EC9D3 Advanced Microeconomics, Part I: Lecture 6

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August, 2020

Pure Exchange Economy

A general pure exchange economy with I consumers is characterized by the following elements:

- i's endowment vectors:

$$\omega^i = \begin{pmatrix} \omega_1^i \\ \vdots \\ \omega_L^i \end{pmatrix};$$

- i's (locally-non-satiated) preferences represented by a utility function

$$u_i(\cdot).$$

Pure Exchange Economy (2)

- Denote the total endowment of each commodity l as

$$\bar{\omega}_l = \sum_{i=1}^{I} \omega_l^i \qquad \forall l \in \{1, \ldots, L\}$$

- Denote consumer i's *excess demand vector* for any given distribution of endowments $\omega = \{\omega^1, \ldots, \omega^I\}$ to be:

$$z^i(p) = \begin{pmatrix} x_1^i(p) - \omega_1^i \\ \vdots \\ x_L^i(p) - \omega_L^i \end{pmatrix}$$

Pure Exchange Economy (3)

- Denote the vector of *aggregate excess demands* as

$$Z(p) = \begin{pmatrix} Z_1(p) = \sum_{i=1}^{I} z_1^i(p) \\ \vdots \\ Z_L(p) = \sum_{i=1}^{I} z_L^i(p) \end{pmatrix}$$

- In this pure exchange economy we can define a *Walrasian equilibrium* by means of the vector of aggregate excess demands in the following manner.

Pure Exchange Economy (4)

Definition (Walrasian equilibrium)

It is defined by a vector of prices p^* and an induced allocation $x^* = \{x^{1,*}(p^*), \ldots, x^{I,*}(p^*)\}$ such that all *markets clear:*

$$Z(p^*) = 0$$

or for every $l = 1, \ldots, L$:

$$Z_l(p^*) = \sum_{i=1}^{I} \left(x_l^{i,*}(p^*) - \omega_l^i \right) = 0$$

These L equations are not all independent, the reason being *Walras Law*.

Pure Exchange Economy (4)

- Indeed, each consumer Marshallian demand $x^{i,*}(p)$ will be such that the consumer's budget constraint will be binding:

$$p^* x^{i,*}(p^*) = p^* \omega^i$$

- If we sum these budget constraint across the consumers we get:

$$\sum_{i=1}^{I} p^* x^{i,*}(p^*) = \sum_{i=1}^{I} p^* \omega^i$$

or

$$p^* \; Z(p^*) = 0$$

- This condition introduces a degree of freedom in the equilibrium price determination: if $L-1$ markets clear the L-th market also clears.

Walrasina Equilibrium in a Pure Exchange Economy

- An old approach to general equilibrium analysis consisted in counting equations and unknowns.
- A modern approach is the one introduced by Debreu (1959).
- It starts from an *alternative definition of Walrasian equilibrium.*

Definition (Walrasian Equilibrium)

A *Walrasian equilibrium* is a vector of prices p^* and an allocation of resources x^* associated to p^* such that:

$$Z(p^*) \leq 0$$

Walrasina Equilibrium in a Pure Exchange Economy (2)

Given the definition above we can prove the following Lemma.

Lemma

The Walrasian equilibrium price is such that $p_l \geq 0 \ \forall l \in \{1, \ldots, L\}$.

Proof: Assume by way of contradiction that there exists l such that $p_l < 0$. The utility maximization problem is then:

$$\max_{x} \quad u(x)$$

$$\text{s.t.} \quad \sum_{h \neq l} p_h \, x_h \leq m - p_l \, x_l$$

If $x_l > 0$ then $p_l \, x_l < 0$ therefore by increasing x_l we do not decrease the objective function $u(x)$.

Walrasina Equilibrium in a Pure Exchange Economy (3)

We can then increase x_h, $h \neq l$ also unboundedly and $u(x) \to +\infty$.

