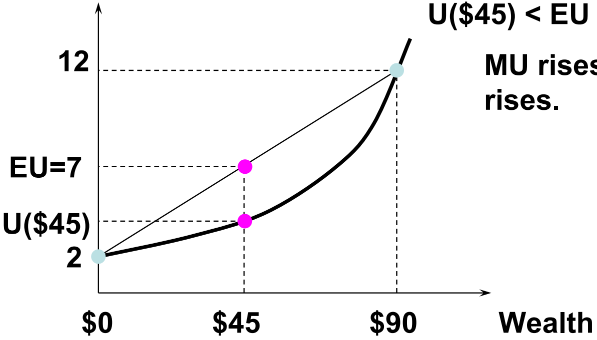
Preferences Under Uncertainty



U($45) > EU ⇒ risk-aversion.

MU declines as wealth rises.

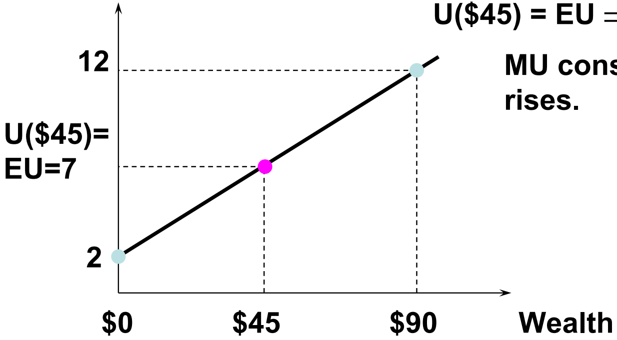
Preferences Under Uncertainty



U($45) < EU ⇒ risk-loving.

MU rises as wealth rises.

Preferences Under Uncertainty



U($45) = EU ⇒ risk-neutrality.

MU constant as wealth rises.

Certainty Equivalent and Risk Premium

- We can evaluate the degree of someone's risk aversion by asking how much they would be prepared to pay to avoid a gamble.
- The certainty equivalent to a monetary gamble, M, is the amount which if received with certainty would give the same expected utility as the actual gamble: $v(M) = Ev(y)$.
- The risk premium, $m = Ey - M$, is the difference between the expected monetary value of the gamble and the certainty equivalent.
- The risk premium is what one would pay to avoid the gamble.

Coefficient of Absolute Risk Aversion

- Taking Taylor expansions:

$$v\left(Ey-m\right)\approx v\left(Ey\right)-mv'\left(Ey\right).$$

$$\begin{aligned}
Ev\left(y\right) &\approx v\left(Ey\right)+E\left(y-Ey\right)v'\left(Ey\right)+\frac{1}{2}E\left(y-Ey\right)^{2}v''\left(Ey\right)\\
&= v\left(Ey\right)+\frac{1}{2}Var\left(y\right)v''\left(Ey\right)\\
\text{so that } m &\approx -\frac{1}{2}\frac{v''\left(Ey\right)}{v'\left(Ey\right)}Var\left(y\right).
\end{aligned}$$

- The ratio $R(y)=-v''\left(Ey\right)/v'\left(Ey\right)$ is the Arrow-Pratt coefficient of absolute risk aversion.

Risk Aversion (3)

- Three equivalent definitions of risk aversion ranking:

  1. the more risk averse individual has a Bernoulli utility function which is an increasing concave transformation of that of the other

  2. the more risk averse individual has a higher risk premium for any gamble

  3. the more risk averse individual has a higher coefficient of absolute risk aversion at all y.

- The coefficient of relative risk aversion is $r(y) = R(y)y$.

- Just as $m \approx \frac{1}{2} R(y) \, Var(y)$, it is also the case that $m/Ey = \frac{1}{2} r(y) \, Var(y) / (Ey)^2$.