A contradiction to the existence of a solution to the utility maximization problem. □

Lemma

Let $\{p^*, x^*\}$ be a Walrasian equilibrium then:

1. *if $p_l^* > 0$ then $Z_l(p^*) = 0$;*
2. *if $Z_l(p^*) < 0$ then $p_l^* = 0$.*

Walrasina Equilibrium in a Pure Exchange Economy (4)

Proof: Walras Law implies that

$$p^* \; Z(p^*) = 0.$$

or

$$\sum_{l=1}^{L} p_l^* \; Z_l(p^*) = 0.$$

By the previous lemma $p_l^* \geq 0$ while by the definition of Walrasian equilibrium we have

$$Z_l(p^*) \leq 0$$

From here the result. □

We address next the problem of *existence of a general equilibrium.*

Existence of General Equilibrium

Definition (Fixed Point)

Consider a mapping $F : \mathbb{R}^L \to \mathbb{R}^L$, any x^* such that

$$x^* = F(x^*)$$

is a *fixed point* of the mapping F.

Theorem (Brouwer Fixed Point Theorem)

Let S be a compact and convex set, and

$$F : S \to S$$

a continuous mapping from S into itself. Then the mapping F has at least one fixed point in S.

Existence of General Equilibrium (2)

Consider a *pure exchange economy* without any *externality*.

Let $Z(p)$ be the vector of excess demands that satisfies the following assumptions on $Z(p)$:

1. $Z(p)$ is *single valued* (it is a function).
2. $Z(p)$ is *continuous*.
3. $Z(p)$ is *bounded*.
4. $Z(p)$ is *homogeneous of degree 0*.
5. *Walras Law:* $p\,Z(p) = 0$.

Existence of General Equilibrium (3)

Theorem (Existence Theorem of Walrasian Equilibrium)

Under assumptions 1–5 there exists a Walrasian Equilibrium price vector p^* and an allocation x^* such that

$$Z(p^*) \leq 0.$$

Proof: Let us normalize the set of prices we consider (Walras Law leaves us a degree of freedom in solving for the WE price vector p^*).

Consider the prices in the L dimensional Simplex:

$$S = \left\{ p \mid p \geq 0, \sum_{l=1}^{L} p_l = 1 \right\}$$

Existence of General Equilibrium (4)

Notice that S is *compact* and *convex*. The strategy of the reminder of the proof is then:

- Define a continuous mapping from the *Simplex* S into itself.
- Use Brower Fixed Point Theorem to obtain a fixed point of such mapping.
- Show that such a fixed point is indeed a Walrasian Equilibrium price vector.

Existence of General Equilibrium (5)

Let $\beta > 0$ and define

$$t_l(p) = \max\{0,\ p_l + \beta\ Z_l(p)\}$$

which we normalize to be in S:

$$q_l(p) = \frac{t_l}{\sum_{l=1}^{L} t_l}$$

The mapping from p into q is *continuous* by construction.

Existence of General Equilibrium (6)

Indeed,

- the mapping from p to $t(p)$ is continuous:
    - $p_l + \beta \, Z_l(p)$ is continuous in p by assumption 2;
    - a constant function is clearly continuous;
    - the maximum of two continuous functions is also continuous.
- the mapping from t to $q(p)$ is continuous provided that $\sum_{l=1}^{L} t_l \neq 0$.

Existence of General Equilibrium (7)

Lemma

It is the case that

$$\sum_{l=1}^{L} t_l \neq 0.$$

Proof: Notice that by construction $t_l \geq 0$ for every $l = 1, \ldots, L$.

Therefore $\sum_{l=1}^{L} t_l = 0$ if and only if $t_l = 0$ for every $l = 1, \ldots, L$.

Assume that this is the case.

Recall that

$$t_l(p) = \max\{0,\ p_l + \beta\ Z_l(p)\}$$

Existence of General Equilibrium (8)

From the very first Lemma above we know that $p_l \geq 0$ therefore

- for every l such that $p_l = 0$ for $t_l = 0$ we need $Z_l(p) \leq 0$.
- for every l such that $p_l > 0$ for $t_l = 0$ we need $Z_l(p) < 0$.

However, the latter case contradicts Walras Law:

Denote

$$A(p) = \{l \leq L \mid p_l = 0\},$$

and

$$B(p) = \{l \leq L \mid p_l > 0\},$$

Existence of General Equilibrium (9)

By Walras Law:

$$0 = \sum_{l=1}^{L} p_l Z_l(p) = \sum_{l \in A(p)} p_l Z_l(p) + \sum_{l \in B(p)} p_l Z_l(p)$$

Since by definition of $A(p)$

$$\sum_{l \in A(p)} p_l Z_l(p) = 0$$

Walras Law implies:

$$\sum_{l \in B(p)} p_l Z_l(p) = 0.$$

This is a contradiction of $p_l > 0$ and $Z_l(p) < 0$ for every $l \in B(p)$. □

Existence of General Equilibrium (10)

Therefore the mapping from p into q is continuous and maps a compact and convex set in itself.

Brower Fixed Point Theorem applies which means that *there exists a fixed point p^* such that $q(p^*) = p^*$.*

We still need to show that such a point is a Walrasian Equilibrium price vector.

Consider first $l \in A(p^*)$ then $p_l^* = 0$ by definition of $A(p^*)$.

Further, being p^* a fixed point $q_l(p^*) = p_l^* = 0$ which implies by definition of $t_l(p^*)$ and boundedness of $Z(p)$ that $t_l(p^*) = 0$, hence $Z_l(p^*) \leq 0$.

Existence of General Equilibrium (11)

Therefore $Z_l(p^*) \leq 0$ for every $l \in A(p^*)$.

Consider now $l \in B(p^*)$ then $p_l^* > 0$ by definition of $B(p^*)$.

Therefore by definition of $t_l(p^*)$:

$$q_l(p^*) = p_l^* = \frac{p_l^* + \beta Z_l(p^*)}{\sum_{l \in B(p^*)} t_l(p^*)}$$

multiplying both sides by $Z_l(p^*)$ we get:

$$p_l^* \; Z_l(p^*) = \frac{p_l^* \; Z_l(p^*) + \beta [Z_l(p^*)]^2}{\sum_{l \in B(p^*)} t_l(p^*)}$$

Existence of General Equilibrium (12)

which summed over $l \in B(p^*)$ gives:

$$\sum_{l \in B(p^*)} p_l^* \; Z_l(p^*) = \frac{\sum_{l \in B(p^*)} p_l^* \; Z_l(p^*) + \beta \sum_{l \in B(p^*)} [Z_l(p^*)]^2}{\sum_{l \in B(p^*)} t_l(p^*)}.$$

Walras Law

$$\sum_{l \in B(p^*)} p_l^* \; Z_l(p^*) = 0 \quad \Rightarrow \quad \frac{\beta \sum_{l \in B(p^*)} [Z_l(p^*)]^2}{\sum_{l \in B(p^*)} t_l(p^*)} = 0$$

From Lemma 2 and $t_l(p^*) = 0$ for every $l \in A(p^*)$

$$\sum_{l=1}^{L} t_l = \sum_{l \in A(p^*)} t_l + \sum_{l \in B(p^*)} t_l = \sum_{l \in B(p^*)} t_l \neq 0 \quad \Rightarrow \quad \sum_{l \in B(p^*)} [Z_l(p^*)]^2 = 0$$

or $Z_l(p^*) = 0$ for every $l \in B(p^*)$. □

Existence of General Equilibrium (13)

In other words, we have proved that under assumptions 1–5 there exists a Walrasian Equilibrium price vector p^* and an allocation $x^*(p^*)$ such that:

- for every $l \in A(p^*)$ — for every l such that $p_l^* = 0$ — we have that

$$Z_l(p^*) \leq 0$$

- while for every $l \in B(p^*)$ — for every l such that $p_l^* > 0$ — we have that

$$Z_l(p^*) = 0$$

Notice that in equilibrium there exist excess demands only of commodities that are free (whose equilibrium price is zero).

Properties of Walrasian Equilibrium

Recall that $x = \{x^1, \ldots, x^I\}$ denotes an allocation.

Definition

An allocation x *Pareto dominates* an alternative allocation $\bar{x}$ if and only if:

$$u_i(x^i) \geq u_i(\bar{x}^i) \quad \forall i \in \{1, \ldots, I\}$$

and for some i:

$$u_i(x^i) > u_i(\bar{x}^i).$$

Pareto Efficiency

In other words, the allocation x makes no one worse-off and someone strictly better-off.

Definition

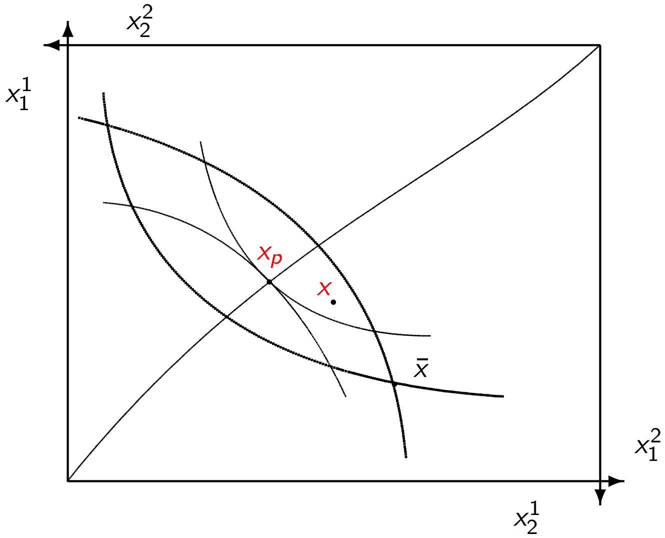
An allocation x is *feasible* in a pure exchange economy if and only if:

$$\sum_{i=1}^{I} x_l^i \leq \bar{\omega}_l \qquad \forall l \in \{1, \ldots, L\}.$$

Definition

An allocation x is *Pareto efficient* if and only if it is *feasible* and there does *not* exist an other feasible allocation that Pareto-dominates x.

Pareto Efficiency (2)



Pareto Efficiency (3)

A standard way to identify *a Pareto-efficient allocation* is to introduce a *benevolent central planner* that has the authority to re-allocate resources across consumers so as to exhaust any gains-from-trade available.

Result

An allocation x^* is Pareto-efficient if there exists a vector of weights $\lambda = (\lambda^1, \ldots, \lambda^I)$ such that x^* solves the following problem:

$$\begin{aligned} \max_{x^1,\ldots,x^I} \quad & \sum_{i=1}^{I} \lambda^i \, u_i(x^i) \\ s.t \quad & \sum_{i=1}^{I} x^i \leq \bar{\omega} \end{aligned} \tag{1}$$

Pareto Efficiency (4)

Proof: We start from the *only if:*

Assume by way of contradiction that the allocation $\hat{x}$ that solves (1) is not Pareto efficient.

Then there exists a feasible allocation $\tilde{x}$ and at least an individual i such that

$$u_i(\tilde{x}^i) > u_i(\hat{x}^i), \qquad u_j(\tilde{x}^i) \geq u_j(\hat{x}^j) \; \forall j \neq i$$

If then follows that, given $(\lambda^1, \ldots, \lambda^I)$, the allocation $\tilde{x}$ is feasible in problem (1) and achieves a higher maximand.

This observation contradicts the assumption that $\hat{x}$ solves problem (1). $\square$

We come back to the *if* later on.

First Welfare Theorem

Theorem (First Fundamental Theorem of Welfare Economics)

Consider a pure exchange economy such that consumers' preferences are weakly monotonic.

Assume that this economy is such that there exists a Walrasian equilibrium $\{p^*, x^*\}$.

Then the allocation x^* *is a Pareto-efficient allocation.*

Proof: Assume that the theorem is *not* true.

First Welfare Theorem (2)

Contradiction hypothesis: There exists an allocation x such that

$$\sum_{i=1}^{I} x^i \leq \bar{\omega}$$

and

$$u_i(x^i) \geq u_i(x^{i,*}) \quad \forall i \leq I$$

and for some $i \leq I$

$$u_i(x^i) > u_i(x^{i,*})$$

First Welfare Theorem (3)

Claim

Then

$$p^* x^i \geq p^* x^{i,*} \quad \forall i \leq I.$$

Proof: Assume that this is not true and there exists $i \leq I$ such that

$$p^* x^i < p^* x^{i,*}$$

From

$$p^* x^{i,*} = p^* \omega^i$$

we then get

$$p^* x^i < p^* \omega^i$$

First Welfare Theorem (4)

This implies that there exists $\varepsilon > 0$ such that if we denote e^T the vector $e^T = (1, \ldots, 1)$

$$p^* \left(x^i + \varepsilon\, e\right) < p^* \omega^i.$$

Monotonicity of preferences then implies that

$$u_i(x^i + \varepsilon\, e) > u_i(x^i)$$

which together with the contradiction hypothesis gives:

$$u(x^i + \varepsilon\, e) > u_i(x^{i,*})$$

This contradicts $x^{i,*} = x^i(p^*)$. □

First Welfare Theorem (5)

Claim

Since for some i we have $u_i(x^i) > u_i(x^{i,*})$ then for the same i

$$p^* \, x^i > p^* x^{i,*}.$$

Proof: Assume this is not the case.

Then there exists a consumption bundle x^i which is affordable for i:

$$p^* x^i \leq p^* x^{i,*} = p^* \, \omega^i$$

and yields a higher level of utility: $u_i(x^i) > u_i(x^{i,*})$.

This is a contradiction of the hypothesis $x^{i,*} = x^i(p^*)$. □

First Welfare Theorem (6)

Adding up these conditions across consumers we obtain:

$$\sum_{i=1}^{I} p^* x^i > \sum_{i=1}^{I} p^* x^{i,*}$$

or

$$\sum_{i=1}^{I} p^* x^i > \sum_{i=1}^{I} p^* x^{i,*} = p^* \bar{\omega}$$

a contradiction of the feasibility of the allocation x. □

Notice that the hypothesis necessary for this Theorem are not enough to guarantee the existence of a Walrasian equilibrium.

The Converse Question

So far we assumed:

- perfectly competitive markets;
- every commodity has a corresponding market (no-externalities).

Consider now the converse question.

Suppose you have a pure exchange economy and you want the consumer to achieve a given Pareto-efficient allocation.

Is there a way to achieve this allocation in a fully decentralized (hands-off) way?

Answer: *redistribution of endowments.*

Separating Hyperplane Theorem

Theorem (Separating Hyperplane Theorem)

Let A and B be two disjoint and convex set in $\mathbb{R}^N$. Then there exists a vector $p \in \mathbb{R}^N$ such that

$$p\,x \geq p\,y$$

for every $x \in A$ and every $y \in B$.

In other words there exists an hyperplane identified by the vector p that separates the set A and the set B.

Second Welfare Theorem

Theorem (Second Fundamental Theorem of Welfare Economics)

Consider a pure exchange economy with (weakly) convex, continuos and strongly monotonic consumers' preferences.

Let x^* be a Pareto-efficient allocation such that $x_l^{i,*} > 0$ for every $l \leq L$ and every $i \leq I$. Then there exists an endowment re-allocation ω' such that:

$$\sum_{i=1}^{I} \omega'^{i} = \sum_{i=1}^{I} \omega^{i}$$

and for some p^* the vector $\{p^*, x^*\}$ is a Walrasian equilibrium given ω'.

Second Welfare Theorem (2)

Proof: Consider

$$B^i = \left\{ x^i \in \mathbb{R}^L_+ \mid u_i(x^i) > u_i(x^{i,*}) \right\}$$

Notice that B^i is convex since preferences are convex by assumption (utility function is quasi-concave).

Let

$$B = \sum_{i=1}^{I} B^i = \left\{ z \in \mathbb{R}^L_+ \mid z = \sum_{i=1}^{I} x^i, x^i \in B^i \right\}$$

Second Welfare Theorem (3)

Claim

B is convex.

Proof: Take $z, z' \in B$.

Now $z \in B$ implies $z = \sum_{i=1}^{I} x^i$ and $z' \in B$ implies $z' = \sum_{i=1}^{I} x'^i$.

Therefore

$$
\begin{aligned}
[\lambda z + (1-\lambda) z'] &= \lambda \sum_{i=1}^{I} x^i + (1-\lambda) \sum_{i=1}^{I} x'^i \\
&= \sum_{i=1}^{I} [\lambda x^i + (1-\lambda) x'^i] \in B
\end{aligned}
$$

since $[\lambda x^i + (1-\lambda) x'^i] \in B^i$ by convexity of B^i. □

Second Welfare Theorem (4)

Claim

$$v = \sum_{i=1}^{I} x^{i,*} \notin B$$

Proof: Assume that this is not the case: $v \in B$.

This means that there exist I consumption bundles $\hat{x}^i \in B^i$ such that

$$v = \sum_{i=1}^{I} x^{i,*} = \sum_{i=1}^{I} \hat{x}^i.$$

Now, Pareto-efficiency of x^* implies that v is feasible:

$$v = \sum_{i=1}^{I} \hat{x}^i = \sum_{i=1}^{I} \omega^i$$

and by definition of B^i

$$u_i(\hat{x}^i) > u_i(x^{i,*})$$

for every $i \leq I$.

This contradicts the Pareto-efficiency of x^*. □

Second Welfare Theorem (6)

Claim

There exists a p^* such that:

$$p^* z \geq p^* v = p^* \sum_{i=1}^{I} x^{i,*} = p^* \sum_{i=1}^{I} \omega^i \qquad \forall z \in B$$

Proof: It follows directly from the Separating Hyperplane Theorem.

Indeed, the sets $\{v\}$ and the set B satisfy the assumptions of the theorem. □

We still need to show that the p^* we have obtained is indeed a Walrasian equilibrium.

Second Welfare Theorem (7)

Claim

$$p^* \geq 0$$

Proof: Denote $e_n^T = (0, \ldots, 0, 1, 0, \ldots, 0)$ where the digit 1 is in the n-th position, $n \leq L$.

Notice that strict monotonicity of preferences implies:

$$v + e_n \in B$$

therefore from Claim 3 we have that:

$$p^* (v + e_n) \geq p^* v$$

Second Welfare Theorem (8)

In other words:

$$p^* \ (v + e_n - v) \geq 0$$

or

$$p^* \ e_n \geq 0$$

which is equivalent to:

$$p_n^* \geq 0 \quad \square$$

Second Welfare Theorem (9)

Claim

For every consumer $i \leq I$

$$u_i(x^i) > u_i(x^{i,*})$$

implies

$$p^* \, x^i \geq p^* x^{i,*}$$

Proof: Let $\theta \in (0, 1)$. Consider the allocation

$$z^i = x^i \, (1 - \theta)$$

and

$$z^h = x^{h,*} + \frac{x^i \, \theta}{I - 1} \qquad \forall h \neq i$$

the allocation z is a redistribution of resources from i to every h.

For a small θ by strict monotonicity we have that z is Pareto-preferred to x^*. Hence by the previous Claim:

$$p^* \sum_{i=1}^{I} z^i \geq p^* \sum_{i=1}^{I} x^{i,*}$$

or

$$p^* \left[x^i(1-\theta) + \sum_{h \neq i} x^{h,*} + x^i \theta \right] = p^* \left[x^i + \sum_{h \neq i} x^{h,*} \right] \geq p^* \sum_{i=1}^{I} x^{i,*}$$

which implies $p^* x^i \geq p^* x^{i,*}$. □

Second Welfare Theorem (11)

Claim

For some agent i

$$u_i(x^i) > u_i(x^{i,*})$$

implies

$$p^* x^i > p^* x^{i,*}$$

Proof: From the previous Claim we have $p^* x^i \geq p^* x^{i,*}$. Therefore we just have to rule out $p^* x^i = p^* x^{i,*}$. Continuity of preferences implies that for some scalar ξ close to 1 we have

$$u_i(\xi\, x^i) > u_i(x^{i,*})$$

and by the previous Claim $p^* \xi\, x^i \geq p^* x^{i,*}$. If now $p^* x^i = p^* x^{i,*} > 0$ from $p^* > 0$ and $x^{i,*} > 0$ it follows that $p^* \xi\, x^i < p^* x^{i,*}$: a contradiction. □

Second Welfare Theorem (12)

The previous Claims imply that whenever $u_i(x^i) > u_i(x^{i,*})$ then $p^* x^i > p^* x^{i,*}$ with a strict inequality for some i.

This implies that the consumption bundles $x^{i,*}$ maximizes consumer i's utility subject to budget constraint.

Indeed $\sum_{i=1}^{I} p^* x^{i,*} = \sum_{i=1}^{I} p^* \omega^i$

Let now $\omega'^i = x^{i,*}$. This concludes the proof of the SWT. □

Notice that the assumptions of the Second Welfare Theorem are the same that guarantee the existence of a Walrasian equilibrium.